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ON STARLIKE FUNCTIONS ASSOCIATED WITH CARDIOID DOMAIN

Saira Zainab, Mohsan Raza, Janusz Sokół, and Sarfraz Nawaz Malik

Abstract. Analytic functions are characterized by the geometry of their image domains. That's why, geometry of image domain is of substantial importance to have a comprehensive study of analytic functions. To introduce and study new geometrical structures as image domain and to define their subsequent analytic functions is an ongoing part of research in geometric function theory. We introduced a new domain named as cardioid domain and defined the corresponding analytic function, see [**14**]. Here we further study the cardioid domain, to define and study starlike functions associated with cardioid domain.

1. Introduction and preliminaries

Let $\mathcal{A}$ be the class of functions of the form

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n, \tag{1.1}$$

which are analytic in the open unit disk $\mathbb{U} = \{z : |z| < 1\}$ and $\mathcal{S}$ be the class of functions from $\mathcal{A}$ which are univalent in open unit disk $\mathbb{U}$. The function f is said to be subordinate to the function g, written symbolically as $f \prec g$, if there exists a function w such that

$$f(z) = g(w(z)), \quad z \in \mathbb{U}, \tag{1.2}$$

where $w(0) = 0$, $|w(z)| < 1$ for $z \in \mathbb{U}$. The class $\mathcal{S}^*$ of starlike univalent functions is defined to be the set of functions $f \in \mathcal{S}$ such that

$$\frac{zf'(z)}{f(z)} \prec p(z), \tag{1.3}$$

where $p(z) \in \mathcal{P} = \{h : h(0) = 1,\ \operatorname{Re} h(z) > 0,\ z \in \mathbb{U}\}$.

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Using the concept of subordination, several subclasses of analytic functions were defined by condition (1.3) on the basis of geometrical interpretation of the image domain $p(\mathbb{U})$. Some interesting geometrical structures are like right half plane [**7**], circular disk [**9**], conic domain [**1, 10, 11, 13**], cardioid domain [**14, 22**], generalized conic domains [**16**], oval and petal type domains [**17**], leaf-like domain [**2, 3, 21, 26**], Bernoulli lemniscate [**12, 24**] and the most concerning one is the shell-like curve [**4–6, 8, 19, 20, 23, 25**]. A general approach to the classes of functions defined by subordination (1.3) can be found in [**15**].

The shell-like curve is caused by the function $p_\tau(z) = \frac{1+\tau^2 z^2}{1-\tau z-\tau^2 z^2}$, where $\tau = \frac{1-\sqrt{5}}{2}$. The image of unit circle under the function p_τ gives the conchoid of Maclaurin, also named as shell-like curve. That is,

$$p_\tau(e^{i\varphi}) = \frac{\sqrt{5}}{2(3-2\cos\varphi)} + i\frac{\sin\varphi(4\cos\varphi - 1)}{2(3-2\cos\varphi)(1+\cos\varphi)}, \quad 0 \leqslant \varphi < 2\pi.$$

The function p_τ, has the following series representation

$$p_\tau(z) = 1 + \sum_{n=1}^{\infty}(u_{n-1} + u_{n+1})\tau^n z^n, \quad \text{where} \quad u_n = \frac{(1-\tau)^n - \tau^n}{\sqrt{5}}.$$

This generates a series of coefficient constants which made it closer to Fibonacci numbers.

Getting inspiration from the concept of shell-like curves and circular disk, we define a class of analytic functions as follows.

Definition 1.1. Let $\mathcal{CP}[A,B]$ be the class of functions $p(z)$ which are defined by the subordination relation $p(z) \prec \tilde{p}(A,B;z)$, where $\tilde{p}(A,B;z)$ is defined by

$$\tilde{p}(A,B;z) = \frac{2A\tau^2 z^2 + (A-1)\tau z + 2}{2B\tau^2 z^2 + (B-1)\tau z + 2}, \tag{1.4}$$

with $-1 < B < A \leqslant 1$ and $\tau = (1-\sqrt{5})/2$, $z \in \mathbb{U}$.

For in-depth understanding of the class $\mathcal{CP}[A,B]$, it would be worthwhile here to have a geometrical description of the function $\tilde{p}(A,B;z)$ defined by (1.4). If we denote $\operatorname{Re}\tilde{p}(A,B;e^{i\theta}) = u$ and $\operatorname{Im}\tilde{p}(A,B;e^{i\theta}) = v$, then the image $\tilde{p}(A,B;e^{i\theta})$ of the unit circle is a cardioid-like curve defined by the following parametric form as

$$\begin{aligned} u(\theta) &= \frac{4 + (A-1)(B-1)\tau^2 + 4AB\tau^4 + 2\lambda\cos\theta + 4(A+B)\tau^2\cos 2\theta}{4 + (B-1)^2\tau^2 + 4B^2\tau^4 + 4(B-1)(\tau + B\tau^3)\cos\theta + 8B\tau^2\cos 2\theta}, \\ v(\theta) &= \frac{2(A-B)[(\tau - \tau^3)\sin\theta + 2\tau^2\sin 2\theta]}{4 + (B-1)^2\tau^2 + 4B^2\tau^4 + 4(B-1)(\tau + B\tau^3)\cos\theta + 8B\tau^2\cos 2\theta}, \end{aligned} \tag{1.5}$$

where $\lambda = (A+B-2)\tau + (2AB - A - B)\tau^3$, and $0 \leqslant \theta < 2\pi$.

Furthermore, we note that $\tilde{p}(A,B;0) = 1$ and

$$\tilde{p}(A,B;1) = \frac{AB + 9(A+B) + 1 + 4(B-A)\sqrt{5}}{B^2 + 18B + 1}.$$

The cusp of the cardioid-like curve, defined by (1.5), is given by

$$\gamma(A,B) = \tilde{p}(A,B;e^{\pm i \arccos(1/4)}) = \frac{2AB - 3(A+B) + 2 + (A-B)\sqrt{5}}{2(B^2 - 3B + 1)}.$$

The above discussed cardioid-like curve with different values of parameters can be seen in Figure 1.

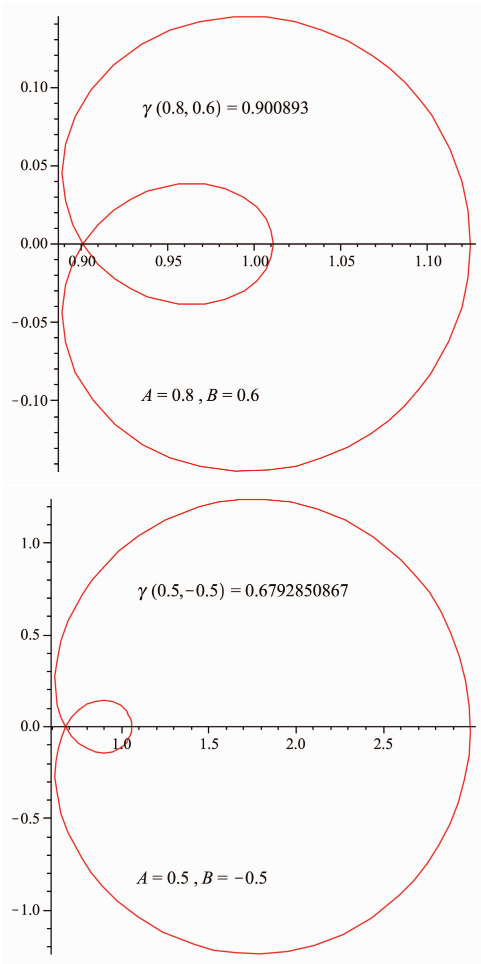

Figure 1. Selected cases of curve (1.5)

If we consider the open unit disk $\mathbb{U}$ as the collection of concentric circles having origin as center, then the image of each inner circle is a nested cardioid-like curve. Therefore, the function $\tilde{p}(A,B;z)$ maps the open unit disk $\mathbb{U}$ onto a cardioid region. That is, $\tilde{p}(A,B;\mathbb{U})$ is a cardioid domain as shown in Figure 2.

For more details, see [**14**].

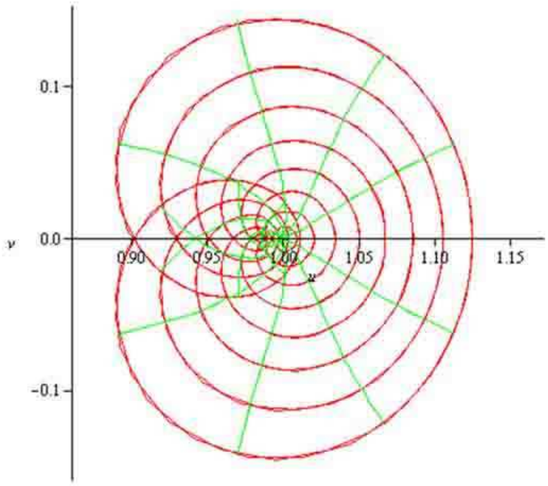

FIGURE 2. The image $\tilde{p}(0.8, 0.6; \mathbb{U})$

LEMMA 1.1. [**14**] *Consider the function* $\tilde{p}(A, B; z)$ *defined by* (1.4). *Then*

- *the function* $\tilde{p}(A, B; z)$ *is univalent in the disk* $|z| < \tau^2 = 0.38\ldots$,
- *if* $p(z) \prec \tilde{p}(A, B; z)$, *then* $\operatorname{Re} p(z) > \alpha$, *where*

$$\alpha = \frac{2(A+B-2)\tau + 2(2AB - A - B)\tau^3 + 16(A+B)\tau^2\eta}{4(B-1)(\tau + B\tau^3) + 32B\tau^2\eta}, \tag{1.6}$$

where $\eta = \frac{4+\tau^2-B^2\tau^2-4B^2\tau^4-(1-B\tau^2)\sqrt{5(2B\tau^2-(B-1)\tau+2)(2B\tau^2+(B-1)\tau+2)}}{4\tau(1+B^2t^2)}$,

- *if* $\tilde{p}(A, B; z) = 1 + \sum_{n=1}^{\infty} p_n z^n$, *then*

$$p_n = \begin{cases} (A-B)\frac{\tau}{2}, & \text{for } n = 1, \\ (A-B)(5-B)\frac{\tau^2}{2^2}, & \text{for } n = 2, \\ \frac{1-B}{2}\tau p_{n-1} - B\tau^2 p_{n-2}, & \text{for } n = 3, 4, 5, \ldots. \end{cases} \tag{1.7}$$

Some further properties of functions p, such that $p \prec \tilde{p}$ can be deduced from a general approach to this subordination, see [**15**].

2. Main Results

Now we define the class of starlike functions associated with cardioid domain.

DEFINITION 2.1. The class of starlike functions associated with cardioid domain, denoted by $\mathcal{CS}^*[A, B]$, is defined to be the set of functions $f \in \mathcal{A}$ such that

$$\frac{zf'(z)}{f(z)} \prec \tilde{p}(A, B; z), \tag{2.1}$$

where $\tilde{p}(A, B; z)$ is defined by (1.4).

In other words, the function f will belong to the class $\mathcal{CS}^*[A,B]$ when the function zf'/f takes all values in the cardioid domain $\tilde{p}(A,B;\mathbb{U})$. Furthermore, it is worthwhile here to note that

(1) The class $\mathcal{CS}^*[1,-1]$ coincides with the class SL of starlike functions connected with Fibonacci numbers, introduced and studied in [**23**].
(2) $\mathcal{CS}^*[A,B] \subset \mathcal{S}^*(\alpha) = \left\{f \in \mathcal{S} : \operatorname{Re} \frac{zf'(z)}{f(z)} > \alpha,\ z \in \mathbb{U}\right\}$, where α is defined by (1.6).

Theorem 2.1. *If $f(z) \in \mathcal{CS}^*[A,B]$, $-1 \leqslant B < A \leqslant 1$ and is of the form* (1.1), *then for $n = 2,3,4,\dots,$ we have*

$$|a_n|^2 \leqslant \frac{1}{4(n-1)^2}\Bigg\{\big[|\tau[(A-1)-(n-1)(B-1)]|^2 - 4(n-2)^2\big]|a_{n-1}|^2 + \sum_{k=1}^{n-2}\big[(|\tau||(A-1)-k(B-1)| + 2\tau^2|A-Bk|)^2 - 4(k-1)^2\big]|a_k|^2\Bigg\}.$$

Proof. For $f(z) \in \mathcal{CS}^*[A,B]$, $-1 \leqslant B < A \leqslant 1$, we have from (2.1) and (1.2),

$$\frac{zf'(z)}{f(z)} = \tilde{p}(A,B;w(z)),$$

where $w(0)=0$, $|w(z)|<1$ for $z \in \mathcal{U}$. This implies that

$$\frac{zf'}{f} = \frac{2A\tau^2w^2 + (A-1)\tau w + 2}{2B\tau^2w^2 + (B-1)\tau w + 2},$$

which reduces to

$$2(zf' - f) = \tau w((A-1)f - (B-1)zf') + 2\tau^2w^2(Af - Bzf').$$

This, along with (1.1) gets the form

$$\sum_{k=1}^{\infty} 2(k-1)a_kz^k = \tau w\sum_{k=1}^{\infty}[(A-1)-k(B-1)]a_kz^k + 2\tau^2w^2\sum_{k=1}^{\infty}(A-Bk)a_kz^k.$$

This implies that

$$\sum_{k=1}^{n} 2(k-1)a_kz^k + \sum_{k=n+1}^{\infty} b_kz^k = \tau w\sum_{k=1}^{n-1}[(A-1)-k(B-1)]a_kz^k + 2\tau^2w^2\sum_{k=1}^{n-2}(A-Bk)a_kz^k, \tag{2.2}$$

where

$$\sum_{k=n+1}^{\infty} b_kz^k = \sum_{k=n+1}^{\infty} 2(k-1)a_kz^k - \tau w\sum_{k=n}^{\infty}[(A-1)-k(B-1)]a_kz^k - 2\tau^2w^2\sum_{k=n-1}^{\infty}(A-Bk)a_kz^k.$$

Now from (2.2), one may have

$$\Big|\sum_{k=1}^{n} 2(k-1)a_k z^k + \sum_{k=n+1}^{\infty} b_k z^k\Big|^2$$
$$= \Big|\tau w \sum_{k=1}^{n-1}[(A-1)-k(B-1)]a_k z^k + 2\tau^2 w^2 \sum_{k=1}^{n-2}(A-Bk)a_k z^k\Big|^2.$$

Now, since $|w|<1$, so we have

$$\Big|\sum_{k=1}^{\infty} d_k z^k\Big|^2 < \Big|\tau[(A-1)-(n-1)(B-1)]a_{n-1}z^{n-1}$$
$$+ \sum_{k=1}^{n-2}[\tau[(A-1)-k(B-1)] + 2\tau^2(A-Bk)w(z)]a_k z^k\Big|^2,$$

where $d_k = \left\{\begin{smallmatrix} 2(k-1)a_k, & 1\leqslant k\leqslant n, \\ b_k, & k>n. \end{smallmatrix}\right.$ Making use of the formula $\frac{1}{2\pi}\int_0^{2\pi} \big|\sum_{k=1}^{\infty} d_k(re^{i\theta})^k\big|^2 \mathrm{d}\theta = \sum_{k=1}^{\infty}|d_k|^2 r^{2k}$, see [**7**], and integrating on $z=re^{i\theta}$, $0<r<1$, $0\leqslant\theta<2\pi$, we have

$$\sum_{k=1}^{\infty}|d_k|^2 r^{2k} < \frac{1}{2\pi}\int_0^{2\pi}\Big|\tau[(A-1)-(n-1)(B-1)]a_{n-1}(re^{i\theta})^{n-1}$$
$$+\sum_{k=1}^{n-2}\big[\tau[(A-1)-k(B-1)] + 2\tau^2(A-Bk)w(re^{i\theta})\big]a_k r^k e^{ik\theta}\Big|^2 \mathrm{d}\theta$$
$$= \frac{1}{2\pi}\int_0^{2\pi 0}\bigg(\tau[(A-1)-(n-1)(B-1)]a_{n-1}r^{n-1}e^{i(n-1)\theta}$$
$$+\sum_{k=1}^{n-2}\big[\tau[(A-1)-k(B-1)] + 2\tau^2(A-Bk)w(re^{i\theta})\big]a_k r^k e^{ik\theta}\bigg)$$
$$\times\bigg(\tau[(A-1)-(n-1)(B-1)]\overline{a_{n-1}}r^{n-1}e^{-i(n-1)\theta}$$
$$+\sum_{l=1}^{n-2}\big[\tau[(A-1)-l(B-1)] + 2\tau^2(A-Bl)\overline{w(re^{i\theta})}\big]\overline{a_l}r^l e^{-il\theta}\bigg)\mathrm{d}\theta.$$

Since the integral of the product with $k\neq l$ gives 0, so consequently, we have

$$\sum_{k=1}^{n}|2(k-1)a_k|^2 r^{2k} + \sum_{k=n+1}^{\infty}|b_k|^2 r^{2k} < |\tau[(A-1)-(n-1)(B-1)]|^2|a_{n-1}|^2 r^{2n-2}$$
$$+\sum_{k=1}^{n-2}|\tau[(A-1)-k(B-1)] + 2\tau^2(A-Bk)w(z)|^2|a_k|^2 r^{2k}.$$

Now, since

$$|\tau[(A-1)-k(B-1)] + 2\tau^2(A-Bk)w(z)$$
$$\leqslant |\tau[(A-1)-k(B-1)]| + |2\tau^2(A-Bk)||w(z)|$$
$$< |\tau||(A-1)-k(B-1)| + 2\tau^2|A-Bk|.$$

Thus, we have

$$\begin{aligned}\sum_{k=1}^{n}|2(k-1)a_k|^2r^{2k}+\sum_{k=n+1}^{\infty}|b_k|^2r^{2k}\\ &<|\tau[(A-1)-(n-1)(B-1)]|^2|a_{n-1}|^2r^{2n-2}\\ &\quad+\sum_{k=1}^{n-2}(|\tau||(A-1)-k(B-1)|+2\tau^2|A-Bk|)^2|a_k|^2r^{2k}\end{aligned}$$

which reduces to

$$\begin{aligned}\sum_{k=1}^{n}(k-1)^2|a_k|^2r^{2k}&<|\tau[(A-1)-(n-1)(B-1)]|^2|a_{n-1}|^2r^{2n-2}\\ &\quad+\sum_{k=1}^{n-2}(|\tau||(A-1)-k(B-1)|+2\tau^2|A-Bk|)^2|a_k|^2r^{2k}.\end{aligned}$$

Letting $r\to 1$, one may have

$$\begin{aligned}\sum_{k=1}^{n-2}4(k-1)^2|a_k|^2+4(n-2)^2|a_{n-1}|^2+4(n-1)^2|a_n|^2\\ &\leqslant|\tau[(A-1)-(n-1)(B-1)]|^2|a_{n-1}|^2\\ &\quad+\sum_{k=1}^{n-2}(|\tau||(A-1)-k(B-1)|+2\tau^2|A-Bk|)^2|a_k|^2\end{aligned}$$

and this leads us to the required result. □

Conjecture 2.1. *If $f(z)\in\mathcal{CS}^*[A,B]$, $-1\leqslant B\leqslant 9-4\sqrt{5}$, $B<A$ and is of the form* (1.1)*, then*

$$\begin{aligned}\frac{|a_n|}{|\tau|^{n-1}}\leqslant\sum_{m=0}^{n-1}\Bigg[\Bigg[\sum_{k=0}^{\lfloor m/2\rfloor}\left(\frac{\left|\frac{B-1}{2}\right|^{m-2k}|B|^k}{(m-2k)!\,k!}\prod_{j=0}^{m-1-k}\left|\frac{A-B}{2B}-(m-1-k)+j\right|\right)\Bigg]\\ \times\Bigg[\sum_{l=0}^{n-1-m}\left(\binom{\delta}{l}\binom{\delta+n-m-l-2}{n-1-m-l}|c|^l|d|^{n-1-m-l}\right)\Bigg]\Bigg],\end{aligned}$$

where $\delta=\frac{(A-B)(1+B)}{2B\sqrt{B^2-18B+1}}$, $c=\frac{1-B-\sqrt{B^2-18B+1}}{4}$ *and* $d=\frac{1-B+\sqrt{B^2-18B+1}}{4}$. *This bound is sharp.*

The above inequality suggests the function

$$\tilde{f}(z)=z\Big[1+\frac{B-1}{2}\tau z+Bt^2z^2\Big]^{\frac{A-B}{2B}}\left[\frac{1-\left(\frac{1-B-\sqrt{B^2-18B+1}}{4}\right)\tau z}{1-\left(\frac{1-B+\sqrt{B^2-18B+1}}{4}\right)\tau z}\right]^{\frac{(A-B)(1+B)}{2B\sqrt{B^2-18B+1}}}\tag{2.3}$$

with $-1 \leqslant B \leqslant 9-4\sqrt{5}$. This function is connected with function $\tilde{p}(A,B;z)$ by the relation

$$\frac{z\tilde{f}'(z)}{\tilde{f}(z)} = \tilde{p}(A,B;z), \qquad (z \in \mathbb{U}).$$

Thus, the function $\tilde{f}(z)$ plays the role of extremal function for the class $\mathcal{CS}^*[A,B]$ as

$$f(z) \in \mathcal{CS}^*[A,B] \quad \Longleftrightarrow \quad \frac{zf'(z)}{f(z)} \prec \frac{z\tilde{f}'(z)}{\tilde{f}(z)} = \tilde{p}(A,B;z), \qquad (z \in \mathcal{U}).$$

The following binomial expansions can be easily formed.

$$(1+az+bz^2)^\beta = \sum_{n=0}^{\infty}\left[\sum_{k=0}^{\lfloor n/2\rfloor}\left(\frac{a^{n-2k}b^k}{(n-2k)!\,k!}\prod_{j=0}^{n-1-k}(\beta-(n-1-k)+j)\right)\right]z^n. \tag{2.4}$$

$$\left(\frac{1-cz}{1-dz}\right)^\delta = \sum_{n=0}^{\infty}\left[\sum_{l=0}^{n}\left((-1)^l\binom{\delta}{l}\binom{\delta+n-l-1}{n-l}c^l d^{n-l}\right)\right]z^n.$$

By making use of above binomial expansions, one can have

$$\begin{aligned} z(1+az+bz^2)^\beta\left(\frac{1-cz}{1-dz}\right)^\delta &= z\left[\sum_{n=0}^{\infty}\epsilon_n z^n\right]\left[\sum_{n=0}^{\infty}\varepsilon_n z^n\right] \\ &= \sum_{n=0}^{\infty}\left(\sum_{m=0}^{n}\epsilon_m\varepsilon_{n-m}\right)z^{n+1} \\ &= \sum_{n=1}^{\infty}\left(\sum_{m=0}^{n-1}\epsilon_m\varepsilon_{n-1-m}\right)z^{n} \\ &= \epsilon_0\varepsilon_0 z + \sum_{n=2}^{\infty}\left(\sum_{m=0}^{n-1}\epsilon_m\varepsilon_{n-1-m}\right)z^{n} \\ &= z + \sum_{n=2}^{\infty}\left(\sum_{m=0}^{n-1}\epsilon_m\varepsilon_{n-1-m}\right)z^{n}, \end{aligned} \tag{2.5}$$

where

$$\begin{aligned} \epsilon_n &= \sum_{k=0}^{\lfloor n/2\rfloor}\left(\frac{a^{n-2k}b^k}{(n-2k)!\,k!}\prod_{j=0}^{n-1-k}(\beta-(n-1-k)+j)\right), \\ \varepsilon_n &= \sum_{l=0}^{n}\left((-1)^l\binom{\delta}{l}\binom{\delta+n-l-1}{n-l}c^l d^{\,n-l}\right). \end{aligned}$$

Now, if the function $\tilde{f}(z)$ is of the form $\tilde{f}(z) = z + \sum_{n=2}^{\infty} b_n z^n$, then from (2.5), one can have

$$b_n = \sum_{m=0}^{n-1}\left[\left[\sum_{k=0}^{\lfloor m/2\rfloor}\left(\frac{\left(\frac{B-1}{2}\right)^{m-2k}B^k\tau^m}{(m-2k)!\,k!}\prod_{j=0}^{m-1-k}\left(\frac{A-B}{2B}-(m-1-k)+j\right)\right)\right]\right.$$

$$\times\Bigg[\sum_{l=0}^{n-1-m}\Bigg((-1)^l\binom{\delta}{l}\binom{\delta+n-m-l-2}{n-1-m-l}c^l d^{n-1-m-l}\tau^{n-1-m}\Bigg)\Bigg]\Bigg]$$

$$=\tau^{n-1}\sum_{m=0}^{n-1}\Bigg[\Bigg[\sum_{k=0}^{\lfloor m/2\rfloor}\Bigg(\frac{\big(\frac{B-1}{2}\big)^{m-2k}B^k}{(m-2k)!\,k!}\prod_{j=0}^{m-1-k}\Big(\frac{A-B}{2B}-(m-1-k)+j\Big)\Bigg)\Bigg]$$

$$\times\Bigg[\sum_{l=0}^{n-1-m}\Bigg((-1)^l\binom{\delta}{l}\binom{\delta+n-m-l-2}{n-1-m-l}c^l d^{n-1-m-l}\Bigg)\Bigg]\Bigg],$$

where

$$\delta=\frac{(A-B)(1+B)}{2B\sqrt{B^2-18B+1}},$$

$$c=\frac{1-B-\sqrt{B^2-18B+1}}{4}, \tag{2.6}$$

$$d=\frac{1-B+\sqrt{B^2-18B+1}}{4}. \tag{2.7}$$

with $-1\leqslant B\leqslant 9-4\sqrt{5}$. Therefore, if $f\in\mathcal{CS}^*[A,B]$ and is of the form (1.1), then

$$\frac{|a_n|}{|\tau^{n-1}|}\leqslant\sum_{m=0}^{n-1}\Bigg[\Bigg[\sum_{k=0}^{\lfloor m/2\rfloor}\Bigg(\frac{\big|\frac{B-1}{2}\big|^{m-2k}|B|^k}{(m-2k)!\,k!}\prod_{j=0}^{m-1-k}\Big|\frac{A-B}{2B}-(m-1-k)+j\Big|\Bigg)\Bigg]$$

$$\times\Bigg[\sum_{l=0}^{n-1-m}\Bigg(\binom{\delta}{l}\binom{\delta+n-m-l-2}{n-1-m-l}|c|^l|d|^{n-1-m-l}\Bigg)\Bigg]\Bigg].$$

Corollary 2.1. *If we take $A=1$ and $B=-1$, then the extremal function $\tilde{f}(z)$ defined by (2.3) takes the form $\tilde{f}(z)=\frac{z}{1-\tau z-\tau^2z^2}$. Taylor series of this function can be obtained by using (2.4) as follows.*

$$\begin{aligned}\tilde{f}(z)&=z(1-\tau z-\tau^2z^2)^{-1}\\&=z\sum_{n=0}^{\infty}\Bigg[\sum_{k=0}^{\lfloor n/2\rfloor}\Bigg(\frac{(-\tau)^{n-2k}(-\tau^2)^k}{(n-2k)!\,k!}\prod_{j=0}^{n-1-k}(-1-(n-1-k)+j)\Bigg)\Bigg]z^n\\&=\sum_{n=0}^{\infty}\Bigg[\sum_{k=0}^{\lfloor n/2\rfloor}\Bigg(\frac{(-1)^{n-k}\tau^n}{(n-2k)!\,k!}\prod_{j=0}^{n-1-k}(-1)(n-k-j)\Bigg)\Bigg]z^{n+1}\\&=\sum_{n=0}^{\infty}\Bigg[\sum_{k=0}^{\lfloor n/2\rfloor}\Bigg(\frac{(-1)^{n-k}\tau^n}{(n-2k)!\,k!}(-1)^{n-k}\prod_{j=0}^{n-1-k}(n-k-j)\Bigg)\Bigg]z^{n+1}\\&=\sum_{n=0}^{\infty}\Bigg[\sum_{k=0}^{\lfloor n/2\rfloor}\Bigg(\frac{\tau^n}{(n-2k)!\,k!}\prod_{j=0}^{n-1-k}(n-k-j)\Bigg)\Bigg]z^{n+1}\\&=\sum_{n=0}^{\infty}\Bigg[\sum_{k=0}^{\lfloor n/2\rfloor}\Bigg(\frac{1}{(n-2k)!\,k!}(n-k)!\Bigg)\Bigg]\tau^nz^{n+1}\end{aligned}$$

$$= \sum_{n=0}^{\infty}\left[\sum_{k=0}^{\lfloor n/2\rfloor}\binom{n-k}{k}\right]\tau^n z^{n+1}$$
$$= z+\sum_{n=2}^{\infty}\left[\sum_{k=0}^{\lfloor\frac{n-1}{2}\rfloor}\binom{n-1-k}{k}\right]\tau^{n-1}z^n = z+\sum_{n=2}^{\infty}u_n\tau^{n-1}z^n$$
$$= z+\tau z^2+2\tau^2z^3+3\tau^3z^4+5\tau^4z^5+8\tau^5z^6+\dots,$$

where $\{u_n\} = \left\{\sum_{k=0}^{\lfloor\frac{n-1}{2}\rfloor}\binom{n-1-k}{k}\right\} = \{1,1,2,3,5,8,13,21,\dots\}$ *is the sequence of Fibonacci numbers. This coefficient bound* $|a_n| \leqslant |\tau|^{n-1}u_n$ *for the class* SL *of starlike functions connected with Fibonacci numbers is found in* [**5**, **23**].

Theorem 2.2. *The function* $h(z) = z+cz^n$ *does not belong to the class* $\mathcal{CS}^*[A,B]$ *with* $-1 \leqslant B \leqslant (3-\sqrt{5})/2$, $B < A \leqslant 1$ *if*

$$|c| > \frac{(A-B)(3-\sqrt{5}-2B)}{2n(B^2-3B+1)-2AB+3(A+B)-(A-B)\sqrt{5}-2}. \tag{2.8}$$

Proof. Consider that
$$H(z) = \frac{zh'(z)}{h(z)} = \frac{1+ncz^{n-1}}{1+cz^{n-1}}.$$
The image domain $H(\mathbb{U})$ is a disk with diameter end points $\mathcal{D}_1 = \frac{1-n|c|}{1-|c|}$ and $\mathcal{D}_2 = \frac{1+n|c|}{1+|c|}$, If (2.8) is satisfied, then one of $\mathcal{D}_i$ would satisfy $\mathcal{D}_i < \gamma(A,B)$ which results the negation of inclusion relation $H(\mathbb{U}) \subset \tilde{p}(A,B;\mathbb{U})$. Thus, $H(z) \not\prec \tilde{p}(A,B;z)$ and this proves our proposition. □

For $A = 1$, $B = -1$, the above result reduces to the following one, proved in [**5**].

Corollary 2.2. *The function* $h(z) = z+cz^n$ *does not belong to the class* SL *if* $|c| > \frac{\sqrt{5}-1}{n\sqrt{5}-1}$.

Let $z = re^{i\theta}$, $0 \leqslant \theta < 2\pi$. Then we have
$$\tilde{p}(A,B;re^{i\theta}) = \frac{2A\tau^2r^2e^{2i\theta}+(A-1)\tau re^{i\theta}+2}{2B\tau^2r^2e^{2i\theta}+(B-1)\tau re^{i\theta}+2}$$
$$= \frac{4+(A-1)(B-1)\tau^2r^2+4AB\tau^4r^4+2\lambda_r\cos\theta+4(A+B)\tau^2r^2\cos 2\theta}{|2B\tau^2r^2e^{2i\theta}+(B-1)\tau re^{i\theta}+2|^2}$$
$$+ i2(A-B)\frac{(\tau r-\tau^3r^3)\sin\theta+2\tau^2r^2\sin 2\theta}{|2B\tau^2r^2e^{2i\theta}+(B-1)\tau re^{i\theta}+2|^2},$$
where $\lambda_r = (A+B-2)\tau r+(2AB-A-B)\tau^3r^3$. From above representation, one may have

$$\left|\frac{\operatorname{Im}\tilde{p}(A,B;re^{i\theta})}{\operatorname{Re}\tilde{p}(A,B;re^{i\theta})}\right| \tag{2.9}$$
$$= \left|\frac{2(A-B)\tau r(1-\tau^2r^2+4\tau r\cos\theta)\sin\theta}{4+(A-1)(B-1)\tau^2r^2+4AB\tau^4r^4+2\lambda_r\cos\theta+4(A+B)\tau^2r^2(2\cos^2\theta-1)}\right|$$

$$\begin{aligned}&\leqslant \frac{-2(A-B)\tau r(1-\tau^2r^2-4\tau r)}{4+(A-1)(B-1)\tau^2r^2+4AB\tau^4r^4-2\lambda_r+4(A+B)\tau^2r^2}\\&=\frac{-2(A-B)\tau r(1-\tau^2r^2-4\tau r)}{(2+(1-B)\tau r+2B\tau^2r^2)(2+(1-A)\tau r+2A\tau^2r^2)}:=\psi(A,B;r)\end{aligned}$$

The radius of univalency for the function $\tilde{p}(A,B;z)$ is $r_u=(3-\sqrt{5})/2$. That is, for such r, the curve $\tilde{p}(A,B;re^{i\theta})$, $\theta\in[0,2\pi)\setminus\{\pi\}$ has no loops, see [**14**].

Theorem 2.3. *If* $f(z)\in\mathcal{CS}^*[A,B]$, *then* $\left|\arg\frac{zf'(z)}{f(z)}\right|<\arctan\psi(A,B;r)$, *where* $\psi(A,B;r)$ *is given by* (2.9).

This theorem says that if $f(z)\in\mathcal{CS}^*[A,B]$, then f is strongly starlike of order $\beta=\frac{2}{\pi}\arctan\psi(A,B;r)$ in the disk $|z|<r$, whenever $r<r_u=(3-\sqrt{5})/2$.

Lemma 2.1. *A function f belongs to the class $\mathcal{CS}^*[A,B]$ if and only if there exists an analytic function q, $q\prec\tilde{p}(A,B;z)$, such that* $f(z)=z\exp\int_0^z\frac{q(t)-1}{t}\mathrm{d}t$.

Theorem 2.4. *If a function f belongs to the class $\mathcal{CS}^*[A,B]$, then*

$$f(z)=z\Big\{\frac{g(z)}{z}\Big\}^{\alpha}\Big\{\frac{h(z)}{z}\Big\}^{\beta}$$

for some $g\in\mathcal{S}^*(1/(1+c\tau))$ *and* $h\in\mathcal{S}^*(1/(1+d\tau))$, *where the class* $\mathcal{S}^*(\alpha)$ *was defined immediately before Theorem* 2.1, *and* c, d *are defined by* (2.6) *and by* (2.7) *respectively.*

Proof. Let $f\in\mathcal{CS}^*[A,B]$, then by Definition 2.1, there exists an analytic function $\omega(z)$ with $\omega(0)=0$ and $|\omega(z)|<1$, $z\in\mathbb{U}$, such that

$$f(z)=z\exp\int_0^z\frac{\tilde{p}(A,B;\omega(t))-1}{t}\mathrm{d}t.$$

Now consider that

$$\tilde{p}(A,B;z)=\frac{2A\tau^2z^2+(A-1)\tau z+2}{2B\tau^2z^2+(B-1)\tau z+2}=\frac{A}{B}+\frac{\alpha}{1-c\tau z}+\frac{\beta}{1-d\tau z},$$

where

$$\alpha=\frac{B-A}{2B}\Big(1+\frac{1+B}{\sqrt{B^2-18B+1}}\Big),\quad \beta=\frac{B-A}{2B}\Big(1-\frac{1+B}{\sqrt{B^2-18B+1}}\Big)$$

with $-1\leqslant B\leqslant 9-4\sqrt{5}$ and with c, d defined by (2.6) and (2.7), respectively. This implies that

$$\begin{aligned}f(z)&=z\exp\int_0^z\frac{\big(\frac{A}{B}+\frac{\alpha}{1-c\tau\omega(t)}+\frac{\beta}{1-d\tau\omega(t)}\big)-1}{t}\mathrm{d}t\\&=z\exp\int_0^z\frac{\big(\frac{\alpha}{1-c\tau\omega(t)}-\alpha\big)+\big(\frac{\beta}{1-d\tau\omega(t)}-\beta\big)}{t}\mathrm{d}t\\&=z\exp\left[\alpha\int_0^z\frac{\frac{1}{1-c\tau\omega(t)}-1}{t}\mathrm{d}t+\beta\int_0^z\frac{\frac{1}{1-d\tau\omega(t)}-1}{t}\mathrm{d}t\right]=z\Big\{\frac{g(z)}{z}\Big\}^{\alpha}\Big\{\frac{h(z)}{z}\Big\}^{\beta},\end{aligned}$$

where $g(z) = z\exp\int_0^z \frac{\frac{1}{1-c\tau\omega(t)}-1}{t}$ and $h(z) = z\exp\int_0^z \frac{\frac{1}{1-d\tau\omega(t)}-1}{t}$. Therefore, we have

$$\operatorname{Re}\frac{zg'(z)}{g(z)} = \operatorname{Re}\frac{1}{1-c\tau\omega(z)} > \frac{1}{(1+c\tau)},$$
$$\operatorname{Re}\frac{zh'(z)}{h(z)} = \operatorname{Re}\frac{1}{1-d\tau\omega(z)} > \frac{1}{(1+d\tau)},$$

with $\omega(0)=0$ and $|\omega(z)|<1$, $z\in\mathbb{U}$. This shows that $g(z) = z\exp\int_0^z \frac{1}{t}\big(\frac{1}{1-c\tau\omega(t)} - 1\big) \in \mathcal{S}^*(1/(1+c\tau))$ and $h(z) = z\exp\int_0^z \frac{1}{t}\big(\frac{1}{1-d\tau\omega(t)} - 1\big) \in \mathcal{S}^*(1/(1+d\tau))$. $\square$

In [**21**] are considered some further coefficients problems in the class $\mathcal{CS}^*[A,B]$.

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School of Electrical Engineering and Computer Science (SEECS)
National University of Sciences and Technology
Islamabad, Pakistan
sairazainab07@yahoo.com

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Government College University
Faisalabad, Pakistan
mohsan976@yahoo.com

College of Natural Sciences
University of Rzeszów
Rzeszów, Poland
jsokol@ur.edu.pl

COMSATS University Islamabad
Wah Campus,
Wah Cantt, Pakistan
snmalik110@yahoo.com