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INTEGER POINTS ENUMERATOR OF HYPERGRAPHIC POLYTOPES

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Abstract. For a hypergraphic polytope there is a weighted quasisymmetric function which enumerates positive integer points in its normal fan and determines its f-polynomial. This quasisymmetric function invariant of hypergraphs extends the Stanley chromatic symmetric function of simple graphs. We consider a certain combinatorial Hopf algebra of hypergraphs and show that universal morphism to quasisymmetric functions coincides with this enumerator function. We calculate the f-polynomial of uniform hypergraphic polytopes.

1. Introduction

The theory of combinatorial Hopf algebras developed by Aguiar, Bergeron and Sottile in the seminal paper [**2**] provides an algebraic framework for symmetric and quasisymmetric generating functions arising in enumerative combinatorics. Extensive studies of various combinatorial Hopf algebras are initiated recently [**3, 4, 9, 10**]. The geometric interpretation of the corresponding (quasi)symmetric functions was first given for matroids [**4**] and then for simple graphs [**6**] and building sets [**7**]. The quasisymmetric function invariants are expressed as integer points enumerators associated to generalized permutohedra. This class of polytopes introduced by Postnikov [**11**] is distinguished with rich combinatorial structure. The comprehensive treatment of weighted integer points enumerators associated to generalized permutohedra is carried out by Grujić et al [**8**]. Here we consider a certain naturally defined non-cocommutative combinatorial Hopf algebra of hypergraphs and show that the derived quasisymmetric function invariant of hypergraphs is integer points enumerator of hypergraphic polytopes (Theorem 4.1).

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2. Combinatorial Hopf algebra of hypergraphs $\mathcal{HG}$

A *combinatorial Hopf algebra* is a pair $(\mathcal{H}, \zeta)$ of a graded connected Hopf algebra $\mathcal{H} = \bigoplus_{n\geqslant 0} \mathcal{H}_n$ over a field $\mathbf{k}$, whose homogeneous components $\mathcal{H}_n$, $n \geqslant 0$ are finite-dimensional, and a multiplicative linear functional $\zeta : \mathcal{H} \to \mathbf{k}$ called *character*. We consider a combinatorial Hopf algebra structure on hypergraphs different from the chromatic Hopf algebra of hypergraphs studied in [**10**]. The difference is in the coalgebra structures based on different combinatorial constructions, which is manifested in (non)co-commutativity. It extends the Hopf algebra of building set studied by Grujić in [**7**]. This Hopf algebra of hypergraphs can be derived from the Hopf monoid structure on hypergraphs introduced in [**1**].

A *hypergraph* $\mathbf{H}$ on the vertex set V is a collection of nonempty subsets $H \subseteq V$, called *hyperedges*. We assume that there are no ghost vertices, i.e., $\mathbf{H}$ contains all singletons $\{i\}$, $i \in V$. A hypergraph $\mathbf{H}$ is *connected* if it can not be represented as a disjoint union of hypergraphs $\mathbf{H}_1 \sqcup \mathbf{H}_2$. Every hypergraph $\mathbf{H}$ splits into its connected components. Let $c(\mathbf{H})$ be the number of connected components of $\mathbf{H}$. Hypergraphs $\mathbf{H}_1$, on the vertex set V_1, and $\mathbf{H}_2$, on the vertex set V_2, are *isomorphic* if there is a bijection $f : V_1 \to V_2$ such that $\mathbf{H}_2 = \{f(H) : H \in \mathbf{H}_1\}$. Let $\mathcal{HG} = \bigoplus_{n\geqslant 0} \mathcal{HG}_n$, where $\mathcal{HG}_n$ is the linear span of isomorphism classes $[\mathbf{H}]$ of hypergraphs on the set $[n]$.

Definition 2.1. For a hypergraph $\mathbf{H}$ on the vertex set $[n]$ and a subset $S \subseteq [n]$ the *restriction* $\mathbf{H}|_S$ and the *contraction* $\mathbf{H}/S$ are defined by

$$\mathbf{H}|_S = \{H \in \mathbf{H} : H \subseteq S\},$$
$$\mathbf{H}/S = \{H \setminus S : H \in \mathbf{H}\}.$$

Define a *product* and a *coproduct* on the linear space $\mathcal{HG}$ by

$$[\mathbf{H}_1] \cdot [\mathbf{H}_2] = [\mathbf{H}_1 \sqcup \mathbf{H}_2],$$
$$\Delta([\mathbf{H}]) = \sum_{S\subset[n]} [\mathbf{H}|_S] \otimes [\mathbf{H}/S].$$

The straightforward checking shows that the space $\mathcal{HG}$ with the above operations together with the unit $\eta : \mathbf{k} \to \mathcal{HG}$ given by $\eta(1) = [\mathbf{H}_\emptyset]$ (the empty hypergraph) and the counit $\epsilon : \mathcal{HG} \to \mathbf{k}$ which is the projection on the component $\mathcal{HG}_0 = \mathbf{k}$, become a graded connected commutative and non-cocommutative bialgebra. Namely, associativity and commutativity of the product follow because $\sqcup$ is associative and commutative up to isomorphism. For a hypergraph $\mathbf{H}$ on the set $[n]$ the following equalities

$$((\Delta \otimes \mathrm{Id}) \circ \Delta)([\mathbf{H}]) = \sum_{\emptyset\subset S_1\subset S_2\subset[n]} [\mathbf{H}|_{S_1}] \otimes [(\mathbf{H}|_{S_2})/S_1] \otimes [\mathbf{H}/S_2],$$
$$((\mathrm{Id} \otimes\Delta) \circ \Delta)([\mathbf{H}]) = \sum_{\emptyset\subset S_1\subset S_2\subset[n]} [\mathbf{H}|_{S_1}] \otimes [(\mathbf{H}/S_1)|_{S_2\setminus S_1}] \otimes [\mathbf{H}/S_2]$$

between $((\Delta \otimes \mathrm{Id}) \circ \Delta)([\mathbf{H}])$ and $((\mathrm{Id} \otimes\Delta) \circ \Delta)([\mathbf{H}])$ provides the coassociativity.

Commutativity of the bialgebra diagram follows since

$$[\mathbf{H}_1|_{S_1}] \sqcup [\mathbf{H}_2|_{S_2}] = [(\mathbf{H}_1 \sqcup \mathbf{H}_2)|_{S_1 \sqcup S_2}],$$
$$[\mathbf{H}_1/S_1] \sqcup [\mathbf{H}_2/S_2] = [(\mathbf{H}_1 \sqcup \mathbf{H}_2)/(S_1 \sqcup S_2)].$$

Since graded connected bialgebras of finite type posses antipodes, $\mathcal{HG}$ is in fact a Hopf algebra. The formula for antipode $S : \mathcal{HG} \to \mathcal{HG}$ is derived from the general Takeuchi formula [**12**]

$$S([\mathbf{H}]) = \sum_{k \geqslant 1} (-1)^k \sum_{\mathcal{L}_k} \prod_{j=1}^{k} ([\mathbf{H}] \mid_{I_j})/I_{j-1},$$

where the inner sum goes over all chains of subsets $\mathcal{L}_k : \emptyset = I_0 \subset I_1 \subset \cdots \subset I_{k-1} \subset I_k = V$. Define a character $\zeta : \mathcal{HG} \to \mathbf{k}$ by $\zeta([\mathbf{H}]) = 1$ if $\mathbf{H}$ is discrete, i.e., contains only singletons and $\zeta([\mathbf{H}]) = 0$ otherwise. This determines the combinatorial Hopf algebra $(\mathcal{HG}, \zeta)$.

3. Integer points enumerator

In this section we review the definition of the integer points enumerator of a generalized permutohedron introduced in [**8**].

For a point $(a_1, a_2, \ldots, a_n) \in \mathbb{R}^n$ with increasing coordinates $a_1 < \cdots < a_n$ let us define the set $\Omega(a_1, a_2, \ldots, a_n)$ by

$$\Omega(a_1, a_2, \ldots, a_n) = \{(a_{\omega(1)}, a_{\omega(2)}, \ldots, a_{\omega(n)}) \ : \ \omega \in \mathfrak{S}_n\},$$

where $\mathfrak{S}_n$ is the permutation group of the set $[n]$. The convex hull of the set $\Omega(a_1, a_2, \ldots, a_n)$ is a *standard* $(n-1)$*-dimensional permutohedron* Pe^{n-1}. The d-dimensional faces of Pe^{n-1} are in one-to-one correspondence with set compositions $\mathcal{C} = C_1|C_2|\cdots|C_{n-d}$ of the set $[n] = \{1, 2, \ldots, n\}$, see [**11**], Proposition 2.6. By this correspondence and the obvious correspondence between set compositions and flags of subsets, we identify d-faces of Pe^{n-1} with flags $\mathcal{F} : \emptyset = F_0 \subset F_1 \subset \cdots \subset F_{n-d} = [n]$. The dimension of a face and the length of the corresponding flag is related by $\dim(\mathcal{F}) = n - |\mathcal{F}|$.

The normal fan $\mathcal{N}(Pe^{n-1})$ of the standard permutohedron is the fan of the braid arrangement $\{x_i = x_j\}_{1 \leqslant i < j \leqslant n}$ in the space $\mathbb{R}^n$. The dimension of the normal cone $C_{\mathcal{F}}$ at the face $\mathcal{F}$ is $\dim(C_{\mathcal{F}}) = |\mathcal{F}|$. The relative interior points $\omega \in C^{\circ}_{\mathcal{F}}$ are characterized by the condition that their coordinates are constant on $F_i \setminus F_{i-1}$ and increase $\omega|_{F_i \setminus F_{i-1}} < \omega|_{F_{i+1} \setminus F_i}$. A positive integer vector $\omega \in \mathbb{Z}^n_+$ belongs to $C^{\circ}_{\mathcal{F}}$ if the weight function $\omega^*(x) = \langle \omega, x \rangle$ is maximized on Pe^{n-1} along a face $\mathcal{F}$.

Definition 3.1. For a flag $\mathcal{F}$ let $M_{\mathcal{F}}$ be the enumerator of interior positive integer points $\omega \in \mathbb{Z}^n_+$ of the corresponding cone $C_{\mathcal{F}}$

$$M_{\mathcal{F}} \ = \sum_{\omega \in \mathbb{Z}^n_+ \cap \, C^{\circ}_{\mathcal{F}}} \mathbf{x}_{\omega},$$

where $\mathbf{x}_{\omega} = x_{\omega_1} x_{\omega_2} \cdots x_{\omega_n}$.

The enumerator $M_{\mathcal{F}}$ is a monomial quasisymmetric function depending only of the composition $\mathrm{type}(\mathcal{F}) = (|F_1|, |F_2 \setminus F_1|, \dots, |F_k \setminus F_{k-1}|)$.

The fan $\mathcal{N}$ is a *coaresement* of $\mathcal{N}(Pe^{n-1})$ if every cone in $\mathcal{N}$ is a union of cones of $\mathcal{N}(Pe^{n-1})$. An $(n-1)$-dimensional *generalized permutohedron* Q is a convex polytope whose normal fan $\mathcal{N}(Q)$ is a coaresement of $\mathcal{N}(Pe^{n-1})$. There is a map $\pi_Q : L(Pe^{n-1}) \to L(Q)$ between face lattices given by

$$\pi_Q(\mathcal{F}) = G \quad \text{if and only if} \quad C^\circ_{\mathcal{F}} \subseteq C^\circ_G,$$

where C°_G in the relative interior of the corresponding normal cone C_G at the face $G \in L(Q)$.

Definition 3.2. For an $(n-1)$-generalized permutohedron Q, let $F_q(Q)$ be the weighted integer points enumerator

$$F_q(Q) \ = \sum_{\omega \in \mathbb{Z}^n_+} q^{\dim(\pi_Q(\mathcal{F}_\omega))} \mathbf{x}_\omega \ = \sum_{\mathcal{F} \in L(Pe^{n-1})} q^{\dim(\pi_Q(\mathcal{F}))} M_{\mathcal{F}},$$

where $\mathcal{F}_\omega$ is a unique face of Pe^{n-1} containing ω in the relative interior.

Remark 3.1. It is shown in [**8**, Theorem 4.4] that the enumerator $F_q(Q)$ contains the information about the f-vector of a generalized permutohedron Q. More precisely, the principal specialization of $F_q(Q)$ gives the f-polynomial of Q

$$f(Q, q) = (-1)^n \mathbf{ps}(F_{-q}(Q))(-1). \tag{3.1}$$

Recall that the principal specialization $\mathbf{ps}(F)(m)$ of a quasisymmetric function F in variables $x_1, x_2, \dots$ is a polynomial in m obtained from the evaluation map at $x_i = 1$, $i = 1, \dots, m$ and $x_i = 0$ for $i > m$.

4. The hypergraphic polytope

For the standard basis vectors $e_i, 1 \leqslant i \leqslant n$ in $\mathbb{R}^n$ let $\Delta_H = \mathrm{conv}\{e_i \ : \ i \in H\}$ be the simplex determined by a subset $H \subset [n]$. The *hypergraphic polytope* of a hypergraph $\mathbf{H}$ on $[n]$ is the Minkowski sum of simplices

$$P_{\mathbf{H}} = \sum_{H \in \mathbf{H}} \Delta_H.$$

As generalized permutohedra can be described as the Minkowski sum of delated simplices (see [**11**]), we have that hypergraphic polytopes are generalized permutohedra. For the following description of $P_{\mathbf{H}}$ see [**5**, Section 1.5] and the references within it. Let $\mathbf{H} = \mathbf{H}_1 \sqcup \mathbf{H}_2 \sqcup \cdots \sqcup \mathbf{H}_k$ be the decomposition into connected components. Then $P_{\mathbf{H}} = P_{\mathbf{H}_1} \times P_{\mathbf{H}_2} \times \cdots \times P_{\mathbf{H}_k}$ and $\dim(P_{\mathbf{H}}) = n - k$. For connected hypergraphs $P_{\mathbf{H}}$ can be described as the intersection of the hyperplane $H_{\mathbf{H}} := \{x \in \mathbb{R}^n : \sum_{i=1}^n x_i = |\mathbf{H}|\}$ with the halfspaces

$$H_{S,\geqslant} := \left\{ x \in \mathbb{R}^n : \sum_{i \in S} x_i \geqslant |\mathbf{H}|_S| \right\}$$

corresponding to all proper subsets $S \subset [n]$. It follows that $P_{\mathbf{H}}$ can be obtained by iteratively cutting the standard simplex $\Delta_{[n]}$ by the hyperplanes $H_{S,\geqslant}$ corresponding to proper subsets S. For instance the standard permutohedron Pe^{n-1} is a hypergraphic polytope $P_{\mathbf{C}_n}$ corresponding to the complete hypergraph $\mathbf{C}_n$ consisting of all subsets of $[n]$.

DEFINITION 4.1. For a connected hypergraph $\mathbf{H}$ the $\mathbf{H}$-*rank* is a map $\mathrm{rk}_{\mathbf{H}} : L(Pe^{n-1}) \to \{0, 1, \dots, n-1\}$ given by

$$\mathrm{rk}_{\mathbf{H}}(\mathcal{F}) = \dim(\pi_{P_{\mathbf{H}}}(\mathcal{F})).$$

Subsequently we deal only with connected hypergraphs. The quasisymmetric function $F_q(P_{\mathbf{H}})$ corresponding to a hypergraphic polytope $P_{\mathbf{H}}$, according to Definitions 3.2 and 4.1, depends only on the rank function

$$F_q(P_{\mathbf{H}}) = \sum_{\mathcal{F} \in L(Pe^{n-1})} q^{\mathrm{rk}_{\mathbf{H}}(\mathcal{F})} M_{\mathcal{F}}. \tag{4.1}$$

We extend the ground field $\mathbf{k}$ to the field of rational function $\mathbf{k}(q)$ in a variable q and consider the Hopf algebra $\mathcal{HG}$ over this extended field. Let $\mathrm{rk}(\mathbf{H}) = n - c(\mathbf{H})$ for hypergraphs on n vertices. Define a linear functional $\zeta_q : \mathcal{HG} \to \mathbf{k}(q)$ with

$$\zeta_q([\mathbf{H}]) = q^{\mathrm{rk}(\mathbf{H})} = q^{n-c(\mathbf{H})},$$

which is obviously multiplicative. By the characterization of the combinatorial Hopf algebra of quasisymmetric functions (QSym, ζ_Q) as a terminal object ([**2**, Theorem 4.1]) there exists a unique morphism of combinatorial Hopf alegbras $\Psi_q\,(\mathcal{HG},\, \zeta_q) \to (\mathrm{QSym},\, \zeta_Q)$ given on monomial basis by

$$\Psi_q([\mathbf{H}]) = \sum_{\alpha \models n} (\zeta_q)_\alpha([\mathbf{H}]) M_\alpha.$$

We determine the coefficients by monomial functions in the above expansion more explicitly. For a hypergraph $\mathbf{H}$ define its splitting hypergraph $\mathbf{H}/\mathcal{F}$ by a flag $\mathcal{F}$ with

$$\mathbf{H}/\mathcal{F} = \bigsqcup_{i=1}^{k} \mathbf{H}|_{F_i}/F_{i-1}.$$

The coefficient corresponding to a composition $\alpha = (\alpha_1, \alpha_2, \dots, \alpha_k) \models n$ is a polynomial in q determined by

$$(\zeta_q)_\alpha([\mathbf{H}]) = \sum_{\mathcal{F}\,:\,\mathrm{type}(\mathcal{F})=\alpha} \prod_{i=1}^{k} q^{\mathrm{rk}(\mathbf{H}|_{F_i}/F_{i-1})} = \sum_{\mathcal{F}\,:\,\mathrm{type}(\mathcal{F})=\alpha} q^{\mathrm{rk}(\mathrm{H}/\mathcal{F})},$$

where the sum is over all flags $\mathcal{F} : \emptyset =: F_0 \subset F_1 \subset \cdots \subset F_k := [n]$ of the type α and

$$\mathrm{rk}(\mathbf{H}/\mathcal{F}) = \sum_{i=1}^{k} \mathrm{rk}(\mathbf{H}|_{F_i}/F_{i-1}) = n - \sum_{i=1}^{k} c(\mathbf{H}|_{F_i}/F_{i-1}). \tag{4.2}$$

By this correspondence, we have

$$\Psi_q([\mathbf{H}]) = \sum_{\mathcal{F}\in L(Pe^{n-1})} q^{\mathrm{rk}(\mathbf{H}/\mathcal{F})} M_{\mathcal{F}}. \tag{4.3}$$

Now we have two quasisymmetric functions associated to hypergraphs whose expansions in monomial bases are given by (4.1) and (4.3). We show that they actually coincide which describes the corresponding hypergraphic quasisymmetric invariant algebraically and geometrically.

Theorem 4.1. *For a connected hypergraph* $\mathbf{H}$ *the integer points enumerator* $F_q(P_{\mathbf{H}})$ *associated to a hypergraphic polytope and the quasisymmetric function* $\Psi_q([\mathbf{H}])$ *coincide*

$$F_q(P_{\mathbf{H}}) = \Psi_q([\mathbf{H}]).$$

Proof. Let $\mathbf{H}$ be a connected hypergraph on the set $[n]$ and $\mathcal{F} : \emptyset = F_0 \subset F_1 \subset F_2 \subset \cdots \subset F_m = [n]$ be a flag of subsets of $[n]$. It is sufficient to prove that

$$\mathrm{rk}_{\mathbf{H}}(\mathcal{F}) = \mathrm{rk}(\mathbf{H}/\mathcal{F}). \tag{4.4}$$

For this we need to determine the face G of the hypergraphic polytope $P_{\mathbf{H}}$ along which the weight function ω^* is maximized for an arbitrary $\omega \in C^\circ_{\mathcal{F}}$. Since $P_{\mathbf{H}}$ is the Minkowski sum of simplices Δ_H for $H \in \mathbf{H}$ the face G is itself a Minkowski sum of the form $G = \sum_{H\in\mathbf{H}}(\Delta_H)_{\mathcal{F}}$ where $(\Delta_H)_{\mathcal{F}}$ is a unique face of Δ_H along which the weight function ω^* is maximized for $\omega \in C^\circ_{\mathcal{F}}$. Let $\omega = (\omega_1, \ldots, \omega_n)$ where $\omega_i = j$ if $i \in F_j \setminus F_{j-1}$ for $i = 1, \ldots, n$. Then $\omega \in C^\circ_{\mathcal{F}}$ and we can convince that $(\Delta_H)_{\mathcal{F}} = \Delta_{H\setminus F_{j-1}}$ where $j = \min\{k \mid H \subset F_k\}$. Denote by $\mathbf{H}_j$ the collection of all $H \in \mathbf{H}$ with $j = \min\{k \mid H \subset F_k\}$ for $j = 1, \ldots, m$. We can represent the face G as $G = \sum_{j=1}^m \sum_{H\in\mathbf{H}_j} \Delta_{H\setminus F_{j-1}}$, which shows that G is precisely a hypergraphic polytope corresponding to the splitting hypergraph

$$G = P_{\mathbf{H}/\mathcal{F}}.$$

Equation (4.4) follows from the fact that $\dim P_{\mathbf{H}/\mathcal{F}} = \mathrm{rk}(\mathbf{H}/\mathcal{F})$, which is given by (4.2). ☐

As a corollary, by Remark 3.1 and equation (3.1) within it, we can derive the f-polynomial of a hypergraphic polytope $P_{\mathbf{H}}$ in a purely algebraic way.

Corollary 4.1. *The f-polynomial of a hypergraphic polytope* $P_{\mathbf{H}}$ *is determined by the principal specialization*

$$f(P_{\mathbf{H}}, q) = (-1)^n\, \mathbf{ps}\,(\Psi_{-q}([\mathbf{H}]))\,(-1).$$

We proceed with some examples and calculations.

Example 4.1. Let $\mathbf{U}_{n,k}$ be the k-uniform hypergraph containing all k-elements subsets of $[n]$ with $k > 1$. Divide flags into two families depending on whether they contain a k-elements subset. Let $\circ$ be a bilinear operation on quasisymmetric functions given on the monomial bases by concatenation $M_\alpha \circ M_\beta = M_{\alpha\cdot\beta}$. The flags that contain k-elements subset contribute to $\Psi([\mathbf{U}_{n,k}])$ with

$$\sum_{i=1}^{k} \binom{n}{k-i, i, n-k} q^{i-1} M_{(1)}^{k-i} \circ M_{(i)} \circ \Psi_q([\mathbf{C}_{n-k}]).$$

The contribution to $\Psi([\mathbf{U}_{n,k}])$ of the remaining flags is

$$\sum_{0\leqslant a<k<n-b\leqslant n} \binom{n}{a, b, n-a-b} q^{n-a-b-1} M_{(1)}^{a} \circ M_{(n-a-b)} \circ \Psi_q(\mathbf{C}_b).$$

By Corollary 4.1 since the principal specialization respects the operation $\circ$, it follows from $P_{\mathbf{C}_m} = Pe^{m-1}$ that

$$f(P_{\mathbf{U}_{n,k}}, q) = \sum_{i=1}^{k} \binom{n}{k-i, i, n-k} q^{i-1} f(Pe^{n-k-1}, q) \\ + \sum_{0\leqslant a<k<n-b\leqslant n} \binom{n}{a, b, n-a-b} q^{n-a-b-1} f(Pe^{b-1}, q).$$

Example 4.2. The hypergraphic polytope PS^{n-1} corresponding to the hypergraph $\{[1], [2], \dots, [n]\}$ is known as the Pitman–Stanley polytope. It is combinatorially equivalent to the $(n-1)$-cube [**11**, Proposition 8.10]. The following recursion is satisfied

$$F_q(PS^n) = F_q(PS^{n-1})M_{(1)} + (q-1)(F_q(PS^{n-1}))_{+1},$$

where $_{+1}$ is given on monomial bases by $(M_{(i_1,i_2,\dots,i_k)})_{+1} = M_{(i_1,i_2\dots,i_k+1)}$. It can be seen by dividing flags into two families according to the position of the element n. To a flag $\mathcal{F} : \emptyset = F_0 \subset F_1 \subset \cdots \subset F_m = [n]$, we associate the flag $\widetilde{\mathcal{F}} : \emptyset = F_0 \subset F_1 \setminus \{n\} \subset \cdots \subset F_m \setminus \{n\} = [n-1]$. If $n \in F_k$ for some $k < m$, then $\mathrm{rk}_{PS^n}(\mathcal{F}) = \mathrm{rk}_{PS^{n-1}}(\mathcal{F})$ and if $n \notin F_k$ for $k < m$, then $\mathrm{rk}_{PS^n}(\mathcal{F}) = \mathrm{rk}_{PS^{n-1}}(\mathcal{F})+1$. The principal specialization of the previous recursion formula gives

$$f_q(PS_n) = (2+q)f_q(PS_{n-1}),$$

consequently $f_q(PS_n) = (2+q)^n$ which reflects the fact that PS^n is an n-cube.

Example 4.3. If Γ is a simple graph, the corresponding hypergraphic polytope P_Γ is the graphic zonotope

$$P_\Gamma = \sum_{\{i,j\}\in\Gamma} \Delta_{e_i,e_j}.$$

Simple graphs generate the Hopf subalgebra of $\mathcal{HG}$ which is isomorphic to the chromatic Hopf algebra of graphs. Therefore $F_q(P_\Gamma)$ is the q-analogue of the Stanley chromatic symmetric function of graphs introduced in [**6**].

Example 4.4. Simplicial complexes generate another Hopf subalgebra of $\mathcal{HG}$ which is isomorphic to the Hopf algebra of simplicial complexes introduced in [**10**] and studied more extensively in [**3**]. It is shown in [**1**, Lemma 21.2] that hypergraphic polytopes P_K and P_{K^1} corresponding to a simplicial complex K and its 1-skeleton K^1 are normally equivalent and therefore have the same enumerators

$$F_q(P_K) = F_q(P_{K^1}).$$

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