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REDUCTION OF AUTOMATA IN LABYRINTHS AND UNIVERSAL TRAPS

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Abstract. We consider the problem of searching labyrinths by collectives of independent automata and show that for such collectives it is possible to construct, by using reduction of automata, traps of various types even in the class of all finite mosaic labyrinths.

1. Introduction

The considered labyrinths lie in the plane, all their "corridors" (edges) are parallel to x- or y-axis, they are integer (their vertices lie in integer points) and all their "corridors" are of length 1; such labyrinths are called mosaic labyrinths [**1**]. We consider collectives of automata in mosaic labyrinths.

Each automaton $\mathfrak{A}$ of the given collective of automata $\mathcal{A}$, which in a moment t is in a state q and in a vertex x of the given mosaic labyrinth L, chooses the direction determined by an edge (x, x') that exits from x and passes into a new state q', and that choice (of vertex x' and state q') depends on its internal state q, on the information about which automata of the collective are located in x and about their internal states in moment t, and on the directions of the edges that exit from x; in moment $t+1$ the automaton $\mathfrak{A}$ passes into state q' and moves to vertex x'. If for every automaton $\mathfrak{A}$ of the collective $\mathcal{A}$ this next state and next position in L (in moment $t+1$) do not depend on the states of the other automata of $\mathcal{A}$ that find themselves in moment t in the same vertex of L as $\mathfrak{A}$ does, then we say that $\mathcal{A}$ is a collective of independent automata.

Labyrinth L is a universal trap for a collective $\mathcal{A}$ if for any choice of initial state of automata from $\mathcal{A}$ and any choice of initial vertices in which these automata are placed in moment 0, there exists a vertex of L which is never visited by any automaton of the collective $\mathcal{A}$. Then the main result of the paper can be formulated in the following way: for any collective of independent automata there exists a universal trap in the class of all finite mosaic labyrinths. As one of direct consequences

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of this result, we also get the main theorem from [**2**]: for any initial (acceptable) automaton there exists a finite mosaic trap.

The paper also considers infinite universal trap for the class of all acceptable initial automata. The proof that such trap exists was first given in [**3**]. Here we give a new proof of this fact. In all the constructions given here we use the method of reduction of automata presented in [**1**]. This paper is a continuation of the research conducted in [**1**], therefore we shall often refer to the terminology and results given there. Still, the definition of certain important notions and notations are given anew, either as in [**1**], or in an equivalent form. This makes it possible to read the present paper practically without looking in [**1**].

2. Basic Notions and Results

The set of all subsets [nonempty subsets] of a set X is denoted by $\mathcal{P}(X)$ $[\mathcal{P}_0(X)]$. Let $X_1, \dots, X_n$ be arbitrary sets. For every $1 \leqslant i \leqslant n$, by pr_i, we denote the projection map of the Cartesian product $X_1 \times \dots \times X_n$ onto X_i. The set of all words over an alphabet A is denoted by A^*; by Λ we denote the empty word.

In a digraph, arcs (arrows or directed edges) are called simply edges. A simple digraph is a digraph that has no loops and multiple edges. A simple digraph is symmetric if for every its edge there is a corresponding inverted edge. In a symmetric simple digraph an edge and its inverted edge make a pair of opposite edges.

Let $\mathfrak{D} = \{\mathbf{e}, \mathbf{n}, \mathbf{w}, \mathbf{s}\}$ (we interpret the elements of the set $\mathfrak{D}$ as the corresponding cardinal points: east, north, west, and south). Define on $\mathfrak{D}$ a unary operation $\cdot^{-1}$ such that: $\mathbf{e}^{-1} = \overline{\mathbf{e}} = \mathbf{w}$, $\mathbf{n}^{-1} = \overline{\mathbf{n}} = \mathbf{s}$, $\mathbf{w}^{-1} = \overline{\mathbf{w}} = \mathbf{e}$, and $\mathbf{s}^{-1} = \overline{\mathbf{s}} = \mathbf{n}$. Extend the operation $\cdot^{-1}$ on the set $\mathfrak{D}^*$ in the following way: if $\alpha = \omega_1 \dots \omega_n \in \mathfrak{D}^*$, $n \geqslant 1$, is an nonempty word over $\mathfrak{D}$, then $\alpha^{-1} = \omega_n^{-1} \dots \omega_1^{-1}$; also, take that $\Lambda^{-1} = \Lambda$.

A connected edge-labeled symmetric simple digraph (L, f), $L = (V, E)$, where V is the set of vertices, E is the set of edges, and $f\colon E \to \mathfrak{D}$ is an edge labeling of L, is a *rectangular labyrinth* (or simply a *labyrinth*) if $f[(y, x)] = (f[(x, y)])^{-1}$ for every $(x, y) \in E$, and if $f(u) \neq f(v)$ for every $u, v \in E$, $u \neq v$, satisfying $\mathrm{pr}_1(u) = \mathrm{pr}_1(v)$.

Let $|u|_L = f(u)$ for each $u \in E$. Also, for every $x \in V$, let $[x]_L = \{|u|_L \mid u \in E \text{ and } \mathrm{pr}_1(u) = x\}$. When the labyrinth L is unambiguous, instead of $|u|_L$ and $[x]_L$ write $|u|$ and $[x]$ respectively. Add to $\mathfrak{D}$ a new element which is denoted by $\mathbf{0}$, and extend the definition of f on the pairs (x, x), $x \in V$, taking that $|(x, x)| = \mathbf{0}$ for every $x \in V$.

Further on, we shall omit f in the designation of a labyrinth (L, f), considering that in every concrete case f is determined. Sometimes, the set of all vertices and the set of all edges of a labyrinth L are labeled by $V(L)$ and $E(L)$ respectively.

A labyrinth L is *finite* if $V(L)$ is a finite set; otherwise L is *infinite*. All labyrinths in the sequel will be finite if it is not stated otherwise.

In a labyrinth L, we can mark a vertex x' [two different vertices x' and x''] as an *entrance* [as an *entrance* and an *exit* respectively]. Denote the fact that L is a labyrinth with an entrance x' [with an entrance x' and an exit x''] by $(L; x')$ (or by $(V(L), E(L); x')$) $[(L; x', x'')$ (or by $(V(L), E(L); x', x''))]$. A labyrinth with an entrance (and possible with an exit) is called an 1-*initial labyrinth.* If L is an 1-initial labyrinth with an entrance x' and an exit x'', then by L^{-1} we denote the same labyrinth but, with the entrance x'' and the exit x'.

Generalize the notion of a 1-initial labyrinth. Fix an integer $n > 1$. Let L be a labyrinth, let $\vec{x}_0 = (x_1^{(0)}, \dots, x_n^{(0)}) \in [V(L)]^n$, and let $x_1 \in V(L)$ be a vertex of L such that $x_1 \neq x_i^{(0)}$ for every $1 \leqslant i \leqslant n$. The pair $(L; \vec{x}_0)$ [The triple $(L; \vec{x}_0, x_1)$] is an n-*initial labyrinth* L with the *entrance* $\vec{x}_0$ [with the *entrance* $\vec{x}_0$ and the *exit* x_1], and in this case, especially when it is not so important to point the entrance of L exactly or when the entrance is known from the context, we sometimes simply say that L is an *initial* labyrinth (an labyrinth having an entrance) or L is an *initial* labyrinth with *valence* n (the length of the entrance). For the sake of shortness, we sometimes denote $(L; \vec{x}_0)$ by $L_{\vec{x}_0}$.

In n-initial labyrinth L the entrance and the exit (if it exists) are sometimes denoted by $x_{\mathrm{s}}(L)$ and $x_{\mathrm{f}}(L)$ respectively.

Let L be a labyrinth. For every walk ρ in L, define the word $|\rho| \in \mathfrak{D}^*$ in the following way: if $\rho = x_0, u_1, x_1, \dots, u_n, x_n$, $n \geqslant 1$, is a nonempty walk in L, then $|\rho| = |u_1| \dots |u_n|$; if ρ is an empty walk in L, then $|\rho| = \Lambda$. For every $x \in V(L)$ and $\alpha \in \mathfrak{D}^*$, if in L there exists a walk ρ starting at x such that $|\rho| = \alpha$, then by $(x\alpha)_L$ or by $x\alpha$ (when the labyrinth L is unambiguous) we denote the end vertex of ρ. Thus $x\Lambda = x$ for every $x \in V(L)$.

Replacing each pair of opposite edges in a labyrinth L with the corresponding nonlabeled undirected edge, we obtain a graph $G(L)$. A labyrinth L is a *tree* if the graph $G(L)$ is a tree. A labyrinth $(L; x', x'')$ is an $\omega_1\omega_2$-*tree*, $\omega_1, \omega_2 \in \mathfrak{D}$, if L is a tree, $[x'] = \{\omega_1\}$, and $[x''] = \{\overline{\omega_2}\}$; an $\omega_1\omega_2$-tree L is an ω-*tree* if $\omega_1 = \omega_2 = \omega$. A vertex x of L is a *leave* of L if it is a leave of $G(L)$.

Let M and N, $M \neq N$, be some points of the plane. By $\overline{MN}$ denote the line segment which is defined by the given points, and by $|\overline{MN}|$ — its length. Let $\vec{\imath}$ and $\vec{\jmath}$ be the unit vectors in the direction of the x-axis and y-axis of the rectangular coordinate system respectively. The vector $\overrightarrow{MN} = \alpha_1\vec{\imath} + \alpha_2\vec{\jmath}$ *goes in the direction*: (1) $\mathbf{e}$ if $\alpha_1 > 0$ and $\alpha_2 = 0$, (2) $\mathbf{n}$ if $\alpha_1 = 0$ and $\alpha_2 > 0$, (3) $\mathbf{w}$ if $\alpha_1 < 0$ and $\alpha_2 = 0$ and (4) $\mathbf{s}$ if $\alpha_1 = 0$ and $\alpha_2 < 0$.

A set T of line segments in the plane $\mathbf{R}^2$ is called a *configuration* (*of line segments*) if any two different line segments of the set T can have not more than one common point, and if such a point exists, it must be an end point for both the line segments.

A labyrinth $L = (V, E)$, $V \subseteq \mathbf{R}^2$, is *plane* if the set $T = \{\overline{xy} \mid (x, y) \in E\}$ is a configuration of line segments, the vector $\overrightarrow{xy}$ goes in the direction $|(x, y)|$ for every $(x, y) \in E$, and $|D \cap V| < +\infty$ for every open disk D in $\mathbf{R}^2$. If L is plane, and, in addition, it holds that $|\overline{xy}| = 1$ for every $(x, y) \in E$, then we say that L is a *mosaic* labyrinth. Moreover, a mosaic labyrinth M is a *maze* if it satisfies that for every $x, y \in V(M)$ from $|\overline{xy}| = 1$ it follows that $(x, y) \in E(M)$.

For every plane labyrinth L, the set $\overline{L} = \bigcup_{(x,y)\in E(L)} \overline{xy}$ is (*geometric*) *realization* of L. A plane labyrinth L is *bounded* if $\operatorname{diam} \overline{L} < +\infty$; otherwise it is *unbounded*.

The notion of isomorphism of labyrinths is introduced in [**1**] in a quite natural way, as isomorphism of edge-labeled digraphs. Remember only that if labyrinths L_1 and L_2 are isomorphic, then we write $L_1 \cong L_2$. By $[L]$ denote the set of all labyrinths which are isomorphic to L.

A plane [mosaic] n-initial labyrinth L, $n \geqslant 1$, with an exit $x_1 \in V(L)$ is *regular* [*perfect*] if there exists an unbounded plane [mosaic] labyrinth L_1 such that $\overline{L} \cap \overline{L_1} = \{x_1\}$ and $x_1 \in V(L_1)$.

By (finite) automaton $\mathfrak{A}$ we mean a quintuple (A, Q, B, φ, ψ), where the finite nonempty sets A, Q and B are the input alphabet, the set of states and the output alphabet of the automaton respectively, $\psi\colon Q \times A \to B$ is its output function, and $\varphi : Q \times A \to Q$ is its state-transition function. If a state q_0 is marked in Q, we get an initial automaton $\mathfrak{A}_{q_0} = (A, Q, B, \varphi, \psi, q_0)$ (in other words, $\mathfrak{A}_{q_0}$ is a Mealy machine). For the given initial or noninitial automaton $\mathfrak{A}$ we sometimes denote the input alphabet, the set of states, the output alphabet, the output function, and the transition function by $A_{\mathfrak{A}}$, $Q_{\mathfrak{A}}$, $B_{\mathfrak{A}}$, $\psi_{\mathfrak{A}}$, and $\varphi_{\mathfrak{A}}$ respectively.

An automaton (initial or noninitial) $\mathfrak{A}$ is *acceptable* if $A_{\mathfrak{A}} = \mathcal{P}(\mathfrak{D})$, $B_{\mathfrak{A}} = \mathfrak{D} \cup \{\mathbf{0}\}$, and $\psi_{\mathfrak{A}}(q, a) \in a \cup \{\mathbf{0}\}$ for all $q \in Q_{\mathfrak{A}}$ and $a \in A_{\mathfrak{A}}$. If $\mathfrak{A}$ [$\mathfrak{A}_{q_0}$] is an acceptable automaton, we write, for the sake of shortness, $\mathfrak{A} = (Q, \varphi, \psi)$ [$\mathfrak{A}_{q_0} = (Q, \varphi, \psi, q_0)$] instead of $\mathfrak{A} = (A, Q, B, \varphi, \psi)$ [$\mathfrak{A}_{q_0} = (A, Q, B, \varphi, \psi, q_0)$].

An acceptable automaton $\mathfrak{A}$ is *trivial* if $\psi_{\mathfrak{A}}(q, a) = \mathbf{0}$ for every $q \in Q_{\mathfrak{A}}$ and $a \in \mathcal{P}(\mathfrak{D})$.

Let $L = (V, E; x_0)$ be a labyrinth and $\mathfrak{A}_{q_0} = (Q, \varphi, \psi, q_0)$ be an initial acceptable automaton.

The sequence $\pi(\mathfrak{A}_{q_0}, L) = (q_0, x_0), (q_1, x_1), \ldots$ is the *behavior* of $\mathfrak{A}_{q_0}$ in L if for every $i \geqslant 0$ it holds that $(q_i, x_i) \in Q \times V$, $(x_i, x_{i+1}) \in E$ or $x_{i+1} = x_i$, $q_{i+1} = \varphi(q_i, [x_i])$, and $\psi(q_i, [x_i]) = |(x_i, x_{i+1})|$; the sequence $\tau(\mathfrak{A}_{q_0}, L) = x_0, x_1, \ldots$ is called the *trajectory* of $\mathfrak{A}_{q_0}$ in L. Let $\pi_i(\mathfrak{A}_{q_0}, L) = (q_i, x_i)$ and $\tau_i(\mathfrak{A}_{q_0}, L) = x_i$ for every $i \geqslant 0$. Also, let $\mathrm{Int}(\mathfrak{A}_{q_0}, L) = \{x_i \mid i \geqslant 0\}$. If L has an exit y_0, we say that $\mathfrak{A}_{q_0}$ *goes out of* the labyrinth $(L; x_0, y_0)$ if $y_0 \in \mathrm{Int}(\mathfrak{A}_{q_0}, L)$; otherwise we say that $(L; x_0, y_0)$ is a *trap* for $\mathfrak{A}_{q_0}$. If $\mathfrak{A}_{q_0}$ goes out of $(L; x_0, y)$ for every $y \in V \smallsetminus \{x_0\}$, we say that $\mathfrak{A}_{q_0}$ *searches* $(L; x_0)$.

Let $V' \subseteq V$. Throwing out of $\pi(\mathfrak{A}_{q_0}, L)$ all the pairs (q_i, x_i) for which $x_i \notin V'$, we get a finite (empty or nonempty) or infinite sequence $(q_{i_0}, x_{i_0}), (q_{i_1}, x_{i_1}), \ldots$ which is called the V'-*behavior* of $\mathfrak{A}_{q_0}$ in $(L; x_0)$. For example, $\pi(\mathfrak{A}_{q_0}, L)$ is the $V(L)$-behavior of $\mathfrak{A}_{q_0}$ in $(L; x_0)$. The sequence $x_{i_0}, x_{i_1}, \ldots$ is the V'-*trajectory* of $\mathfrak{A}_{q_0}$ in $(L; x_0)$.

For every $V_1 \subseteq V$, determine the values $\mathrm{st}(\pi, V_1)$ and $\mathrm{pl}(\pi, V_1)$, where $\pi = \pi(\mathfrak{A}_{q_0}, L)$, in the following way. If there exists $t > 0$ such that $x_t \in V_1$ and for every t', $0 < t' < t$, it holds that $x_{t'} \notin V_1$, then $\mathrm{st}(\pi, V_1) = q_t$ and $\mathrm{pl}(\pi, V_1) = x_t$; otherwise $\mathrm{st}(\pi, V_1)$ and $\mathrm{pl}(\pi, V_1)$ are not determined.

An ordered n-tuple $\mathcal{A} = (\mathfrak{A}_1, \ldots, \mathfrak{A}_n)$, where $\mathfrak{A}_i = (A_i, Q_i, B_i, \varphi_i, \psi_i)$ is an automaton for every $1 \leqslant i \leqslant n$, is said to be an *acceptable collective of automata* (or simply a *collective of automata*) if for every $1 \leqslant i \leqslant n$,

$$A_i = \{a \in \mathcal{P}(\mathfrak{D}) \times \hat{Q}_1 \times \cdots \times \hat{Q}_n \mid \mathrm{pr}_{i+1}(a) = \theta\},$$

where $\theta \notin \sum_{i=1}^{n} Q_i$ is a fixed element and $\hat{Q}_i = Q_i \cup \{\theta\}$ for every $1 \leqslant i \leqslant n$, $B_i = \mathfrak{D} \cup \{\mathbf{0}\}$, and $\psi_i(q, a) \in \mathrm{pr}_1(a) \cup \{\mathbf{0}\}$ for every $q \in Q_i$ and $a \in A_i$. If an n-tuple $\vec{q} = (q_1, \ldots, q_n) \in Q_{\mathcal{A}}$, where $Q_{\mathcal{A}} = Q_1 \times \cdots \times Q_n$, is marked off, we get a *collective of initial automata* $\mathcal{A}_{\vec{q}}$; sometimes, instead of $\mathcal{A}_{\vec{q}}$ we write $((\mathfrak{A}_1)_{q_1}, \ldots, (\mathfrak{A}_n)_{q_n})$. For every $1 \leqslant i \leqslant n$, by $\mathcal{A}_i$ denote the automaton $\mathfrak{A}_i$. Also, by $|\mathcal{A}|$ denote the number of automata in $\mathcal{A}$, i.e., the number n.

For every $\Delta \subseteq \mathfrak{D}$, by $(\Delta)_\theta$ denote $n+1$-tuple $(\Delta, \theta, \ldots, \theta)$. If for every $q \in Q_i$, $a \in A_i$, and $1 \leqslant i \leqslant n$, it holds that $\varphi_i(q, a) = \varphi_i(q, (\mathrm{pr}_1(a))_\theta)$ and $\psi_i(q, a) = \psi_i(q, (\mathrm{pr}_1(a))_\theta)$, we say that $\mathcal{A}$ is a *collective of independent automata*. For every $1 \leqslant i \leqslant n$, by $\hat{\mathfrak{A}}_i = (Q_i, \hat{\varphi}_i, \hat{\psi}_i)$ denote an acceptable automaton defined by $\hat{\varphi}_i(q, a) = \varphi_i(q, (a)_\theta)$ and $\hat{\psi}_i(q, a) = \psi_i(q, (a)_\theta)$ for every $q \in Q_i$ and $a \subseteq \mathfrak{D}$.

Let L be a labyrinth and $\mathcal{A} = (\mathfrak{A}_1, \ldots, \mathfrak{A}_n)$ be a collective of automata. For every $\vec{x} = (x_1, \ldots, x_n) \in [V(L)]^n$, $\vec{q} = (q_1, \ldots, q_n) \in Q_{\mathcal{A}}$, and $1 \leqslant i \leqslant n$, by $a_i(\mathcal{A}, \vec{q}; L, \vec{x})$ denote the ordered $n+1$-tuple $([x_i]_L, a_1^{(i)}, \ldots, a_n^{(i)})$, where

$$a_j^{(i)} = \begin{cases} q_j, & \text{if } x_j = x_i \text{ and } j \neq i; \\ \theta, & \text{if } x_j \neq x_i \text{ or } j = i \end{cases}$$

for every $1 \leqslant j \leqslant n$; note that always $a_i(\mathcal{A}, \vec{q}; L, \vec{x}) \in A_{\mathfrak{A}_i}$.

Let $L_{\vec{x}_0}$ be an n-initial labyrinth and $\mathcal{A}_{\vec{q}_0}$ be a collective of n initial automata, where $\mathcal{A}_i = (A_i, Q_i, B_i, \varphi_i, \psi_i, q_i^{(0)})$ for every $1 \leqslant i \leqslant n$; here $\vec{x}_0 = (x_1^{(0)}, \ldots, x_n^{(0)})$ and $\vec{q}_0 = (q_1^{(0)}, \ldots, q_n^{(0)})$. The *behavior* of $\mathcal{A}_{\vec{q}_0}$ in $L_{\vec{x}_0}$ is a sequence

$$\pi(\mathcal{A}_{\vec{q}_0}, L_{\vec{x}_0}) = (\vec{q}_0, \vec{x}_0), \ldots, (\vec{q}_t, \vec{x}_t), \ldots$$

such that for every $t \geqslant 0$ and $1 \leqslant i \leqslant n$, it holds that $(x_i^{(t)}, x_i^{(t+1)}) \in E(L)$ or $x_i^{(t)} = x_i^{(t+1)}$, $q_i^{(t+1)} = \varphi_i(q_i^{(t)}, \alpha_i^{(t)})$, and $\psi_i(x_i^{(t)}, \alpha_i^{(t)}) = |(x_i^{(t)}, x_i^{(t+1)})|$, where $\vec{x}_t = (x_1^{(t)}, \ldots, x_n^{(t)})$, $\vec{q}_t = (q_1^{(t)}, \ldots, q_n^{(t)})$, and $\alpha_i^{(t)} = a_i(\mathcal{A}, \vec{q}_t; L, \vec{x}_t)$. The sequence $\vec{x}_0, \vec{x}_1, \ldots$ is called *trajectory* of $\mathcal{A}_{\vec{q}_0}$ in $L_{\vec{x}_0}$ and it is denoted by $\tau(\mathcal{A}_{\vec{q}_0}, L_{\vec{x}_0})$. Also, for every $1 \leqslant i \leqslant n$, let

$$\mathrm{Int}_i(\mathcal{A}_{\vec{q}_0}, L_{\vec{x}_0}) = \{x_i^{(j)} \mid j \geqslant 0\} \quad \text{and} \quad \mathrm{Int}(\mathcal{A}_{\vec{q}_0}, L_{\vec{x}_0}) = \bigcup_{i=1}^{n} \mathrm{Int}_i(\mathcal{A}_{\vec{q}_0}, L_{\vec{x}_0}).$$

Let us agree upon the following: in the further text, by the behavior of a collective $\mathcal{A}$ of initial automata in an 1-initial labyrinth L_{x_0}, $x_0 \in V(L)$, we mean the behavior of $\mathcal{A}$ in $|\mathcal{A}|$-initial labyrinth $L_{\vec{y}_0}$, where $\vec{y}_0$ is the $|\mathcal{A}|$-tuple $(x_0, x_0, \ldots, x_0)$.

PROPOSITION 2.1. *Let $L_{\vec{x}_0}$, $\vec{x}_0 = (x_1^{(0)}, \ldots, x_n^{(0)})$, be an n-initial labyrinth and $\mathcal{A} = (\mathfrak{A}_1, \ldots, \mathfrak{A}_n)$ be a collective of n independent automata, $n \geqslant 1$. Then for every $\vec{q}_0 = (q_1^{(0)}, \ldots, q_n^{(0)}) \in Q_{\mathfrak{A}_1} \times \cdots \times Q_{\mathfrak{A}_n}$, it holds that*

$$\mathrm{Int}(\mathcal{A}_{\vec{q}_0}, L_{\vec{x}_0}) = \bigcup_{i=1}^{n} \mathrm{Int}((\hat{\mathfrak{A}}_i)_{q_i^{(0)}}, (L; x_i^{(0)})).$$

PROOF. The assertion follows from the fact that if $(\vec{q}_0, \vec{x}_0), (\vec{q}_1, \vec{x}_1), \ldots$ is the behavior of $\mathcal{A}_{\vec{q}_0}$ in $L_{\vec{x}_0}$, then $(q_i^{(0)}, x_i^{(0)}), (q_i^{(1)}, x_i^{(1)}), \ldots$ is the behavior of $\hat{\mathfrak{A}}_i$ in $(L; x_i^{(0)})$ for every $1 \leqslant i \leqslant n$. □

An acceptable collective of automata which consists of only one automaton can, apparently, be understood as an acceptable automaton, and vice versa. In the sequel, all (individual) automata will be acceptable, and because of that, we just say 'automaton' instead of 'acceptable automaton' for the sake of brevity.

Let L be a finite (noninitial, 1-initial, or n-initial) labyrinth and $\mathcal{A}$ be a collective of n (noninitial or initial) automata. Contrary to the case of only one automaton, in the case of a collective of automata, we can introduce the whole spectrum of different kinds of traps. This spectrum of traps for the given L and $\mathcal{A}$ can be described by the phrase 'L is a γ $\beta\alpha$-trap for $\mathcal{A}$', where the parameter $\alpha \in \{\mathrm{I}, \mathrm{A}\}$ is associated with $\mathcal{A}$, the parameter $\beta \in \{\mathrm{I}, \mathrm{A}, \mathrm{I}_0, \mathrm{A}_0\}$ is associated with L, and the value of the parameter γ is either the word 'weak', or Λ, or 'strong'. The given phrase covers 24 different types of traps. To decode every possible case let us agree that the letters I and A stand for the words 'initial' and 'all' respectively. Further, index 0 in notations for possible values of β means that L is an 1-initial labyrinth, the absence of the index means that L is an n-initial labyrinth (if $n = 1$, there is no difference between I_0 and I, and between A_0 and A, and in that case we prefer to use I and A instead of I_0 and A_0 respectively). Also, if $\gamma =$ 'strong', then $\mathrm{Int}(\mathcal{A}_{\vec{q}_0}, L_{\vec{x}_0}) \neq V(L)$; if $\gamma = \Lambda$, then $\mathrm{Int}_i(\mathcal{A}_{\vec{q}_0}, L_{\vec{x}_0}) \neq V(L)$ for every $1 \leqslant i \leqslant n$, and if $\gamma =$ 'weak', then $\mathrm{Int}_i(\mathcal{A}_{\vec{q}_0}, L_{\vec{x}_0}) \neq V(L)$ for an $1 \leqslant i \leqslant n$.

Let us explain the above "coding" by several examples. Let $\vec{q}_0$ be an element of $Q_{\mathcal{A}}$ and x_0 be a vertex of L. We say that L_{x_0} is strong I_0A-trap for $\mathcal{A}$ if $\mathrm{Int}(\mathcal{A}_{\vec{q}}, L_{x_0}) \neq V(L)$ for every $\vec{q} \in Q_{\mathcal{A}}$; L is AA-trap for $\mathcal{A}$ if $\mathrm{Int}_i(\mathcal{A}_{\vec{q}}, L_{\vec{x}}) \neq V(L)$ for every $1 \leqslant i \leqslant n$, $\vec{q} \in Q_{\mathcal{A}}$ and $\vec{x} \in [V(L)]^n$; L_{x_0} is I_0I-trap for $\mathcal{A}_{\vec{q}_0}$ if $\mathrm{Int}_i(\mathcal{A}_{\vec{q}_0}, L_{x_0}) \neq V(L)$ for every $1 \leqslant i \leqslant n$.

In any concrete case, if we have a labyrinth and a collective of automata, the parameters α and β can be omitted as their values are determined by the context. For example, in the first of the above cases, for a given labyrinth L_{x_0} and a collective of noninitial automata $\mathcal{A}$, we can just say that L_{x_0} is a strong trap for $\mathcal{A}$ (here we obviously deal with a strong I_0A-trap as the letter I_0 is associated with the given labyrinth and the letter A is associated with the collective because it is noninitial). A trap of "the strongest type" for a collective of automata $\mathcal{A}$, i.e., a strong AA-trap for $\mathcal{A}$, is also called a *universal trap* for $\mathcal{A}$.

Let $\mathfrak{A}_1$ and $\mathfrak{A}_2$ be initial automata, and let $(L_1; x_1')$ and $(L_2; x_2')$ be labyrinths. For a $V_1 \subseteq V(L_1)$, let $(q_{i_0}^{(1)}, x_{i_0}^{(1)}), (q_{i_1}^{(1)}, x_{i_1}^{(1)}), \ldots$ be the V_1-behavior of $\mathfrak{A}_1$ in L_1, and for a $V_2 \subseteq V(L_2)$, let $(q_{j_0}^{(2)}, x_{j_0}^{(2)}), (q_{j_1}^{(2)}, x_{j_1}^{(2)}), \ldots$ be the V_2-behavior of $\mathfrak{A}_2$ in L_2. We say that the V_1-behavior of $\mathfrak{A}_1$ in L_1 and the V_2-behavior of $\mathfrak{A}_2$ in L_2 are *isomorphic* if: (1) for every $k \geqslant 0$, $(q_{i_k}^{(1)}, x_{i_k}^{(1)})$ exists iff $(q_{j_k}^{(2)}, x_{j_k}^{(2)})$ exists; and (2) there exist bijections $g\colon Q_{\mathfrak{A}_1} \to Q_{\mathfrak{A}_2}$ and $h\colon V_1 \to V_2$ such that $(g(q_{i_m}^{(1)}), h(x_{i_m}^{(1)})) = (q_{j_m}^{(2)}, x_{j_m}^{(2)})$ for every $m \geqslant 0$ for which $(q_{i_m}^{(1)}, x_{i_m}^{(1)})$ exists. For example, for every initial automaton $\mathfrak{A}_{q_0}$, if $(L_1; x_1') \cong (L_2; x_2')$, then $\pi(\mathfrak{A}_{q_0}, L_1)$ and $\pi(\mathfrak{A}_{q_0}, L_2)$ are isomorphic.

As the behaviors of an automaton in isomorphic labyrinths are isomorphic and, consequently, as it is not important for the problems we investigate here which of isomorphic labyrinths is taken, we do not differentiate isomorphic labyrinths and we adopt the following convention. In the sequel, we introduce some binary operations on labyrinths, which are partially defined and which satisfy the following condition: if $*$ is one of such operations, and L_1 and L_2 some labyrinths, then the labyrinth [edge-labeled digraph] $L * L'$ belongs to the same class of isomorphic labyrinths [edge-labeled digraphs] $[L_1 * L_2]$ for every $L \in [L_1]$ and $L' \in [L_2]$ for which it is defined. In fact, we will consider these operations as operations on corresponding classes of isomorphic labyrinths, and when we say 'given a labyrinth [edge-labeled digraph] $L_1 * L_2$', we mean, in fact, that is given a labyrinth [edge-labeled digraph] from the class $[L_1 * L_2]$, and, consequently, the result of the application of operation $*$ may exist even in the case when $L_1 * L_2$ does not exist.

If by applying some operation on some labyrinths the new edges do not appear and we do not change the original labels of the remaining edges which they had in given labyrinths, we do not describe the edge labeling function of the resulting labyrinth [edge-labeled digraph] for the sake of shortness.

Let $L_1 = (V_1, E_1)$ and $L_2 = (V_2, E_2)$ be arbitrary labyrinths such that $V_1 \cap V_2 = \emptyset$. By $L_1 \dot{\cup} L_2$ denote the disjoint union of labyrinths L_1 and L_2, i.e., $L_1 \dot{\cup} L_2 = (V_1 \cup V_2, E_1 \cup E_2)$.

Let L be a labyrinth, and let x and y, $x \neq y$, be its vertices (not obviously adjacent). Denote the labyrinth $(V(L), E(L) \setminus \{(x,y),(y,x)\})$ by $L - \langle x, y \rangle$.

Let x and y, $x \neq y$, be vertices of a labyrinth $L = (V, E)$ that are not adjacent. Assume that $[x] \cap [y] = \emptyset$ and that $x\omega_1 \neq y\omega_2$ for every $\omega_1, \omega_2 \in \mathfrak{D}$. By $\mathrm{vi}(L, x, y)$ denote the labyrinth

$$(V \setminus \{y\}, [E \setminus ((\{y\} \times V) \cup (V \times \{y\}))] \cup \overleftarrow{E}(x,y) \cup \overrightarrow{E}(x,y)),$$

where $\overleftarrow{E}(x,y) = \{(y\omega, x) \mid \omega \in [y]\}$, $\overrightarrow{E}(x,y) = \{(x, y\omega) \mid \omega \in [y]\}$, and $|(x, y\omega)| = \omega$ and $|(y\omega, x)| = \overline{\omega}$ for every $\omega \in [y]$ (we do not change the labels of the other edges).

Let L and $(L_1; x_1', x_1'')$ be labyrinths such that $V(L) \cap V(L_1) = \emptyset$, and let x and y be different vertices of L. Suppose that $[x]_{L-\langle x,y\rangle} \cap [x_1']_{L_1} = [y]_{L-\langle x,y\rangle} \cap [x_1'']_{L_1} = \emptyset$. By $L\,{}_x{+}_y\,L_1$ denote the labyrinth

$$\mathrm{vi}(\mathrm{vi}((L - \langle x, y \rangle) \dot{\cup} L_1, x, x_1'), y, x_1'').$$

The idea of the described operation is the following: labyrinth L_1 is "put" in L between the vertices x and y.

Let $(L_1; x_1', x_1'')$ and $(L_2; x_2', x_2'')$ be labyrinths such that $V(L_1) \cap V(L_2) = \emptyset$ and $[x_1'']_{L_1} \cap [x_2']_{L_2} = \emptyset$. Denote the labyrinth $(\mathrm{vi}(L_1 \dot{\cup} L_2, x_1'', x_2'); x_1', x_2'')$ by $L_1 L_2$. For given labyrinths $(L_i; x_i', x_i'')$, $1 \leqslant i \leqslant n$, by $L_1 \ldots L_n$ denote the expression $(\ldots((L_1 L_2) L_3) \ldots L_{n-1}) L_n$; denote the entrance x_1' [the exit x_n''] of this labyrinth by $(L_1 \ldots L_n; 0)$ $[(L_1 \ldots L_n; n)]$, and for every $1 \leqslant i \leqslant n-1$, by $(L_1 \ldots L_n; i)$ denote, now in $L_1 \ldots L_n$, the vertex x_i''. If $L_1 \cong \ldots \cong L_n \cong L$, we can write L^n instead of $L_1 \ldots L_n$.

3. Universal traps for collectives of independent automata

For every $a = \{\omega_1, \ldots, \omega_k\} \in \mathcal{P}_0(\mathfrak{D})$, let $V'(a) = \{x_{\omega_1}, \ldots, x_{\omega_k}\}$, $V(a) = \{x_{\mathbf{0}}\} \cup V'(a)$, and let $L(a) = (V(a), E(a); x_{\mathbf{0}})$, where $E(a) = (V'(a) \times \{x_{\mathbf{0}}\}) \cup$

$(\{x_{\mathbf{0}}\} \times V'(a))$ and $(|(x_{\mathbf{0}}, x_\omega)|, |(x_\omega, x_{\mathbf{0}})|) = (\omega, \overline{\omega})$ for all $\omega \in a$. For every $\omega \in \mathfrak{D}$, let $\langle\omega\rangle = (V(\{\omega\}), E(\{\omega\}); x_{\mathbf{0}}, x_\omega)$.

For every set $a = \{\omega_1, \dots, \omega_k\} \in \mathcal{P}_0(\mathfrak{D})$, every function $f : a \to \mathbf{N}$ and every positive integer $\lambda \in \mathbf{N}$, by $L(a; \lambda; f)$ denote the labyrinth

$$((\dots(L(a)\ {}_{x_{\mathbf{0}}}{+}_{x_{\omega_1}} < \omega_1 >^{\lambda i_1})\ {}_{x_{\mathbf{0}}}{+}_{x_{\omega_2}} \dots)\ {}_{x_{\mathbf{0}}}{+}_{x_{\omega_k}} < \omega_k >^{\lambda i_k}; x_{\mathbf{0}}),$$

where $i_j = f(\omega_j)$ for every $1 \leqslant j \leqslant k$. The following assertion is a modification of the corresponding assertion from [**4**].

LEMMA 3.1. *Given an automaton $\mathfrak{A} = (Q, \varphi, \psi)$, a set $a \in \mathcal{P}_0(\mathfrak{D})$, a state $q \in Q$, and two functions $f : a \to \mathbf{N}$ and $f' : a \to \mathbf{N}$, let $\pi = \pi(\mathfrak{A}_q, L(a; \lambda; f))$ and $\pi' = \pi(\mathfrak{A}_q, L(a; \lambda; f'))$, where $\lambda = \max\{|Q|!, 4\}$. Now if $\mathrm{st}(\pi', V(a))$ is defined, then $\mathrm{st}(\pi, V(a)) = \mathrm{st}(\pi', V(a))$ and $\mathrm{pl}(\pi, V(a)) = \mathrm{pl}(\pi', V(a))$.*

PROOF. Suppose that $a = \{\omega_1, \dots, \omega_k\}$ for some $1 \leqslant k \leqslant 4$, and that $i_j = f(\omega_j)$ and $i'_j = f'(\omega_j)$ for every $1 \leqslant j \leqslant k$. It suffices to show that this assertion holds for $i'_1 = \dots = i'_k = 1$. Obviously, if $\mathrm{pl}(\pi', V(a)) = x_{\mathbf{0}}$, then $\mathrm{pl}(\pi, V(a)) = x_{\mathbf{0}}$ and $\mathrm{st}(\pi, V(a)) = \mathrm{st}(\pi', V(a))$. Now assume that $\mathrm{st}(\pi', V(a)) = q'$ and $\mathrm{pl}(\pi', V(a)) = x_{\omega_{k_0}}$ for some $1 \leqslant k_0 \leqslant k$. Put $q_1^{m'} = \mathrm{st}(\pi', \{x_{\mathbf{0}}\omega_{k_0}^{m'}\})$, $1 \leqslant m' \leqslant \lambda$, and $q_2^{m''} = \mathrm{st}(\pi, \{x_{\mathbf{0}}\omega_{k_0}^{m''}\})$, $1 \leqslant m'' \leqslant \lambda i_{k_0}$ (so far we cannot claim that all the values $q_2^{m''}$, $0 \leqslant m'' \leqslant \lambda i_{k_0}$, are determined, but it will be clear later on). It is obvious that $q_2^m = q_1^m$ for every $1 \leqslant m \leqslant \lambda$, and there exist m_1 and m_2, $1 \leqslant m_1 < m_2 \leqslant |Q| + 1 \leqslant \lambda - 1$, such that $q_1^{m_1} = q_1^{m_2}$. Hence $q_2^{m_1} = q_2^{m_2}$. Since $\mathrm{pl}(\pi', V(a)) \neq x_0$, we get that

$$q_2^{m_1+i} = q_2^{m_1+j(m_2-m_1)+i} \tag{3.1}$$

for all nonnegative integers i and j satisfying $m_1 + j(m_2 - m_1) + i \leqslant \lambda i_{k_0}$. Thus, (3.1) holds for $i = \lambda - m_1$ and $j = \lambda(i_{k_0} - 1)/(m_2 - m_1)$ (j is a natural number, since $m_2 - m_1 | \lambda$), and we get that $q_1^\lambda = q_2^\lambda = q_2^{\lambda i_{k_0}}$, that is, $\mathrm{st}(\pi, V(a)) = q_1^\lambda = q'$ and $\mathrm{pl}(\pi, V(a)) = x_{\omega_{k_0}}$, which proves the statement. □

For every word $\alpha \in \mathfrak{D}^*$ by $\nu(\alpha)$ denote the word obtained from α by replacing in it, until it is possible, each subword of the form $\omega\omega^{-1}$, $\omega \in \mathfrak{D}$, with the empty word (for example, if $\alpha =$ **wwnsesnnn**, then $\nu(\alpha) =$ **wnn**). A nonempty word $\alpha \in \mathfrak{D}^*$ is a *simple word* over $\mathfrak{D}$ if $\alpha = \nu(\alpha)$; by $\mathrm{Sim}(\mathfrak{D})$ denote the set of all simple words over $\mathfrak{D}$. For every $\alpha = \omega_1 \dots \omega_n \in \mathrm{Sim}(\mathfrak{D})$ by $\langle\alpha\rangle$ denote a labyrinth $\langle\omega_1\rangle \dots \langle\omega_n\rangle$. A labyrinth L is *snakelike* if $L \cong \langle\alpha\rangle$ for an $\alpha \in \mathrm{Sim}(\mathfrak{D})$; for a given snakelike labyrinth L the corresponding simple word α is unique and we denote it by $\alpha(L)$.

Let $L = (V, E)$ be a labyrinth. If there exists an injective mapping $\mu \colon V \to \mathbf{R}^2$ such that the labyrinth $\mu(L) = (\mu(V), \mu(E))$ is plane, where

$$\mu(E) = \{(\mu(x), \mu(y)) \mid (x, y) \in E\} \quad \text{and} \quad |(\mu(x), \mu(y))|_{\mu(L)} = |(x, y)|_L$$

for every $(x, y) \in E$, then L is *embeddable* and μ is an *embedding* of L. Obviously, if μ is an embedding of L, then $L \cong \mu(L)$. If μ is an embedding of L and $\vec{x}_0 = (x_1^{(0)}, \dots, x_n^{(0)}) \in V^n$ [$x_1 \in V$] is the entrance [exit] of L, we get that $\mu(\vec{x}_0) =$

$(\mu(x_1^{(0)}),\ldots,\mu(x_n^{(0)}))\in[\mu(V)]^n$ $[\mu(x_1)]$ is the entrance [exit] of $\mu(L)$. In the sequel, by an embedding we sometimes mean $\mu(L)$ or even $\overline{\mu(L)}$.

If μ is an embedding of a labyrinth $L=(V,E;\vec{x}_0,x_1)$ and $\mu(L)$ is regular, then L is said to be *perfectly embeddable* and μ is a *perfect embedding* of L. For example, if $(L;x',x'')$ is a tree, then it is embeddable; moreover, if $[x'']_L\neq\mathfrak{D}$, then L is perfectly embeddable.

A labyrinth $(L;x',x'')$ is called a *regular trap* for an initial automaton $\mathfrak{A}$ if $(L;x',x'')$ is a trap for $\mathfrak{A}$ and if it is perfectly embeddable. A labyrinth $(L;x')$ is a *regular trap* for an initial automaton $\mathfrak{A}$ if there exists $x''\in V(L)$ such that $(L;x',x'')$ is a regular trap for $\mathfrak{A}$.

A labyrinth $L=(V,E;x',x'')$ is called a ω*-labyrinth*, $\omega\in\{\mathbf{e},\mathbf{n}\}$, if $[x']=\{\omega\}$, $[x'']=\{\overline{\omega}\}$, and if there exists an embedding μ of L such that $\mathrm{pr}_{k_1}(\mu(x'))=\mathrm{pr}_{k_1}(\mu(x''))$, $\mathrm{pr}_{k_2}(\mu(x''))-\mathrm{pr}_{k_2}(\mu(x'))=r>0$ and

$$|\mathrm{pr}_{k_2}(\mu(z))-\mathrm{pr}_{k_2}(\mu(x'))|+|\mathrm{pr}_{k_2}(\mu(x''))-\mathrm{pr}_{k_2}(\mu(z))|=r$$

for every $z\in V$, where $(k_1,k_2)=(2,1)$ if $\omega=\mathbf{e}$, and $(k_1,k_2)=(1,2)$ if $\omega=\mathbf{n}$; such embedding μ of L is called a *standard embedding* of ω-labyrinth L. It follows from the definition that for every two positive real numbers r_1 and r_2 there exists a standard embedding μ of L such that $|\overline{\mu(x')\mu(x'\omega)}|>r_1$, $|\overline{\mu(x'')\mu(x''\overline{\omega})}|>r_1$ and $\mathrm{diam}(V\setminus\{x',x''\})<r_2$.

Let $L=(V,E)$ be an labyrinth and $V_1\subseteq V$. For every $x\in V$, let $V_x=\{x\}\times(\mathfrak{D}\setminus[x]_L)$ and $E_x=\{(x,y)\,|\,y\in V_x\}\cup\{(y,x)\,|\,y\in V_x\}$. By $\mathrm{Cross}(L,V_1)$ denote the labyrinth $(V\cup(\bigcup_{x\in V_1}V_x),E\cup(\bigcup_{x\in V_1}E_x))$ for which $(|(x,(x,\omega))|,|((x,\omega),x)|)=(\omega,\overline{\omega})$ for every $x\in V_1$ and $\omega\in\mathfrak{D}\setminus[x]_L$, and $|(x,y)|=|(x,y)|_L$ for all $(x,y)\in E$. If $V_1=V$, then instead of $\mathrm{Cross}(L,V)$ write $\mathrm{Cross}(L)$.

Suppose that $\alpha=\omega_1\ldots\omega_n\in\mathrm{Sim}(\mathfrak{D})$, that $L=\langle\alpha\rangle$, and suppose that $x_i=(\langle\alpha\rangle;i)$ for every $0\leqslant i\leqslant n$. Let

$$\begin{aligned}&\dashv\alpha\vdash=\mathrm{Cross}(L,\{x_i\,|\,1\leqslant i\leqslant n-1\}),\quad \dashv\alpha\dashv=\mathrm{Cross}(L,\{x_i\,|\,1\leqslant i\leqslant n\}),\\ &\vdash\alpha\vdash=\mathrm{Cross}(L,\{x_i\,|\,0\leqslant i\leqslant n-1\}),\quad \vdash\alpha\dashv=\mathrm{Cross}(L,\{x_i\,|\,0\leqslant i\leqslant n\}).\end{aligned}$$

Suppose that L is a labyrinth, L_1 is an $\mathbf{e}$-labyrinth and L_2 is an $\mathbf{n}$-labyrinth. Arrange in a sequence $(x_1,y_1),\ldots,(x_m,y_m)$ all edges $(x,y)\in E(L)$ for which $|(x,y)|\in\{\mathbf{e},\mathbf{n}\}$. By $\Delta(L;L_1,L_2)$ denote the labyrinth

$$\big(\ldots\big(L\ {}_{x_1}{+}_{y_1}\ L_{\kappa(x_1,y_1)}\big)\ {}_{x_2}{+}_{y_2}\ \ldots\big)\ {}_{x_m}{+}_{y_m}\ L_{\kappa(x_m,y_m)},$$

and by $\Sigma(L;L_1,L_2)$ – the labyrinth $\Delta(\mathrm{Cross}(L);L_1,L_2)$; here, if $|(x,y)|=\mathbf{e}$, then $\kappa(x,y)=1$, and if $|(x,y)|=\mathbf{n}$, then $\kappa(x,y)=2$.

Let L be a labyrinth, L_1 be an $\mathbf{e}$-labyrinth, L_2 be an $\mathbf{n}$-labyrinth, and let $V_1\subseteq V(L)$. Let us agree that if $\vec{x}_0\in[V(L)]^n$ $[x_1\in V(L)]$ is the entrance [exit] of L, then $\vec{x}_0$ $[x_1]$ is the entrance [exit] of everyone of the labyrinths $\mathrm{Cross}(L,V_1)$, $\Delta(L;L_1,L_2)$, and $\Sigma(L;L_1,L_2)$, unless otherwise stated. It is obvious that the following assertion holds.

PROPOSITION 3.1. [**1**] *Let L_1 be an $\mathbf{e}$-labyrinth and L_2 be an $\mathbf{n}$-labyrinth. If L is an embeddable [a perfectly embeddable] labyrinth, then $\Sigma(L;L_1,L_2)$ $[\Delta(L;L_1,L_2)]$ is an embeddable [a perfectly embeddable] labyrinth, too.*

Let $\mathfrak{A} = (Q, \varphi, \psi)$ be an automaton, L_1 be an $\mathbf{e}$-labyrinth, and L_2 be an $\mathbf{n}$-labyrinth. Construct an automaton $\mathfrak{A}(L_1, L_2) = (Q, \varphi', \psi')$ in the following way. Put $L(a; L_1, L_2) = \Delta(L(a); L_1, L_2)$ for every $a \in \mathcal{P}_0(\mathfrak{D})$. Define the values $\varphi'(q, a)$ and $\psi'(q, a)$ for every $a \in \mathcal{P}_0(\mathfrak{D})$ and $q \in Q$ in the following manner: if the values $q' = \mathrm{st}(\pi(\mathfrak{A}_q; L(a; L_1, L_2)), V'(a))$ and $x_\omega = \mathrm{pl}(\pi(\mathfrak{A}_q; L(a; L_1, L_2)), V'(a))$ are determined, then $\varphi'(q, a) = q'$ and $\psi'(q, a) = \omega$; otherwise $\varphi'(q, a) = q$ and $\psi'(q, a) = \mathbf{0}$. Also, take that $\varphi'(q, \emptyset) = q$ and $\psi'(q, \emptyset) = \mathbf{0}$ for every $q \in Q$. For every $q \in Q$, by $\mathfrak{A}_q(L_1, L_2)$ we denote the automaton $\mathfrak{A}(L_1, L_2)$ with the initial state q.

Let $\mathfrak{A}$ be a noninitial automaton. For every natural number λ and $q \in Q_{\mathfrak{A}}$, let $\mathrm{N}(\mathfrak{A}; \lambda) = \mathfrak{A}(\langle \mathbf{e} \rangle^\lambda, \langle \mathbf{n} \rangle^\lambda)$ and $\mathrm{N}(\mathfrak{A}_q; \lambda) = \mathfrak{A}_q(< \mathbf{e} >^\lambda, < \mathbf{n} >^\lambda)$. The *expanding coefficient* of $\mathfrak{A}$ is the number $\varepsilon(\mathfrak{A}) = \max\{|Q_{\mathfrak{A}}|!, 4\}$.

Let L be an embeddable labyrinth and $\lambda \in \mathbf{N}$. Denote by Z_λ^2 the orthogonal grid which has $(\lambda \mathbf{Z})^2$ as the set of its vertices. Denote by $\mathcal{L}(L; \lambda)$ the set of all mosaic labyrinths L' for which there exists an embedding $\mu_{L'}$ of L such that $\mu_{L'}(V(L)) \subseteq (\lambda \mathbf{Z})^2$ and $\overline{\mu_{L'}(L)} = \overline{L'}$. Let us show that $\mathcal{L}(L; \lambda)$ is nonempty. Choose an embedding μ of L. Let us show that from this embedding we can obtain an embedding μ' of L which satisfies that $\mu'(V(L)) \subseteq (\lambda \mathbf{Z})^2$. Let us mark off all horizontal lines on which the vertices of the embedding μ are situated. These horizontal lines are connected by the vertical line segments of the embedding. By stretching or shrinking these line segments, we can move the horizontal lines so that they will go along the horizontal lines of the grid Z_λ^2. By moving in a similar way the vertical lines on which the vertices of the embedding μ are situated, we obtain a desired embedding μ'. Obviously, there is only one mosaic labyrinth L' satisfying $\overline{\mu'(L)} = \overline{L'}$, and, consequently, we get $L' \in \mathcal{L}(L; \lambda)$. It is clear that the following assertion holds.

PROPOSITION 3.2. *If L is an embeddable labyrinth and $\lambda \geqslant 2$, then every $L' \in \mathcal{L}(L; \lambda)$ is a maze.*

Let L be an embeddable labyrinth and $\lambda \in \mathbf{N}$. For every vertex [set of vertices] $x \in V(L)$ $[V_1 \subseteq V(L)]$ and $L' \in \mathcal{L}(L; \lambda)$ by $x|_{L'}$ $[V_1|_{L'}]$ denote the vertex [set of vertices] $\mu_{L'}(x)$ $[\mu_{L'}(V_1)]$ of L'. It is clear that if $(L; x', x'')$ is a perfectly embeddable labyrinth for some $x', x'' \in V(L)$, then there exists a $L'' \in \mathcal{L}(L; \lambda)$ such that the labyrinth $(L''; x'|_{L''}, x''|_{L''})$ is perfect.

Suppose that $\mathfrak{A}$ is an initial automaton, $(L; x_0)$ is a labyrinth and $W \subseteq V(L)$. Let $x_0, x_1, \ldots$ be the W-trajectory of $\mathfrak{A}$ in $(L; x_0)$. The sequence (finite or infinite) which we obtain by replacing each maximal block of equal elements in $x_0, x_1, \ldots$ with one of these elements (i.e., every finite segment $x, y, \ldots, y, z$ and, if it exists, the infinite segment $x, y, y, \ldots$ in $x_0, x_1, \ldots$ is replaced with x, y, z and x, y respectively; here, $x \neq y$ and $y \neq z$) is called the *cleaned W-trajectory* of $\mathfrak{A}$ in $(L; x_0)$. For example, the cleaned trajectory of an initial trivial automaton $\mathfrak{A}$ in a labyrinth $(L; x_0)$ is the finite sequence x_0 (here we have taken that $W = V(L)$).

Let $\mathfrak{A}_1$ and $\mathfrak{A}_2$ be initial automata and let λ be a natural number. We say that $\mathfrak{A}_1$ is a *λ-interpreter* of $\mathfrak{A}_2$ if for every plane labyrinth $(L; x_0)$ the fact that $x_0, x_1, x_2, \ldots$ is the cleaned $V(L)$-trajectory of $\mathfrak{A}_1$ in $(L; x_0)$ implies the fact that

$x_0|_{L'}, x_1|_{L'}, x_2|_{L'}, \dots$ is the cleaned $V(L)|_{L'}$-trajectory of $\mathfrak{A}_2$ in $(L'; x_0|_{L'})$ for every $L' \in \mathcal{L}(L;\lambda)$. From the definition and Lemma 3.1 we get the following two assertions.

PROPOSITION 3.3. *Let $\mathfrak{A}_1$ and $\mathfrak{A}_2$ be initial automata and let λ be a natural number. If $\mathfrak{A}_1$ is a λ-interpreter of $\mathfrak{A}_2$, then*

$$\mu_{L'}(\mathrm{Int}(\mathfrak{A}_1, L) \cap V') = \mathrm{Int}(\mathfrak{A}_2, L') \cap V'|_{L'}$$

for every $L' \in \mathcal{L}(L;\lambda)$ and every $V' \subseteq V(L)$.

THEOREM 3.1. *For every automaton $\mathfrak{A}$, every $q \in Q_{\mathfrak{A}}$, and every natural number λ, if $\varepsilon(\mathfrak{A})|\lambda$, then $\mathrm{N}(\mathfrak{A}_q;\lambda)$ is a λ-interpreter of $\mathfrak{A}_q$.*

Let $\mathfrak{A}_1, \dots, \mathfrak{A}_m$ be a finite family of automata. The *expanding coefficient* of this family is the number

$$\varepsilon(\mathfrak{A}_1, \dots, \mathfrak{A}_m) = \max\{\varepsilon(\mathfrak{A}_1) \dots, \varepsilon(\mathfrak{A}_m), 12\}.$$

Note that $\varepsilon(\mathfrak{A}_i) \,|\, \varepsilon(\mathfrak{A}_1, \dots, \mathfrak{A}_m)$ for every $1 \leqslant i \leqslant m$. Theorem 3.1 implies the following assertion.

COROLLARY 3.1. *Let $\mathfrak{A}_1, \dots, \mathfrak{A}_m$ be a finite family of automata and let λ be a natural number. If $\varepsilon(\mathfrak{A}_1, \dots, \mathfrak{A}_m)|\lambda$, then $\mathrm{N}((\mathfrak{A}_i)_q;\lambda)$ is a λ-interpreter of $(\mathfrak{A}_i)_q$ for every $q \in Q_{\mathfrak{A}_i}$ and every $1 \leqslant i \leqslant m$.*

Let $(L; x', x'')$ be a labyrinth such that $[x']_L = \{\omega\}$ for an $\omega \in \mathfrak{D}$. If there exists a perfect embedding μ for L such that the ray going out from $\mu(x')$ in the direction $\overline{\omega}$ does not intersect with $\overline{\mu(L)}$, then μ an *extraperfect embedding* for L, and L is an *extraperfectly* (or *ω-extraperfectly*) *embeddable* labyrinth.

Let $\mathfrak{A} = (Q, \varphi, \psi)$ be an automaton, $(L; x', x'')$ be an ω-extraperfectly embeddable labyrinth for an $\omega \in \mathfrak{D}$ and $L_1 = \mathrm{Cross}(L, \{x'\})$. If there exists $q \in Q$ such that $\psi(q, \mathfrak{D}) \in \{\omega, \mathbf{0}\}$ and $\mathrm{pl}(\pi(\mathfrak{A}_q, L_1), \{x', x''\}) \neq x''$, then we say that $\mathfrak{A}$ is an *L-reducible* automaton, or that L *reduces* $\mathfrak{A}$.

An automaton $\mathfrak{A}$ is *reducible* if there exists a labyrinth L reducing $\mathfrak{A}$; otherwise it is *irreducible*. Note that if $\mathfrak{A}$ is irreducible, then $\psi_{\mathfrak{A}}(q, \mathfrak{D}) \neq \mathbf{0}$.

Assume that $\mathfrak{A}_q = (Q, \varphi, \psi, q)$ is an initial automaton such that $\omega_1 = \psi(q, \mathfrak{D}) \neq \mathbf{0}$. Take an $\alpha = \omega_1 \dots \omega_n \in \mathrm{Sim}(\mathfrak{D})$, $n \geqslant 2$. Let $L = \vdash\alpha\vdash$, $\tau(\mathfrak{A}_q, L) = x_0, x_1, \dots$, and $z_i = (\langle\alpha\rangle; i)$ for every $0 \leqslant i \leqslant n$. We say that $\mathfrak{A}_q$ *returns on* α if for some i and j, $0 \leqslant i < j < n$, there exist m_1 and m_2, $0 < m_1 < m_2$, such that $x_{m_2} = z_i$, $x_{m_1} = z_j$ and $x_k \neq z_n$ for every $0 \leqslant k < m_2$.

PROPOSITION 3.4. **[1]** *If $\mathfrak{A}$ is an irreducible automaton, then $\mathfrak{A}_q$ does not return on every $\alpha = \psi_{\mathfrak{A}}(q, \mathfrak{D})\omega_2 \dots \omega_n \in \mathrm{Sim}(\mathfrak{D})$, $n \geqslant 2$, for each $q \in Q_{\mathfrak{A}}$.*

Given two automata $\mathfrak{A}_1$ and $\mathfrak{A}_2$, two labyrinths L_1 and L_2, and a mapping $f\colon Q_{\mathfrak{A}_2} \to Q_{\mathfrak{A}_1}$. We say that the pair $(\mathfrak{A}_1, L_1)$ *f-imitates* the pair $(\mathfrak{A}_2, L_2)$, and we write $(\mathfrak{A}_1, L_1) \leqslant_f (\mathfrak{A}_2, L_2)$ if:

(1) there exists an injection $g : V(L_1) \to V(L_2)$ such that for every $x_0 \in V(L_1)$ and for every $q \in Q_{\mathfrak{A}_2}$ the cleaned $g[V(L_1)]$-trajectory of $(\mathfrak{A}_2)_q$

in $(L_2; g(x_0))$ is either the infinite sequence $g(x_0), g(x_1), \ldots$ or the finite sequence $g(x_0), g(x_1), \ldots, g(x_m)$ for some $m \geqslant 0$, where

$$\tau((\mathfrak{A}_1)_{f(q)}, (L_1; x_0)) = x_0, x_1, \ldots;$$

(2) for an $x_0 \in V(L_1)$ and a $q \in Q_{\mathfrak{A}_2}$, $(L_1; x_0)$ is a regular trap for $(\mathfrak{A}_1)_{f(q)}$, then $(L_2; g(x_0))$ is a regular trap for $(\mathfrak{A}_2)_q$.

Such a function g from this definition is called a *justifying injection* for the relation $(\mathfrak{A}_1, L_1) \leqslant_f (\mathfrak{A}_2, L_2)$. We say that $\mathfrak{A}_1$ *imitates* $\mathfrak{A}_2$, and we write $\mathfrak{A}_1 \leqslant \mathfrak{A}_2$, if there exists a mapping $f : Q_{\mathfrak{A}_2} \to Q_{\mathfrak{A}_1}$ such that for every labyrinth L_1 there exists a labyrinth L_2 satisfying $(\mathfrak{A}_1, L_1) \leqslant_f (\mathfrak{A}_2, L_2)$ (though slightly changed, this definition is equivalent to the corresponding definition from [**1**]).

Suppose that L is a labyrinth, that $L_1^{(0)}, \ldots, L_1^{(m)}$ are some **e**-labyrinths, and that $L_2^{(0)}, \ldots, L_2^{(m)}$ are some **n**-labyrinths for an $m \geqslant 0$. Let $L^{(0)} = \Sigma(L; L_1^{(0)}, L_2^{(0)})$ and $L^{(i)} = \Sigma(L^{(i-1)}; L_1^{(i)}, L_2^{(i)})$ for every $1 \leqslant i \leqslant m$. Note that $V(L) \subseteq V(L^{(m)})$. Let $g(L; L_1^{(0)}, \ldots, L_1^{(m)}; L_2^{(0)}, \ldots, L_2^{(m)}) \colon V(L) \to V(L^{(m)})$ be the function satisfying $g(L; L_1^{(0)}, \ldots, L_1^{(m)}; L_2^{(0)}, \ldots, L_2^{(m)})(v) = v$ for every $v \in V(L)$. Denote the labyrinth $L^{(m)}$ by $\Sigma(L; L_1^{(0)}, \ldots, L_1^{(m)}; L_2^{(0)}, \ldots, L_2^{(m)})$.

The following assertion is, in fact, the assertion of Theorem 3.3 from [**1**] that we reformulate here for our purposes.

THEOREM 3.2. *Let $\mathfrak{A}_1, \ldots, \mathfrak{A}_n$ be arbitrary automata. Then there exist automata $\mathfrak{A}'_1, \ldots, \mathfrak{A}'_n$, mappings $f_k \colon Q_{\mathfrak{A}_k} \to Q_{\mathfrak{A}'_k}$, $1 \leqslant k \leqslant n$, and for some $m \geqslant 0$ there exist* **e**-*labyrinths $L_1^{(0)}, \ldots, L_1^{(m)}$ and* **n**-*labyrinths $L_2^{(0)}, \ldots, L_2^{(m)}$ such that $\mathfrak{A}'_i$ is irreducible or trivial for every $1 \leqslant i \leqslant n$ and $g(L; L_1^{(0)}, \ldots, L_1^{(m)}; L_2^{(0)}, \ldots, L_2^{(m)})$ is a justifying injection for*

$$(\mathfrak{A}'_i, L) \leqslant_{f_i} (\mathfrak{A}_i, \Sigma(L; L_1^{(0)}, \ldots, L_1^{(m)}; L_2^{(0)}, \ldots, L_2^{(m)}))$$

for every labyrinth L and every $1 \leqslant i \leqslant n$.

Suppose that $\mathfrak{A}$ is an initial automaton, $(L; x_0)$ is a labyrinth and $\tau = y_0, y_1, \ldots$ is the cleaned trajectory of $\mathfrak{A}$ in $(L; x_0)$ (τ is finite or infinite).

Let $z \in V(L) \setminus \mathrm{Leaf}(L)$, where $\mathrm{Leaf}(L)$ is the set of all leaves of L. A finite segment $\tau' = y_m, y_{m+1}, \ldots, y_n$ $(0 \leqslant m \leqslant n)$ or an infinite segment $\tau' = y_m, y_{m+1}, \ldots$ $(m \geqslant 0)$ of τ is called a *z-block* of τ if τ' has at least one appearance of z and contains, besides z, only leaves of L. A *z-block* τ' of τ is *regular* if it is maximal, i.e., there is no other, different of it, *z-block* which contains τ'. A segment τ' of τ is a *regular block* if there exists a $z \in V(L)$ such that τ' is a regular z-block.

If τ contains at least one $z \notin \mathrm{Leaf}(L)$, then it has at least one regular block, its regular blocks cover it, and any two different regular blocks of τ are disjoint.

Suppose that τ' is a regular block of τ. Let $z \in V(L) \setminus \mathrm{Leaf}(L)$ be such that τ' is a regular z-block. Perform the following procedure on the elements of τ': replace, as long as it is possible, each segment of τ' (or of the block obtained from τ' after the application of some number of those replacements) of the form z, w, z, where $w \in \mathrm{Leaf}(L)$, by z if on the left of it there is at least one appearance of an exactly the same segment (e.g., this procedure transforms z-block

$z, w_1, z, w_2, z, w_2, z, w_1, z, w_3, z$, where w_1, w_2 and w_3 are different leaves of L, into $z, w_1, z, w_2, z, w_3, z$). Perform the above procedure on each regular block of τ. The obtained sequence is called the *doubly cleaned trajectory* of $\mathfrak{A}$ in $(L; x_0)$.

Let $\sigma_{\rm r}$ and $\sigma_{\rm l}$ be respectively the permutations $(\mathbf{e\,n\,w\,s})$ and $(\mathbf{e\,s\,w\,n})$ of the set $\mathfrak{D}$ (given in cycle notation).

An initial automaton $\mathfrak{A}_{q_0}$ is a *snakelike σ_0-walker*, $\sigma_0 \in \{\sigma_{\rm r}, \sigma_{\rm l}\}$, if $\omega_1 = \psi_{\mathfrak{A}_{q_0}}(q_0, \mathfrak{D}) \neq \mathbf{0}$, and if for every $\alpha = \omega_1 \ldots \omega_n \in \mathrm{Sim}(\mathfrak{D})$, $n \geqslant 2$, the doubly cleaned trajectory $y_0, y_1, \ldots$ of $\mathfrak{A}_{q_0}$ in $\vdash\alpha\vdash$ is such that there exists $i_0 = \min\{j \mid y_j = x_{\rm f}(\vdash\alpha\vdash)\}$, and it holds that $|(y_i, y_{i+1})| = \sigma_0(|(y_i, y_{i-1})|)$ for every $1 \leqslant i \leqslant i_0 - 1$ satisfying y_i is not a leaf. A snakelike σ_0-walker is a *snakelike rightwalker* [*snakelike leftwalker*] if $\sigma_0 = \sigma_{\rm r}$ $[\sigma_0 = \sigma_{\rm l}]$.

By $\mathcal{L}_{\rm plwl}$ denote the class of all plane labyrinths without leaves. An initial automaton $\mathfrak{A}_{q_0}$ is a *σ_0-walker*, $\sigma_0 \in \{\sigma_{\rm r}, \sigma_{\rm l}\}$, if for every $L \in \mathcal{L}_{\rm plwl}$ and every $x_0 \in V(L)$ the doubly cleaned trajectory $x_0 = y_0, y_1, y_2, \ldots$ of $\mathfrak{A}_{q_0}$ in $(\mathrm{Cross}(L); x_0)$ is infinite and $|(y_i, y_{i+1})| = \sigma_0(|(y_i, y_{i-1})|)$ for every $i \geqslant 1$ satisfying $y_i \in V(L)$; we say that a σ_0-walker $\mathfrak{A}_{q_0}$ is a *σ_0-walker with the guiding vector* ω if $|(y_0, y_1)| = \sigma_0(\overline{\omega})$. A σ_0-walker $\mathfrak{A}_{q_0}$ is said to be a *rightwalker* [*leftwalker*] if $\sigma_0 = \sigma_{\rm r}$ $[\sigma_0 = \sigma_{\rm l}]$.

Proposition 3.5. [**1**] *Let $\mathfrak{A}$ be an irreducible automaton, $q_0 \in Q_{\mathfrak{A}}$, and $\alpha = \omega_1 \ldots \omega_n \in \mathrm{Sim}(\mathfrak{D})$, where $\omega_1 = \psi_{\mathfrak{A}}(q_0, \mathfrak{D})$ and $n \geqslant 1$. If $\mathfrak{A}_{q_0}$ is a snakelike σ_0-walker for a $\sigma_0 \in \{\sigma_{\rm r}, \sigma_{\rm l}\}$, then $\mathfrak{A}_{q_1}$, where $q_1 = \mathrm{st}(\pi(\mathfrak{A}_{q_0}, \vdash\alpha\vdash), \{x_{\rm f}(\vdash\alpha\vdash)\})$, is a σ_0-walker with the guiding vector ω_n.*

Note that if an initial automaton $\mathfrak{A}_{q_0}$ is a *σ_0-walker* for some $\sigma_0 \in \{\sigma_{\rm r}, \sigma_{\rm l}\}$, then $\mathfrak{A}_{q_0}$ is also a snakelike σ_0-walker.

Let $\mathfrak{A}$ be an automaton. A state $q \in Q_{\mathfrak{A}}$ *orients* $\mathfrak{A}$ [*with the guiding vector* ω] if $\mathfrak{A}_q$ is an leftwalker or an rightwalker [with the guiding vector ω].

Assume that $\mathfrak{A}_{q_0}$ is an initial automaton, L_1 is a snakelike $\mathbf{e}$-labyrinth, and L_2 is a snakelike $\mathbf{n}$-labyrinth. Let $L_1 \oplus L_2 = L(\mathfrak{D}; \dashv\alpha_1\vdash, \dashv\alpha_2\vdash)$, where $\alpha_1 = \alpha(L_1)$ and $\alpha_2 = \alpha(L_2)$. We say that the pair (L_1, L_2) *orients* $\mathfrak{A}_{q_0}$ if the state $q(\mathfrak{A}_{q_0}; L_1, L_2) = \mathrm{st}(\pi(\mathfrak{A}_{q_0}, L_1 \oplus L_2), V'(\mathfrak{D}))$ exists and orients $\mathfrak{A}$ with the guiding vector ω satisfying $x_\omega = \mathrm{pl}(\pi(\mathfrak{A}_{q_0}, L_1 \oplus L_2), V'(\mathfrak{D}))$.

Theorem 3.3. [**1**] *If an automaton $\mathfrak{A}$ is irreducible, then for every $q \in Q_{\mathfrak{A}}$ there exist a snakelike $\mathbf{e}$-labyrinth L_1 and a snakelike $\mathbf{n}$-labyrinth L_2 such that the pair (L_1, L_2) orients $\mathfrak{A}_q$.*

Let $\mathfrak{A}_1, \ldots, \mathfrak{A}_m$ be a finite family of automata, L_1 be a snakelike $\mathbf{e}$-labyrinth, and L_2 be a snakelike $\mathbf{n}$-labyrinth. We say that the pair (L_1, L_2) *orients* the given family of automata if for every $1 \leqslant i \leqslant m$ and every $q \in Q_{\mathfrak{A}_i}$ the pair (L_1, L_2) orients the automaton $(\mathfrak{A}_i)_q$. Let us generalize the assertion of Theorem 3.3.

Two initial automata $(\mathfrak{A}_1)_{q_1}$ and $(\mathfrak{A}_2)_{q_2}$ are *τ-isomorphic*, $(\mathfrak{A}_1)_{q_1} \cong_\tau (\mathfrak{A}_2)_{q_2}$, if there exists a bijection $\iota\colon Q_{\mathfrak{A}_1} \to Q_{\mathfrak{A}_2}$ such that $\iota(q_1) = q_2$, and $\iota(\varphi_1(q, a)) = \varphi_2(\iota(q), a)$ and $\psi_1(q, a) = \psi_2(\iota(q), a)$ for every $q \in Q_{\mathfrak{A}_1}$ and $a \subseteq \mathfrak{D}$. It is clear that τ-isomorphic automata in an arbitrary initial labyrinth have equal trajectories.

THEOREM 3.4. *For each finite family of irreducible automata $\mathfrak{A}_1, \dots, \mathfrak{A}_m$, $m \geqslant 1$, there exist a snakelike* **e***-labyrinth L_1 and a snakelike* **n***-labyrinth L_2 such that the pair (L_1, L_2) orients the given family.*

PROOF. Let $\mathfrak{A}'_1, \dots, \mathfrak{A}'_n$ be a family of pairwise non-τ-isomorphic initial automata such that for every $1 \leqslant i \leqslant m$ and every $q \in Q_{\mathfrak{A}_i}$ there exists $1 \leqslant j \leqslant n$ such that $\mathfrak{A}'_j \cong_\tau (\mathfrak{A}_i)_q$. Let $\mathfrak{A}'_j = (Q_j, \varphi_j, \psi_j, q_0^{(j)})$ for every $1 \leqslant j \leqslant n$.

Introduce four variables $\beta_{\mathbf{e}}$, $\beta_{\mathbf{w}}$, $\beta_{\mathbf{n}}$ and $\beta_{\mathbf{s}}$ whose values are words from the set $\mathrm{Sim}(\mathfrak{D}) \cup \{\Lambda\}$ and take Λ as their initial value, and introduce an integer variable j and put that $j := 1$. Perform the following procedure:

1° From Theorem 3.3 it follows that there exist a snakelike **e**-labyrinth $L_1^{(1)}$ and a snakelike **n**-labyrinth $L_2^{(1)}$ such that the pair $(L_1^{(1)}, L_2^{(1)})$ orients $\mathfrak{A}'_1$ with the guided vector $\omega_1^{(1)}$. Let $\alpha_{\mathbf{e}}^{(1)} = \alpha(L_1^{(1)})$, $\alpha_{\mathbf{n}}^{(1)} = \alpha(L_2^{(1)})$, $\alpha_{\mathbf{w}}^{(1)} = [\alpha(L_1^{(1)})]^{-1}$, and $\alpha_{\mathbf{s}}^{(1)} = [\alpha(L_2^{(1)})]^{-1}$. Also, let $\beta_{\omega_1^{(1)}} := \beta_{\omega_1^{(1)}} \alpha^{(1)}_{\omega_1^{(1)}}$ and $j := j + 1$. Go to 2°.

2° Finish the procedure if $j > n$; otherwise, let $\omega_0^{(j)} = \psi_j(q_0^{(j)}, \mathfrak{D})$ and $\mathfrak{A}''_j = (Q_j, \varphi_j, \psi_j, q_1^{(j)})$, where $q_1^{(j)} = \mathrm{st}(\pi(\mathfrak{A}'_j, \vdash \beta_{\omega_0^{(j)}} \vdash), x_{\mathrm{f}}(\vdash \beta_{\omega_0^{(j)}} \vdash))$ if $\beta_{\omega_0^{(j)}} \neq \Lambda$ and $q_1^{(j)} = q_0^{(j)}$ if $\beta_{\omega_0^{(j)}} = \Lambda$ (as (Q_j, φ_j, ψ_j) is irreducible, value $q_1^{(j)}$ exists).

From Theorem 3.3 it follows that there exist a snakelike **e**-labyrinth $L_1^{(j)}$ and a snakelike **n**-labyrinth $L_2^{(j)}$ such that the pair $(L_1^{(j)}, L_2^{(j)})$ orients $\mathfrak{A}''_j$ with the guided vector $\omega_1^{(j)}$ (if $\beta_{\omega_0^{(j)}} = \Lambda$, then $\omega_1^{(j)} = \omega_0^{(j)}$). Note that $\omega_1^{(j)} \neq \overline{\omega}_0^{(j)}$, otherwise the noninitial automaton corresponding to $\mathfrak{A}''_j$ is not irreducible (see Proposition 3.4). Let $\alpha_{\mathbf{e}}^{(j)} = \alpha(L_1^{(j)})$, $\alpha_{\mathbf{n}}^{(j)} = \alpha(L_2^{(j)})$, $\alpha_{\mathbf{w}}^{(j)} = [\alpha(L_1^{(j)})]^{-1}$, and $\alpha_{\mathbf{s}}^{(j)} = [\alpha(L_2^{(j)})]^{-1}$. Also, let $\beta_{\omega_0^{(j)}} := \beta_{\omega_0^{(j)}} \alpha^{(j)}_{\omega_1^{(j)}} \gamma^{(j)}$, where $\gamma^{(j)} = \omega_0^{(j)} [\omega_1^{(j)}]^{-1} \omega_0^{(j)}$ if $\omega_1^{(j)} \neq \omega_0^{(j)}$ or $\gamma^{(j)} = \Lambda$ if $\omega_1^{(j)} = \omega_0^{(j)}$, and take that $j := j + 1$. Go to 2°.

Note that after each step of our procedure (one individual implementation of 1° or 2°) it holds that $\beta_{\mathbf{e}}\beta_{\mathbf{w}}^{-1}$ $[\beta_{\mathbf{n}}\beta_{\mathbf{s}}^{-1}]$ is an empty word or $\langle \beta_{\mathbf{e}}\beta_{\mathbf{w}}^{-1} \rangle$ $[\langle \beta_{\mathbf{n}}\beta_{\mathbf{s}}^{-1} \rangle]$ is an **e**-labyrinth [**n**-labyrinth]. Denote the final value of the variable β_ω by $\hat{\beta}_\omega$ for every $\omega \in \mathfrak{D}$, and take $L_1 = \hat{\beta}_{\mathbf{e}} \mathbf{e} \hat{\beta}_{\mathbf{w}}^{-1}$ and $L_2 = \hat{\beta}_{\mathbf{n}} \mathbf{n} \hat{\beta}_{\mathbf{s}}^{-1}$. From Proposition 3.5 we get that the pair (L_1, L_2) satisfies the condition of the theorem. □

Let L be a plane labyrinth. Any connected component of the set $\mathbf{R}^2 \setminus \overline{L}$ is called a *face* of L. If L is finite (as we always suppose if it is not stated otherwise), it has only one unbounded face and $k \geqslant 0$ bounded faces. The unique unbounded face of L is also called the *external face* of L, and we usually denote it by $f_\infty(L)$.

Let f be a face of a plane labyrinth L. The *vertex boundary* of f in L is the set $b(f) = \overline{f} \cap V(L)$ (by $\overline{f}$ we denote the closure of f in the standard topology on $\mathbf{R}^2$). It is clear that the following proposition holds.

PROPOSITION 3.6. *Let $(L; x_0)$ be a labyrinth from $\mathcal{L}_{\mathrm{plwl}}$, $\mathfrak{A}$ be an irreducible automaton, and let L_1 be a snakelike* **e***-labyrinth and L_2 be a snakelike* **n***-labyrinth such that the pair (L_1, L_2) orients $\mathfrak{A}$. Then for every $q \in Q_{\mathfrak{A}}$, there exists a face f*

of L, satisfying $x_0 \in b(f)$, such that

$$\mathrm{Int}(\mathfrak{A}_q, \mathrm{Cross}(\Delta(L; L_1, L_2))) \cap V(L) = b(f).$$

Let $\mathfrak{A}_1, \dots, \mathfrak{A}_n$ be a finite family of automata. For every $1 \leqslant i \leqslant n$, let $\mathfrak{A}'_i = \mathrm{N}(\mathfrak{A}_i; \varepsilon(\mathfrak{A}_1, \dots, \mathfrak{A}_n))$. From Theorem 3.2 we get that for automata $\mathfrak{A}'_1, \dots, \mathfrak{A}'_n$ there exist automata $\mathfrak{A}''_1, \dots, \mathfrak{A}''_n$, mappings $f_k : Q_{\mathfrak{A}'_k} \to Q_{\mathfrak{A}''_k}$, $1 \leqslant k \leqslant n$, and for some $m \geqslant 0$ there exist $\mathbf{e}$-labyrinths $L_1^{(0)}, \dots, L_1^{(m)}$ and $\mathbf{n}$-labyrinths $L_2^{(0)}, \dots, L_2^{(m)}$ such that $\mathfrak{A}''_i$ is irreducible or trivial for every $1 \leqslant i \leqslant n$ and the function

$$g(L; L_1^{(0)}, \dots, L_1^{(m)}; L_2^{(0)}, \dots, L_2^{(m)})$$

is a justifying injection for

$$(\mathfrak{A}''_i, L) \leqslant_{f_i} (\mathfrak{A}'_i, \Sigma(L; L_1^{(0)}, \dots, L_1^{(m)}; L_2^{(0)}, \dots, L_2^{(m)}))$$

for every labyrinth L and every $1 \leqslant i \leqslant n$.

Let M be an arbitrary labyrinth. Define a labyrinth $L'(\mathfrak{A}_1, \dots, \mathfrak{A}_n; M)$ in the following way:

(1) Among $\mathfrak{A}''_1, \dots, \mathfrak{A}''_n$ there are irreducible automata. Let $\mathfrak{A}''_{i_1}, \dots, \mathfrak{A}''_{i_k}$, where $1 \leqslant i_1 < i_2 < \dots < i_k \leqslant n$, be all irreducible automata among them. By Theorem 3.4, there exist a snakelike $\mathbf{e}$-labyrinth L_1 and a snakelike $\mathbf{n}$-labyrinth L_2 such that the pair (L_1, L_2) orients the automata $\mathfrak{A}''_{i_1}, \dots, \mathfrak{A}''_{i_k}$. Take

$$L'(\mathfrak{A}_1, \dots, \mathfrak{A}_n; M) = \mathrm{Cross}(\Delta(M; L_1, L_2)).$$

(2) Among $\mathfrak{A}''_1, \dots, \mathfrak{A}''_n$ there are no irreducible automata. Then let

$$L'(\mathfrak{A}_1, \dots, \mathfrak{A}_n; M) = M.$$

Also let

$$L(\mathfrak{A}_1, \dots, \mathfrak{A}_n; M) = \Sigma(L'(\mathfrak{A}_1, \dots, \mathfrak{A}_n; M); L_1^{(0)}, \dots, L_1^{(m)}; L_2^{(0)}, \dots, L_2^{(m)}).$$

Clearly, labyrinths $L'(\mathfrak{A}_1, \dots, \mathfrak{A}_n; M)$ and $L(\mathfrak{A}_1, \dots, \mathfrak{A}_n; M)$, and, also, automata $\mathfrak{A}''_1, \dots, \mathfrak{A}''_n$, are not uniquely determined, but for us it is important only that these labyrinths and these automata are constructed in the above given way. Further on, to simplify our discussion, we can freely use the notations given here.

Now let us prove that for every collective of independent (initial or noninitial) automata, even in the class of all finite mosaic labyrinth, we can construct traps of all types defined above which are, in principle, admissible for the given collective of automata. Obviously, it is enough to show that the following assertion holds.

THEOREM 3.5. *For every collective of independent automata there exists a mosaic universal trap, i.e., a mosaic strong AA-trap.*

PROOF. Let $\mathcal{A} = (\mathfrak{A}_1, \dots, \mathfrak{A}_n)$ be a collective of independent automata. Suppose that M is a maze for which $V(M) = \{x_{i,j} = (i,j) \in \mathbf{Z}^2 \mid 1 \leqslant i, j \leqslant a\}$ for some $a \geqslant 3$. Consider the family of automata $\hat{\mathfrak{A}}_1, \dots, \hat{\mathfrak{A}}_n$. For every $1 \leqslant i \leqslant n$, let $\hat{\mathfrak{A}}'_i = \mathrm{N}(\hat{\mathfrak{A}}_i; \varepsilon(\mathcal{A}))$, where $\varepsilon(\mathcal{A}) = \varepsilon(\hat{\mathfrak{A}}_1, \dots, \hat{\mathfrak{A}}_n)$. As above, define automata $\hat{\mathfrak{A}}''_1, \dots, \hat{\mathfrak{A}}''_n$, and labyrinths $M' = L'(\hat{\mathfrak{A}}_1, \dots, \hat{\mathfrak{A}}_n; M)$ and $M'' = L(\hat{\mathfrak{A}}_1, \dots, \hat{\mathfrak{A}}_n; M)$.

As we have seen, $\mathcal{L}(M'';\varepsilon(\mathcal{A})) \neq \emptyset$. Take a labyrinth $K \in \mathcal{L}(M'';\varepsilon(\mathcal{A}))$. Also, take arbitrarily $\vec{y} = (y_1, \dots, y_n) \in [V(K)]^n$ and $\vec{\kappa} = (\kappa_1, \dots, \kappa_n) \in Q_{\hat{\mathfrak{A}}_1} \times \cdots \times Q_{\hat{\mathfrak{A}}_n}$. Define sets W_r, $1 \leqslant r \leqslant n$. Fix an $1 \leqslant r \leqslant n$. If $V(M)|_K \cap \operatorname{Int}((\hat{\mathfrak{A}}_r)_{\kappa_r}, K_{y_r}) = \emptyset$, then $W_r = \emptyset$. If $V(M)|_K \cap \operatorname{Int}((\hat{\mathfrak{A}}_r)_{\kappa_r}, K_{y_r}) \neq \emptyset$, then there exists $q_r = \operatorname{st}(\pi((\hat{\mathfrak{A}}_r)_{\kappa_r}, K_{y_r}), V(M)|_K)$ and $z_r = \operatorname{pl}(\pi((\hat{\mathfrak{A}}_r)_{\kappa_r}, K_{y_r}), V(M)|_K)$. Clearly, $z_r = x_{i_r,j_r}|_K$ for some $1 \leqslant i_r, j_r \leqslant a$. Proposition 3.3 and Theorem 3.1 imply that

$$\big|V(M)|_K \cap \operatorname{Int}((\hat{\mathfrak{A}}_r)_{q_r}, K_{z_r})\big| = |V(M) \cap \operatorname{Int}((\hat{\mathfrak{A}}'_r)_{q_r}, M''_{x_{i_r,j_r}})|. \tag{3.2}$$

From the definition of the relation $\leqslant_f$ it follows that

$$V(M) \cap \operatorname{Int}((\hat{\mathfrak{A}}'_r)_{q_r}, M''_{x_{i_r,j_r}}) \subseteq V(M) \cap \operatorname{Int}((\hat{\mathfrak{A}}''_r)_{f_r(q_r)}, M'_{x_{i_r,j_r}}). \tag{3.3}$$

Let $W_r = \operatorname{Int}((\hat{\mathfrak{A}}''_r)_{f_r(q_r)}, M'_{x_{i_r,j_r}}) \cap V(M)$. As $\hat{\mathfrak{A}}''_r$ is irreducible or trivial, we have either that $|W_r| = 1$ or, by Proposition 3.6, that $W_r = b(f)$, where f is a face of M satisfying $x_{i_r,j_r} \in b(f)$. In the latter case, if f is the external face of M, then $|W_r| = 4a - 4$, and if f is an bounded face of M, then $|W_r| = 4$.

Therefore, we get that

$$\Big|\bigcup_{i=1}^{n} W_i\Big| \leqslant \sum_{i=1}^{n} |W_i| \leqslant n(4a-4).$$

Take an a such that $a^2 > 4na - 4n$, i.e., $a > 2(n + \sqrt{n(n-1)})$. Then, $|V(M)| > \sum_{i=1}^{n} |W_i|$, and, consequently, from (3.2), (3.3), and Proposition 2.1 we get our assertion, because

$$\sum_{i=1}^{n} |W_i| \geqslant \sum_{i=1}^{n} \big|V(M)|_K \cap \operatorname{Int}((\hat{\mathfrak{A}}_i)_{\kappa_i}, K_{y_i})\big| \geqslant \big|V(M)|_K \cap \operatorname{Int}(\mathcal{A}_{\vec{\kappa}}, K_{\vec{y}})\big|. \qquad \square$$

Let $\mathcal{A} = (\mathfrak{A}_1, \dots, \mathfrak{A}_n)$ be a collective of automata and $(L; \vec{x}_0, x_1)$ be a perfect n-initial mosaic labyrinth. If $x_1 \notin \operatorname{Int}(\mathcal{A}_{\vec{q}}, (L; \vec{x}_0, x_1))$ for every $\vec{q} \in Q_{\mathcal{A}}$, then the labyrinth $(L; \vec{x}_0, x_1)$ is a *perfect IA-trap* for $\mathcal{A}$. If $x_1 \notin \operatorname{Int}(\mathcal{A}_{\vec{q}_0}, (L; \vec{x}_0, x_1))$ for some $\vec{q}_0 \in Q_{\mathcal{A}}$, then the labyrinth $(L; \vec{x}_0, x_1)$ is a *perfect II-trap* for $\mathcal{A}_{\vec{q}_0}$.

THEOREM 3.6. *For every collective $\mathcal{A} = (\mathfrak{A}_1, \dots, \mathfrak{A}_n)$ of independent automata there exists a perfect IA-trap.*

PROOF. Keeping in mind the above notations, take an a such that $4(a-1) > 3(n-4) + 20$, i.e., $a > 3n/4 + 3$, and an $\vec{x} = (x_{i_1,j_1}|_K, \dots, x_{i_n,j_n}|_K)$ such that $2 \leqslant i_k, j_k \leqslant a-1$ for every $1 \leqslant k \leqslant n$. Also, let $\vec{q}$ be an arbitrary element of $Q_{\mathcal{A}}$. Consider the behavior of $\mathcal{A}_{\vec{q}}$ in $K_{\vec{x}}$. Call an automaton of the collective $\mathcal{A}_{\vec{q}}$ a "corner" automaton if it is placed, in the initial moment of time, in a vertex $x_{i,j}|_K$, where (i,j) is one of the pairs $(2,2)$, $(2,a-1)$, $(a-1,2)$, and $(a-1,a-1)$. Let $W = (V(M) \cap \overline{f_\infty(M)})|_K$. Note that every corner automaton of $\mathcal{A}_{\vec{q}}$ "covers" not more than five vertices and every noncorner automaton of $\mathcal{A}_{\vec{q}}$ "covers" not more than three vertices of the set W during its "work" (note that $\vec{q}$ is an arbitrary element of $Q_{\mathcal{A}}$). That means that all automata of the collective $\mathcal{A}_{\vec{q}}$ "cover" not more than $3(n-4) + 20 = 3n + 8$ vertices altogether. As $|W| = 4(a-1)$ and

$a > 3n/4 + 3$, there is at least one vertex of W in which none of the automata of $\mathcal{A}_{\vec{q}}$ finds itself, and, consequently, the labyrinth $K_{\vec{x}}$ is a perfect IA-trap for $\mathcal{A}_{\vec{q}}$. □

Now the following theorem, the main result of [**2**], known as Budach theorem follows easily from the last theorem and Proposition 3.2.

THEOREM 3.7. *For every initial automaton there exists a perfect II-trap that is a maze.*

An infinite labyrinth L is a universal trap for all collectives of independent automata if for every $n \geqslant 1$, every collective $\mathcal{A}$ of n independent automata, every $\vec{q} = (q_1^{(0)}, \dots, q_n^{(0)}) \in Q_{\mathcal{A}}$, and every $\vec{x} = (x_1^{(0)}, \dots, x_n^{(0)}) \in [V(L)]^n$, the set $\mathrm{Int}(\mathcal{A}_{\vec{q}}, L_{\vec{x}})$ is bound. In conclusion, let us prove the theorem which is equivalent to the main result of [**3**]. First let us give an auxiliary statement that obviously holds.

LEMMA 3.2. *Suppose $\mathfrak{A}$ is an irreducible automaton, and L_1 is an $\mathbf{e}$-labyrinth and L_2 is an $\mathbf{n}$-labyrinth such that the pair (L_1, L_2) orients $\mathfrak{A}$. Let K be the plane labyrinth given in Fig. 1. Further suppose that $(K'; y', y'')$ is an arbitrary $\mathbf{w}$-extraperfectly embeddable labyrinth. Then the labyrinth*

$$K'' = (\mathrm{vi}((\mathrm{Cross}(\Delta(K; L_1, L_2)) - \langle x', (x', \mathbf{w})\rangle)\dot{\cup} K', x', y'); z, x'')$$

is a regular trap for $\mathfrak{A}_q$ for every $z \in V(K') \setminus \{y'\}$ and $q \in Q_{\mathfrak{A}}$.

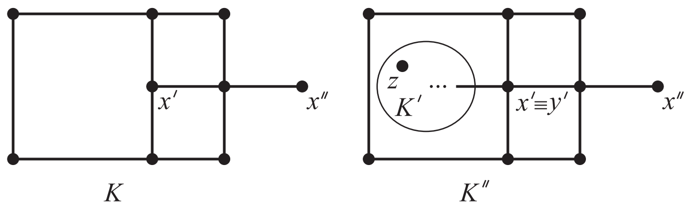

FIGURE 1.

THEOREM 3.8. *There exists a mosaic universal trap for all collectives of independent automata.*

PROOF. Order in some way all pairwise nonisomorphic (acceptable) automata in an array $\tilde{\mathfrak{A}}_1, \tilde{\mathfrak{A}}_2, \dots$. For every $k \geqslant 1$, let $M_k = L(\tilde{\mathfrak{A}}_1, \dots, \tilde{\mathfrak{A}}_k; K)$, where K is the plane labyrinth given in Fig. 1, and let $\varepsilon_k = \varepsilon(\tilde{\mathfrak{A}}_1, \dots, \tilde{\mathfrak{A}}_k)$. For every $k \geqslant 1$, let $n(k)$ be the maximal number satisfying

$$y'_k = (\dots(x', \omega'_1), \dots, \omega'_{n(k)}) \quad \text{and} \quad y''_k = (\dots(x'', \omega''_1), \dots, \omega''_{n(k)}),$$

where $\omega'_1 = \dots = \omega'_{n(k)} = \mathbf{w}$ and $\omega''_1 = \dots = \omega''_{n(k)} = \mathbf{e}$, are vertices in M_k (see the definition of Cross-operation). It is not difficult to see that we can choose

$\hat{M}_k \in \mathfrak{L}(M_k, \varepsilon_k)$ for every $k \geqslant 1$ such that $z'_k = y'_k|_{\hat{M}_k}$ and $z''_k = y''_k|_{\hat{M}_k}$ belong to x-axe, $\mathrm{pr}_1(z''_k) < \mathrm{pr}_1(z'_{k+1})$, and

$$\overline{\hat{M}_k} \cap \overline{\hat{M}_{k+1}} = (\mathrm{pr}_1(z''_k), \mathrm{pr}_1(z'_{k+1})) \cap \overline{\hat{M}_{k+1}} = \emptyset$$

(schematically, we represent $\hat{M}_k$ as it is given in Fig. 2; inside the black area is situated the "basic part" of this labyrinth). Obviously, for every $k \geqslant 2$, the set

$$\bigcup_{i=1}^{k} \overline{\hat{M}_i} \cup \bigcup_{i=1}^{k-1} \overline{z''_i\, z'_{i+1}}$$

is the realization of a mosaic labyrinth M^*_k (see Fig. 2). Also, it is clear that the set $\bigcup_{i=2}^{\infty} \overline{M^*_i}$ is the realization of a mosaic (infinite) labyrinth L^*.

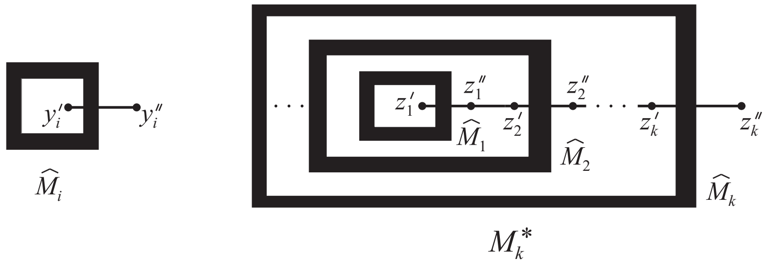

FIGURE 2.

Now suppose that $\mathcal{A} = (\mathfrak{A}_1, \dots, \mathfrak{A}_n)$ is an arbitrary collective of independent automata. Take an arbitrary $\vec{q} = (q_1, \dots, q_n) \in Q_{\mathcal{A}}$ and take an arbitrary $\vec{x} = (x_1, \dots, x_n) \in [V(L^*)]^n$. Obviously, for every $1 \leqslant i \leqslant n$, it holds that $\hat{\mathfrak{A}}_i = \tilde{\mathfrak{A}}_{k_i}$ for a natural number k_i. Take a natural number $n(\mathcal{A})$ such that $M^*_{n(\mathcal{A})-1}$ contains vertices $x_1, \dots, x_n$ and $n(\mathcal{A}) > k_i$ for every $1 \leqslant i \leqslant n$. Now from Lemma 3.2 it follows that the labyrinth $\big(M^*_{n(\mathcal{A})}; x_i, z''_{n(\mathcal{A})}\big)$ is a regular trap for $\mathfrak{A}_{q_i}$ for every $1 \leqslant i \leqslant n$. Therefore the labyrinth $\big(M^*_{n(\mathcal{A})}; \vec{x}, z''_{n(\mathcal{A})}\big)$ is a perfect II-trap for $\mathcal{A}_{\vec{q}}$. $\square$

References

1. G. Kilibarda, *On reduction of automata in labyrinths*, Publ. Inst. Math., Nouv. Sér. **101(115)** (2017), 47–63.
2. L. Budach, *Automata and labirinths*, Math. Nachrichten **86** (1978), 195–282.
3. H. Antelman, L. Budach, H. A. Rollik, *On universal traps*, Elektron. Inform.-verarb. Kybernetik **15**(3) (1979), 123–131.
4. В. Б. Кудрявцев, А. С. Подколзин, Ш. Ушчумлич, *Введение в теорию абстрактных автоматов*, МГУ, Москва, 1985.

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