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ON A CONJECTURE OF SENETA ON REGULAR VARIATION OF TRUNCATED MOMENTS

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Abstract. We prove that $h_\beta(x) = \beta \int_0^x y^{\beta-1}\overline{F}(y)\mathrm{d}y$ is regularly varying with index $\rho \in [0, \beta)$ if and only if $V_\beta(x) = \int_{[0,x]} y^\beta \mathrm{d}F(y)$ is regularly varying with the same index, where $\beta > 0$, $F(x)$ is a distribution function of a nonnegative random variable, and $\overline{F}(x) = 1 - F(x)$. This contains at $\rho = 0$, $\beta = 1$ a result of Rogozin [**8**] on relative stability, and at $\rho = 0$, $\beta = 2$ a new, equivalent characterization of the domain of attraction of the normal law. For $\rho = 0$ and $\beta > 0$ our result implies a recent conjecture by Seneta [**9**].

1. Introduction and results

Let F be the distribution function of a nonnegative random variable, and put $\overline{F}(x) = 1 - F(x)$. For $\beta > 0$ introduce the truncated β-moments as

$$h_\beta(x) = \beta \int_0^x y^{\beta-1}\overline{F}(y)\mathrm{d}y$$

$$V_\beta(x) = \int_{[0,x]} y^\beta \mathrm{d}F(y).$$

Integration by parts gives

$$\begin{aligned} V_\beta(x) &= \beta \int_0^x y^{\beta-1}\overline{F}(y)\mathrm{d}y - x^\beta \overline{F}(x) \\ &= h_\beta(x) - x^\beta \overline{F}(x), \end{aligned} \tag{1.1}$$

the basic relation between h_β and V_β. Note that if h_β or V_β is regularly varying, then its index ρ is at most β. Indeed, as $\overline{F}(x) \leqslant 1$, we have $h_\beta(x) \leqslant x^\beta$, and similarly, $V_\beta(x) \leqslant x^\beta$.

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Seneta [**9**, Theorem 3] proved among other equivalences that if V_β is slowly varying, then h_β is slowly varying and conjectured [**9**, p.1653] that the converse also holds. In this short note, we prove an extension of Seneta's conjecture. The following result is a generalization of Theorem 3 by Seneta [**9**], where he considers the case $\rho = 0$.

In what follows $\mathcal{RV}_\rho$ stands for the class of regularly varying functions with index ρ, and all nonspecified limit relations are meant as $x \to \infty$.

Before the main result, we need some definitions. Let ℓ be a slowly varying function. Then f belongs to the de Haan class Π_ℓ with index $-c$ if for all $\lambda > 0$

$$\lim_{x\to\infty} \frac{f(\lambda x) - f(x)}{\ell(x)} = -c \log \lambda. \tag{1.2}$$

The de Haan class Π consists of all functions f such that $f \in \Pi_\ell$ with nonzero index for some slowly varying ℓ. We note that Π is a proper subclass of the slowly varying functions. For properties on the de Haan class we refer to Chapter 3 in [**1**] and Appendix B in de Haan and Ferreira [**3**].

If $\int_{[0,\infty)} y^\beta \mathrm{d}F(y) < \infty$ then, as $x \to \infty$, $x^\beta \overline{F}(x) \to 0$. In particular, both h_β and V_β converge to the same finite limit. Therefore, we always assume

$$\int_{[0,\infty)} y^\beta \mathrm{d}F(y) = \infty. \tag{1.3}$$

The main novelty in the next result is the relation between the two truncations, h_β and V_β.

Theorem 1.1. *Assume* (1.3) *for some* $\beta > 0$*. For any* $\rho \in (0, \beta)$ *the following are equivalent:*

$$h_\beta \in \mathcal{RV}_\rho; \tag{1.4}$$

$$V_\beta \in \mathcal{RV}_\rho; \tag{1.5}$$

$$\overline{F} \in \mathcal{RV}_{\rho-\beta}; \tag{1.6}$$

$$\lim_{x\to\infty} \frac{\overline{F}(x) x^\beta}{h_\beta(x)} = \frac{\rho}{\beta}; \tag{1.7}$$

$$\lim_{x\to\infty} \frac{V_\beta(x)}{h_\beta(x)} = 1 - \frac{\rho}{\beta}. \tag{1.8}$$

If $\rho = 0$ *then conditions* (1.4)*,* (1.5)*,* (1.7)*, and* (1.8) *are equivalent, and* (1.6) *implies each of them. Furthermore,* (1.6) *is equivalent to* $h_\beta \in \Pi$*.*

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Remark 1.1. For $\beta = 1$ and $\rho = 0$ the result follows from Rogozin's theorem ([**8**], or [**1**, Theorem 8.8.1]), and (1.5) is further equivalent to the relative stability of the corresponding random walk, see [**1**, Section 8.8].

Another important special case is $\beta = 2$, when V_2 is the truncated second moment. A distribution belongs to the domain of attraction of the normal law if and only if V_2 is slowly varying, see [**1**, Section 8.3]. In the corresponding small time setup (asymptotics at 0), for $\beta = 2$, $\rho = 0$, among other results Theorem 2.4 by Maller and Mason [**7**] states that both (1.4) and (1.5) follows from (1.7).

Remark 1.2. For $\rho = 0$ in Theorem 3 [**9**] Seneta showed that (1.5), (1.7) and (1.8) are equivalent, and any of them implies (1.4). The implication (1.4) $\Rightarrow$ (1.5) is the conjecture by Seneta [**9**, (26) on p.1653]. Moreover, in [**9**, Theorem 3] it is also shown that h_β is in fact *slowly varying in the Zygmund sense*, which is, by a result of Bojanić and Karamata [**1**, Theorem 1.5.5], equivalent to h_β being normalized slowly varying.

Remark 1.3. The implications (1.5) $\Rightarrow$ (1.4), (1.6), (1.7), (1.8) are contained in Lemma 5 in Jasiulis-Gołdyn et al. [**5**].

Remark 1.4. For $\rho \in (0, \beta)$ the functions $x^\beta \overline{F}(x)$, $h_\beta(x)$, and $V_\beta(x)$ are all asymptotically equivalent up to a strictly positive finite constant factor. The borderline cases $\rho = 0$ and $\rho = \beta$ are somewhat different.

For $\rho = 0$ we have $h_\beta(x) \sim V_\beta(x)$, and $x^\beta \overline{F}(x) = o(h_\beta(x))$. Moreover, the slow variation of h_β does not imply the slow variation of $x^\beta \overline{F}(x)$. Indeed, if $u(x) := x^\beta \overline{F}(x)$ is a logarithmically periodic function for large x, meaning that for some $p > 1$ we have $u(x) = u(px)$, then $\int_0^x u(y) y^{-1} \mathrm{d}y$ is slowly varying. See Lemma 2.3 by Kevei [**6**], or in a more general setting Proposition 6.7 by Buldygin et al. [**2**]. The simplest such example for $\beta = 1$ is the classical St. Petersburg distribution, with distribution function

$$F(x) = \begin{cases} 1 - 2^{-\lfloor \log_2 x \rfloor} = 1 - \frac{2^{\{\log_2 x\}}}{x}, & \text{for } x \geqslant 2, \\ 0, & \text{otherwise,} \end{cases}$$

where $\log_2$ stands for the logarithm with base 2, $\lfloor \cdot \rfloor$ is the (lower) integer part, and $\{\cdot\}$ is the fractional part; see [**1**, Section 8.8.2], or [**4**, Section VII.7]. It turns out that $x^\beta \overline{F}(x)$ is slowly varying if and only if h_β belong to the de Haan class Π.

On the other hand, if $\rho = \beta$, then $h_\beta \in \mathcal{RV}_\beta$ implies that $\overline{F}(x)$ is slowly varying and $h_\beta(x) \sim \overline{F}(x) x^\beta$. Then $V_\beta(x) = o(h_\beta(x))$, and

$$x^{-\beta} V_\beta(x) = x^{-\beta} \int_0^x \beta y^{\beta - 1} \overline{F}(y) \mathrm{d}y - \overline{F}(x), \tag{1.9}$$

which is not necessarily slowly varying. In fact, by Theorem 3.7.1 in [**1**] (a version of de Haan's theorem) it is slowly varying if and only if $\overline{F}$ belongs to the de Haan class Π.

2. Proof

Equivalence (1.7) $\Leftrightarrow$ (1.8). Clearly, (1.1) implies that (1.7) and (1.8) are equivalent for any $\rho \in [0, \beta]$.

Implications (1.4) $\Rightarrow$ (1.5), (1.6), (1.7). Assume (1.4). Then for $\lambda > 1$

$$\begin{aligned} h_\beta(\lambda x) - h_\beta(x) &= \beta \int_x^{\lambda x} y^{\beta-1}\overline{F}(y)\mathrm{d}y \\ &\geqslant \overline{F}(\lambda x)\left((\lambda x)^\beta - x^\beta\right) \\ &= \overline{F}(\lambda x)(\lambda x)^\beta(1-\lambda^{-\beta}). \end{aligned}$$

Dividing both sides by $h_\beta(\lambda x)$, taking limits as $x \to \infty$ and using that h_β is regularly varying, we obtain

$$\limsup_{x\to\infty} \frac{\overline{F}(\lambda x)(\lambda x)^\beta}{h_\beta(\lambda x)} \leqslant \frac{1-\lambda^{-\rho}}{1-\lambda^{-\beta}}.$$

As $\lambda \downarrow 1$, we have

$$\limsup_{x\to\infty} \frac{\overline{F}(x)x^\beta}{h_\beta(x)} \leqslant \frac{\rho}{\beta}. \tag{2.1}$$

Similarly we derive the lower bound. For $\lambda > 1$

$$\begin{aligned} h_\beta(\lambda x) - h_\beta(x) &\leqslant \overline{F}(x)\left((\lambda x)^\beta - x^\beta\right) \\ &= \overline{F}(x)x^\beta(\lambda^\beta - 1). \end{aligned}$$

Dividing both sides by $h_\beta(x)$ and taking limits as $x \to \infty$, we obtain

$$\liminf_{x\to\infty} \frac{\overline{F}(x)x^\beta}{h_\beta(x)} \geqslant \frac{\lambda^\rho - 1}{\lambda^\beta - 1}.$$

As $\lambda \downarrow 1$, we have

$$\liminf_{x\to\infty} \frac{\overline{F}(x)x^\beta}{h_\beta(x)} \geqslant \frac{\rho}{\beta}. \tag{2.2}$$

Combining (2.1) and (2.2) we obtain (1.7). Furthermore, if $\rho < \beta$ then by (1.8) $V_\beta(x) \sim h_\beta(x)(1-\rho/\beta)$, thus (1.5) also follows. While, if $\rho > 0$ then (1.7) implies (1.6).

Implications (1.5) $\Rightarrow$ (1.4), (1.7). Assume (1.5). Since $\int_{[0,\infty)} y^\beta \mathrm{d}F(y) = \infty$, we may apply Theorem 8.1.2 in [**1**] with $\alpha = 0$, (see also Feller [**4**, Section VIII.9]) and we have

$$\lim_{x\to\infty} \frac{x^\beta \overline{F}(x)}{V_\beta(x)} = \gamma \in [0,\infty], \tag{2.3}$$

and there exists $p \in [0,\beta]$ and a slowly varying function $\ell \in \mathcal{RV}_0$ such that

$$\gamma = \frac{\beta - p}{p}, \quad \lim_{x\to\infty} \frac{V_\beta(x)}{x^{\beta-p}\ell(x)} = p, \quad \lim_{x\to\infty} \frac{x^p\overline{F}(x)}{\ell(x)} = \beta - p. \tag{2.4}$$

If $\gamma \in (0,\infty)$ in (2.3), then the second convergence in (2.4) implies $p = \beta - \rho \in (0,\beta)$, thus $\gamma = \rho/(\beta-\rho)$. Therefore, by (2.4)

$$\frac{h_\beta(x)}{V_\beta(x)} = 1 + \frac{x^\beta\overline{F}(x)}{V_\beta(x)} \to 1 + \frac{\rho}{\beta-\rho} = \frac{\beta}{\beta-\rho},$$

proving both (1.4) and (1.7).

If $\gamma = 0$, then by (2.4) $p = \beta$, which implies $\rho = 0$, $V_\beta(x) \sim \beta\ell(x)$, and $x^\beta \overline{F}(x) = o(\ell(x))$. Thus $h_\beta(x) \sim V_\beta(x)$ by (1.1), implying (1.4) and (1.7).

If $\gamma = \infty$ then $p = 0$, which implies $\rho = \beta$, $\overline{F}(x) \sim \beta\ell(x)$, and $V_\beta(x) = o(x^\beta \ell(x))$. Therefore, $h_\beta(x) \sim x^\beta \overline{F}(x)$, in particular, (1.4) and (1.7) hold.

Implication (1.6) $\Rightarrow$ (1.4). For any $\rho > 0$ this is an immediate consequence of Karamata's theorem ([**1**, Proposition 1.5.8]). If $\rho = 0$ then by Proposition 1.5.9a in [**1**] h_β is slowly varying, and $h_\beta(x)/(x^\beta \overline{F}(x)) \to \infty$.

Implication (1.7) $\Rightarrow$ (1.4). Assume (1.7). Then, since $h'_\beta(x) = \beta x^{\beta-1}\overline{F}(x)$ Lebesgue almost everywhere, we have as $x \to \infty$

$$\delta(x) := \frac{x h'_\beta(x)}{h_\beta(x)} \to \rho. \tag{2.5}$$

Therefore for some $A > 0$, $B > 0$

$$h_\beta(x) = A \exp\left(\int_B^x \frac{\delta(y)}{y} \mathrm{d}y \right) = A B^{-\rho}\, x^\rho \exp\left(\int_B^x \frac{\delta(y) - \rho}{y} \mathrm{d}y \right).$$

By the representation theorem and (2.5) the second factor is a slowly varying function (in fact it is normalized slowly varying), showing that $h_\beta \in \mathcal{RV}_\rho$, i.e., (1.4) holds.

The only remaining parts are the 'furthermore' statements in the borderline cases $\rho = 0$ and $\rho = \beta$. For $\rho = 0$ the result follows from the proof of Theorem 3.6.8 in [**1**]. Indeed, if $\overline{F} \in \mathcal{RV}_{-\beta}$, then $u(x) = x^{\beta-1}\overline{F}(x) = \ell(x)/x$, with a slowly varying ℓ. As in the proof of Theorem 3.6.8 in [**1**], this implies that $h_\beta \in \Pi$ (monotonicity is not used here). For the converse, assume that $h_\beta \in \Pi_\ell$ with some slowly varying ℓ and nonzero index c. Then, using that $\overline{F}$ is nonincreasing, for any $\lambda > 1$

$$\overline{F}(\lambda x) x^\beta (\lambda^\beta - 1) \leqslant h_\beta(\lambda x) - h_\beta(x) \leqslant \overline{F}(x) x^\beta (\lambda^\beta - 1),$$

which, letting $x \to \infty$ implies

$$c \frac{\log \lambda}{\lambda^\beta - 1} \leqslant \liminf_{x\to\infty} \frac{\overline{F}(x) x^\beta}{\ell(x)} \leqslant \limsup_{x\to\infty} \frac{\overline{F}(x) x^\beta}{\ell(x)} \leqslant c \frac{\log \lambda}{1 - \lambda^{-\beta}}.$$

Letting $\lambda \downarrow 1$, the statement follows.

Finally, we show that for $\rho = \beta$ the conditions $V_\beta \in \mathcal{RV}_\beta$ and $\overline{F} \in \Pi$ are equivalent. By (1.9) and Theorem 3.7.1 in [**1**] (a version of de Haan's theorem) we see that $x^{-\beta} V_\beta(x) \sim c\ell(x)/\beta$ for some $c > 0$ and slowly varying ℓ if and only if (1.2) holds. The proof is complete.

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