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# NEW ESTIMATES FOR MEROMORPHIC FUNCTIONS

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ABSTRACT. A boundary version of the Schwarz lemma for meromorphic functions is investigated. For the function $I^n f(z) = \frac{1}{z} + \sum_{k=2}^{\infty} k^n c_{k-2} z^{k-2}$, belonging to the class of $\mathcal{W}$, we estimate from below the modulus of the angular derivative of the function on the boundary point of the unit disc.

## 1. Introduction

One of the most investigated subjects is Schwarz lemma in complex analysis. It is an important result which gives estimates about taking values of the holomorphic functions defined in the unit disc in complex plane and whose image set in the unit disc. In addition, it is a fundamental support to develop geometric function theory, the fixed point theory of holomorphic map, hyperbolic geometry and many areas of analysis. Schwarz lemma, which is a direct application of the maximum modulus principle, is commonly stated as follows.

Let $f$ be a holomorphic function in the unit disc $U = \{z : |z| < 1\}$, $f(0) = 0$ and $|f(z)| < 1$ for $|z| < 1$. For any point $z$ in the unit disc $U$, we have $|f(z)| \leqslant |z|$ and $|f'(0)| \leqslant 1$. Equality in these inequalities (in the first one, for $z \neq 0$) occurs only if $f(z) = \lambda z$, $|\lambda| = 1$ [**6**, p. 329]. For historical background about the Schwarz lemma and its applications on the boundary of the unit disc, we refer to (see [**2, 20**]). Also, Mateljević [**16**] give an approach to Hyperbolic geometry via the Schwarz lemma.

In proving our main results, we shall need the following result due to Jack [**7**].

LEMMA 1.1 (Jack's lemma). *Let $f(z)$ be a holomorphic function in the unit disc $U$ with $f(0) = 0$. If $|f(z)|$ attains its maximum value on the circle $|z| = r$ at the point $z_0$, then $z_0 f'(z_0) = k f(z_0)$ where $k \geqslant 1$ is a real number.*

Let $\mathcal{T}$ denote the class of functions $f(z) = \frac{1}{z} + c_0 + c_1 z + c_2 z^2 + \cdots$ that are holomorphic in the punctured disc $\mathbb{E} = \{z \in \mathbb{C} : 0 < |z| < 1\}$. Define $I^0 f(z) = f(z)$,

$$I^1 f(z) = \frac{1}{z} + 2c_0 + 3c_1 z + 4c_2 z^2 + \cdots = \frac{(z^2 f(z))'}{z}, \quad I^2 f(z) = I^1(I^1 f(z)),$$

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and for $n = 1, 2, 3, \dots$

$$I^n f(z) = I^1(I^{n-1} f(z)) = \frac{1}{z} + 2^n c_0 + 3^n c_1 z + 4^n c_2 z^2 + \dots = \frac{1}{z} + \sum_{k=2}^{\infty} k^n c_{k-2} z^{k-2}.$$

Also, let $\mathcal{W}$ be the class of $\mathcal{T}$ consisting of all the functions $f(z)$ satisfying

$$\operatorname{Re}[-z^2 (I^n f(z))'] > 0, \quad |z| < 1. \tag{1.1}$$

Let $f(z) \in \mathcal{W}$ and consider the function

$$\psi(z) = \frac{1 + z^2 (I^n H(z))'}{1 - z^2 (I^n H(z))'}, \tag{1.2}$$

where $H(z) = \frac{c}{z^{c+1}} \int_0^z t^c f(t) dt$ and $\operatorname{Re} c > 0$, which is holomorphic in $U$ and $\psi(0) = 0$, that is

$$I^0 H(z) = H(z), \quad I^1 H(z) = \frac{1}{z} + \frac{2c}{c+1} c_0 + \frac{3c}{c+2} c_1 z + \frac{4c}{c+3} c_2 z^2 + \dots,$$

$$I^2 H(z) = I^1(I^1 H(z)), \quad I^n H(z) = \frac{1}{z} + \sum_{k=2}^{\infty} k^n \frac{c}{c+k-1} c_{k-2} z^{k-2},$$

$$\begin{aligned}\psi(z) &= \frac{3^n \frac{c}{c+2} c_1 z^2 + 2 \cdot 4^n \frac{c}{c+3} c_2 z^3 + 3 \cdot 5^n \frac{c}{c+3} c_3 z^4 + \dots}{2 - 3^n \frac{c}{c+2} c_1 z^2 - 2 \cdot 4^n \frac{c}{c+3} c_2 z^3 - 3 \cdot 5^n \frac{c}{c+3} c_3 z^4 - \dots} \\ &= 3^n \frac{c}{2(c+2)} c_1 z^2 + 2 \cdot 4^n \frac{c}{2(c+3)} c_2 z^3 + \dots\end{aligned}$$

Since $f(z) \in \mathcal{W}$, we have

$$z(I^n H(z))' + (1+c) I^n H(z) = c I^n f(z). \tag{1.3}$$

Differentiating (1.3), we obtain $z(I^n H(z))'' + (c+2)(I^n H(z))' = c(I^n f(z))'$. Differentiating (1.2), we get $z(I^n H(z))'' + 2(I^n H(z))' = \frac{2\psi'(z)}{c(1+\psi(z))^2}$. Thus, we have

$$-z^2 (I^n f(z))' = -z^2 (I^n H(z))' - \frac{2z\psi'(z)}{c(1+\psi(z))^2}.$$

Now, we show that $|\psi(z)| < 1$ for $|z| < 1$. If there exists a point $z_0 \in U$ such that $\max_{|z| \leqslant |z_0|} |\psi(z)| = |\psi(z_0)| = 1$, then Jack's lemma gives us that $\psi(z_0) = e^{i\theta}$ and $z_0 \psi'(z_0) = k\psi(z_0)$. Thus, we have

$$\begin{aligned}-z_0^2 (I^n f(z_0))' &= -z_0^2 (I^n H(z_0))' - \frac{2z\psi'(z_0)}{c(1+\psi(z_0))^2} \\ &= \frac{1-\psi(z_0)}{1+\psi(z_0)} - \frac{2k\psi(z_0)}{c(1+\psi(z_0))^2} \\ &= \frac{1-e^{i\theta}}{1+e^{i\theta}} - \frac{2ke^{i\theta}}{c(1+e^{i\theta})^2}.\end{aligned}$$

and

$$\operatorname{Re}(-z_0^2 (I^n f(z_0))') = \operatorname{Re}\left(\frac{1-e^{i\theta}}{1+e^{i\theta}} - \frac{2ke^{i\theta}}{c(1+e^{i\theta})^2}\right).$$

Therefore, we obtain

$$\begin{aligned}\mathrm{Re}(-z_0^2(I^n f(z_0))') &= \mathrm{Re}\left(\frac{1-(\cos\theta+i\sin\theta)}{1+\cos\theta+i\sin\theta}-\frac{2ke^{i\theta}}{c(1+e^{i\theta})^2}\right)\\ &= \mathrm{Re}\left(\frac{1-(\cos\theta+i\sin\theta)}{1+\cos\theta+i\sin\theta}-2k\frac{1}{c(2+e^{i\theta}+e^{-i\theta})}\right)\\ &= \mathrm{Re}\left(\frac{1-(\cos\theta+i\sin\theta)}{1+\cos\theta+i\sin\theta}-\frac{2k}{2c(1+\cos\theta)}\right)\\ &= -k\,\mathrm{Re}\left(\frac{1}{c(1+\cos\theta)}\right)\leqslant 0.\end{aligned}$$

This contradicts (1.1). Thus, there is no point $z_0 \in U$ such that $|\psi(z_0)| = 1$ for all $z_0 \in U$. Consequently, we conclude that $|\psi(z)| < 1$ for $|z| < 1$. Thus, by the Schwarz lemma, we obtain

$$|c_1| \leqslant \frac{2}{3^n}|\frac{c+2}{c}|. \tag{1.4}$$

Moreover, the equality in (1.4) occurs for the solution of the equation

$$(I^n H(z))' = \frac{z^2-1}{z^2(1+z^2)},$$

with the condition $\lim_{z\to 0} z^2H(z) = 0$ at $z = 0$. In particular, for $n = 1$, we have

$$(I^1 H(z))' = \frac{z^2-1}{z^2(1+z^2)}, \tag{1.5}$$

with the condition $\lim_{z\to 0} z^2H(z) = 0$ at $z = 0$ Thus, from (1.5), we obtain

$$f(z) = \frac{1}{z} + \frac{2}{3}\frac{c+2}{c}z - \frac{2}{15}\frac{c+4}{c}z^3 + \frac{2}{35}\frac{c+6}{c}z^5 - \cdots$$

Now, we can find equality condition in (1.4).

$$I^n H(z) = \frac{1}{z} + 2^n\frac{c}{c+1}c_0 + 3^n\frac{c}{c+2}c_1 z + 4^n\frac{c}{c+3}c_2 z^2 + 5^n\frac{c}{c+4}c_3 z^3 + \cdots,$$

$$(I^n H(z))' = \frac{-1}{z^2} + 3^n\frac{c}{c+2}c_1 + 2\cdot 4^n\frac{c}{c+3}c_2 z + 3\cdot 5^n\frac{c}{c+4}c_3 z^2 + \cdots$$

Therefore, we obtain

$$(I^n H(z))' = \frac{z^2-1}{z^2(1+z^2)},$$

$$\frac{-1}{z^2} + 3^n\frac{c}{c+2}c_1 + 2\cdot 4^n\frac{c}{c+3}c_2 z + 3\cdot 5^n\frac{c}{c+4}c_3 z^2 + \cdots = \frac{z^2-1}{z^2(1+z^2)},$$

$$-1 + 3^n\frac{c}{c+2}c_1 z^2 + 2\cdot 4^n\frac{c}{c+3}c_2 z^3 + 3\cdot 5^n\frac{c}{c+4}c_3 z^4 + \cdots = \frac{z^2-1}{1+z^2},$$

$$3^n\frac{c}{c+2}c_1 z^2 + 2\cdot 4^n\frac{c}{c+3}c_2 z^3 + 3\cdot 5^n\frac{c}{c+4}c_3 z^4 + \cdots = \frac{z^2-1}{1+z^2} + 1 = \frac{2z^2}{1+z^2},$$

$$3^n\frac{c}{c+2}c_1 + 2\cdot 4^n\frac{c}{c+3}c_2 z + 3\cdot 5^n\frac{c}{c+4}c_3 z^2 + \cdots = \frac{2}{1+z^2}$$

and passing to the limit in the last inequality yields

$$\lim_{z\to 0} 3^n \frac{c}{c+2}c_1 + 2\cdot 4^n \frac{c}{c+3}c_2 z^z + 3\cdot 5^n \frac{c}{c+4}c_3 z^2 + \cdots = \lim_{z\to 0}\frac{2}{1+z^2},$$
$$3^n \frac{c}{c+2}c_1 = 2,\ \ |c_1| = \frac{2}{3^n}|\frac{c+2}{c}|.$$

Osserman [**19**] offered the following boundary refinement of the classical Schwarz lemma. It is very much in the spirit of the sort of result, we wish to consider here. In other words,

$$|f'(b)| \geqslant p + \frac{1-|c_p|}{1+|c_p|}, \tag{1.6}$$

$$|f'(b)| \geqslant p \tag{1.7}$$

under the assumption $f(0) = 0$, where $f$ is a holomorphic function mapping of the unit disc into itself and $b$ is a boundary point which $f$ extends continuosly, and $|f(b)| = 1$. In addition, equality in (1.7) holds if and only if $f(z) = z^p e^{i\theta}$, where $\theta$ is a real number. Also, equality in (1.6) holds if and only if $f$ is of the form $f(z) = -z^p \frac{\xi - z}{1-\xi z}$, $\forall z \in U$, for some constant $\xi \in (-1, 0]$.

The following set is called a Stolz angle at $b \in \partial U$

$$\triangle = \left\{z \in U : |\arg(1-\bar{b}z)| < \alpha,\ |z-b| < \rho\right\}, \quad \left(0 < \alpha < \frac{\pi}{2},\ \rho < 2\cos\alpha\right).$$

Let $f$ be a function from $U$ to $\bar{\mathbb{C}}$. It is said that $f$ has an angular limit $\varsigma \in \bar{\mathbb{C}}$ at $b \in \partial U$ if $f(z) \to \varsigma$ as $z \to b$, $z \in \triangle$ for each Stolz angle $\triangle$ at $b$. The number $2\alpha$ which is length of $\triangle$ can be any number less than $\pi$. It is said that $f$ has the unrestricted limit $\varsigma \in \bar{\mathbb{C}}$ at $b$ if $f(z) \to \varsigma$ as $z \to b$, $z \in U$. Clearly, in the last fact, if the function $f$ which is continuous in $U$ is defined at the point $b$ as $f(b) = a$, then $f$ becomes continuous in $U \cup \{b\}$.

Let $f$ be a function from $U$ to $U$ and $\gamma$ be its angular limit at the point $b$. If there exists a point $\beta$ such that $\lim_{z\to b,\ z\in\triangle} \frac{f(z)-\gamma}{z-b} = \beta$ for every Stolz anagle $\triangle$ at the point $b$, then $\beta$ is called the angular derivative of the function $f$ at $b$ and $\beta$ is shown with $f'(b)$.

In proving our main results, we shall need the following lemma due to Julia–Wolff and Corollary 3 [**21**].

Lemma 1.2 (Julia–Wolff lemma). *Let $f$ be a holomorphic function in $U$, $f(0) = 0$ and $f(U) \subset U$. If, in addition, the function $f$ has an angular limit $f(b)$ at $b \in \partial U$, $|f(b)| = 1$, then the angular derivative $f'(b)$ exists and $1 \leqslant |f'(b)| \leqslant \infty$.*

Corollary 1.1. *The holomorphic function $f$ has a finite angular derivative $f'(b)$ if and only if $f'$ has the finite angular limit $f'(b)$ at $b \in \partial U$.*

Inequality (1.7) and its generalizations have important applications in geometric theory of functions (see, e.g., [**6, 21**]). Therefore, the interest to such type of results have not vanished (see, e.g., [**1, 2, 4, 5, 11, 12, 16, 17, 19, 20, 22**] and references therein).

Dubinin has continued this line and has made a refinement on the boundary Schwarz lemma under the assumption that $f(z) = c_p z^p + c_{p+1} z^{p+1} + \cdots$, with a zero set $\{z_k\}$ [**4**].

Krantz and Burns [**10**] and Chelst [**3**] studied the uniqueness part of the Schwarz lemma. Similar types of results which are related with the subject of the paper can be found in [**13, 14, 15**]. Also, Mateljević's [**16**] give an approach to hyperbolic geometry via Schwarz lemma.

In addition, Jeong [**9**] showed some inequalities at a boundary point for different form of holomorphic functions. He also found the condition for equality. In [**8**], a holomorphic selfmap was defined on the closed unit disc with fixed points only on the boundary of the disc. Wail and Shah [**21**] established some results by using a boundary refinement of the classical Schwarz lemma.

## 2. Main Results

In this section, the meromorphic function $I^n f(z) = \frac{1}{z} + \sum_{k=2}^{\infty} k^n c_{k-2} z^{k-2}$, belonging to $\mathcal{W}$, is estimated from below by the modulus of the angular derivative of the function on the boundary point of the unit disc.

Theorem 2.1. *Let $f(z) \in \mathcal{W}$. Assume that, for some $b \in \partial U$, $(I^n f(z))'$ has angular limit $(I^n f(z))'_{z=b}$ at $b$ and $(I^n f(z))_{z=b} = \frac{c+1}{c}(I^n(\frac{c}{z^{c+1}}\int_0^z t^c f(t)dt))_{z=b}$. Then we have*

$$|(I^n f(z))'_{z=b}| \geqslant \frac{1}{|c|}. \tag{2.1}$$

*The inequality* (2.1) *is sharp.*

Proof. Consider the function

$$\psi(z) = \frac{1 + z^2(I^n H(z))'}{1 - z^2(I^n H(z))'},$$

where $H(z) = \frac{c}{z^{c+1}}\int_0^z t^c f(t)dt$ and $\operatorname{Re} c > 0$. The function $\psi(z)$ is holomorphic in the unit disc $U$ and $\psi(0) = 0$. From Jack's lemma and since $f(z) \in \mathcal{W}$, we have $|\psi(z)| < 1$ for $|z| < 1$. Also, we have $|\psi(b)| = 1$ for $b \in \partial U$. That is, since $(I^n f(z))_{z=b} = \frac{c+1}{c}\big(I^n(\frac{c}{z^{c+1}}\int_0^z t^c f(t)dt)\big) = \frac{c+1}{c}(I^n H(z))_{z=b}$, we have

$$b(I^n H(z))'_{z=b} + (1+c)I^n H(z)_{z=b} = cI^n f(z)_{z=b},$$

$$b(I^n H(z))'_{z=b} + (1+c)I^n H(z)_{z=b} = c\frac{c+1}{c}(I^n H(z))_{z=b}$$

and $(I^n H(z))'_{z=b} = 0$. Therefore, we take $|\psi(b)| = 1$ for $b \in \partial U$.

For $p = 2$, from (1.7), we obtain

$$\begin{aligned} 2 \leqslant |\psi'(b)| = \Bigg| &\frac{(2b(I^n H(z))'_{z=b} + b^2(I^n H(z))''_{z=b})(1 - b^2(I^n H(z))'_{z=b})}{(1 - b^2(I^n f(z))'_{z=b})^2} \\ &+ \frac{(2b(I^n H(z))'_{z=b} + b^2(I^n H(z))''_{z=b})(1 + b^2(I^n H(z))'_{z=b})}{(1 - b^2(I^n H(z))'_{z=b})^2}\Bigg|, \end{aligned}$$

$$2 \leqslant \frac{|4b(I^nH(z))'_{z=b} + 2b^2(I^nH(z))''_{z=b}|}{(1-b^2(I^nH(z))'_{z=b})^2} = |2(I^nH(z))''_{z=b}|,$$

and $|(I^nH(z))''_{z=b}| \geqslant 1$. Since $z(I^nH(z))'' + (c+2)(I^nH(z))' = c(I^nf(z))'$ and $(I^nH(z))'_{z=b} = 0$, we have $b(I^nH(z))''_{z=b} + (c+2)(I^nH(z))'_{z=b} = c(I^nf(z))'_{z=b}$ and $b(I^nH(z))''_{z=b} = c(I^nf(z))'_{z=b}$. Thus, we obtain $|(I^nf(z))'_{z=b}| \geqslant 1/|c|$.

Now, we show that the inequality (2.1) is sharp. Let

$$(I^nH(z))' = \frac{z^2-1}{z^2(1+z^2)}$$

with the condition $\lim_{z\to 0} z^2H(z) = 0$ at $z=0$. Then, we have

$$(I^nH(z))'' = \frac{2z(z^4+z^2)-(4z^3+2z)(z^2-1)}{(z^4+z^2)^2}, \quad (I^nH(z))''_{z=1} = 1.$$

We know that

$$b(I^nH(z))''_{z=b} + (c+2)(I^nH(z))'_{z=b} = c(I^nf(z))'_{z=b}$$

and since $(I^nH(z))''_{z=1} = 1$ and $(I^nH(z))'_{z=1} = 0$, we obtain $(I^nH(z))''_{z=1} = c(I^nf(z))'_{z=1}$. Thus, we get $|(I^nf(z))'_{z=1}| = \frac{1}{|c|}$. □

Theorem 2.2. *Under the same assumptions as in Theorem 4, we have*

$$|(I^nf(z))'_{z=b}| \geqslant \frac{1}{2|c|}\left(\frac{6|c+2|+3^n|c||c_1|}{2|c+2|+3^n|c||c_1|}\right). \tag{2.2}$$

*The inequality (2.2) is sharp.*

Proof. Let $\psi(z)$ be as in the proof of Theorem 4. For $n=2$, using the inequality (1.6) for the function $\psi(z)$, we obtain

$$2 + \frac{1-|d_2|}{1+|d_2|} \leqslant |\psi'(b)| = 2|(I^nH(z))''_{z=b}|$$

Since $|d_2| = \frac{|\psi''(0)|}{2} = \frac{3^n}{2}\left|\frac{c}{c+2}\right||c_1|$, where $d_2$ is the coefficient in the Taylor expansion of the function $\psi(z)$, then we have

$$2 + \frac{1-\frac{3^n}{2}\left|\frac{c}{c+2}\right||c_1|}{1+\frac{3^n}{2}\left|\frac{c}{c+2}\right||c_1|} = 2 + \frac{2|c+2|-3^n|c||c_1|}{2|c+2|+3^n|c||c_1|} \leqslant 2|(I^nH(z))''_{z=b}|,$$

$$|(I^nH(z))''_{z=b}| \geqslant \frac{1}{2}\left(2 + \frac{2|c+2|-3^n|c||c_1|}{2|c+2|+3^n|c||c_1|}\right) = \frac{1}{2}\left(\frac{6|c+2|+3^n|c||c_1|}{2|c+2|+3^n|c||c_1|}\right).$$

To show that the inequality (2.2) is sharp, take the holomorphic function

$$\psi(z) = \frac{1+z^2(I^nH(z))'}{1-z^2(I^nH(z))'} = z^2\frac{z+\varrho}{1+\varrho z}, \quad (0 \leqslant \varrho < 1)$$

with the condition $\lim_{z\to 0} z^2H(z) = 0$ at $z=0$. Then

$$\psi'(z) = 2\frac{2z(I^nH(z))' + z^2(I^nH(z))''}{(1-z^2(I^nH(z))')^2} = \frac{(3z^2+2\varrho z)(1+\varrho z)-\varrho(z^3+\varrho z^2)}{(1+\varrho z)^2},$$

$$\psi'(1) = 2\frac{2(I^nH(z))'_{z=1} + (I^nH(z))''_{z=1}}{(1-(I^nH(z))'_{z=1})^2} = \frac{3+\varrho}{1+\varrho}$$

and $\left|(I^nH(z))''_{z=1}\right| = \frac{1}{2}\left(\frac{3+\varrho}{1+\varrho}\right)$. Since $\varrho = \frac{3^n}{2}\left|\frac{c}{c+2}\right||c_1|$ (see, (1.4)) and $(I^nH(z))''_{z=1} = c(I^nf(z))'_{z=1}$, (2.2) is satisfied with equality. $\square$

THEOREM 2.3. *Under the same assumptions as in Theorem 4, we have*

$$|(I^nf(z))'_{z=1}| \geqslant \frac{1}{|c|}\left(1+\frac{|c+3|(2|c+2|-3^n|c||c_1|)^2}{|c+3|(4|c+2|^2-3^{2n}|c|^2|c_1|^2)-4^{n+1}|c||c+2|^2|c_2|}\right). \tag{2.3}$$

*The inequality* (2.3) *is sharp.*

PROOF. Let $\psi(z)$ be as in the proof of Theorem 4. By the maximum principle, for each $z \in U$, we have $|\psi(z)| \leqslant |z^2|$. So, $\varphi(z) = \psi(z)/z^2$ is a holomorphic function in $U$ and $|\varphi(z)| < 1$ for $|z| < 1$. In particular, we have

$$|\varphi(0)| = \frac{3^n}{2}\left|\frac{c}{c+2}\right||c_1| \leqslant 1 \tag{2.4}$$

and $|\varphi'(0)| = 4^n\left|\frac{c}{c+3}\right||c_2|$. Moreover, one can see that

$$\frac{b\psi'(b)}{\psi(b)} = |\psi'(b)| \geqslant |(b^2)'| = \frac{b(b^2)'}{b^2}.$$

The function

$$\begin{aligned}\Phi(z) = \frac{\varphi(z)-\varphi(0)}{1-\overline{\varphi(0)}\varphi(z)} &= \frac{3^n\frac{c}{2(c+2)}c_1 + 2\cdot 4^n\frac{c}{2(c+3)}c_2z+\cdots - 3^n\frac{c}{2(c+2)}c_1}{1-\overline{3^n\frac{c}{2(c+2)}c_1}\left(3^n\frac{c}{2(c+2)}c_1+2\cdot 4^n\frac{c}{2(c+3)}c_2z+\cdots\right)}\\ &= \frac{4^n\frac{c}{(c+3)}c_2z+\cdots}{1-\overline{3^n\frac{c}{2(c+2)}c_1}\left(3^n\frac{c}{2(c+2)}c_1+2\cdot 4^n\frac{c}{2(c+3)}c_2z+\cdots\right)}\end{aligned}$$

is holomorphic in the unit disc $U$, $|\Phi(z)| < 1$ for $|z| < 1$, $\Phi(0) = 0$ and $|\Phi(b)| = 1$ for $b \in \partial U$. From (1.6), we obtain

$$\begin{aligned}\frac{2}{1+|\Phi'(0)|} \leqslant |\Phi'(b)| &= \frac{1-|\varphi(0)|^2}{|1-\overline{\varphi(0)}\varphi(b)|^2}|\varphi'(b)|\\ &\leqslant \frac{1+|\varphi(0)|}{1-|\varphi(0)|}|\varphi'(b)| = \frac{1+|\varphi(0)|}{1-|\varphi(0)|}\{|\psi'(b)|-2\}.\end{aligned}$$

Since $\Phi'(z) = \frac{1-|\varphi(0)|^2}{(1-\overline{\varphi(0)}\varphi(z))^2}\varphi'(z)$, we have

$$|\Phi'(0)| = \frac{|\varphi'(0)|}{1-|\varphi(0)|^2} = \frac{4^n\left|\frac{c}{c+3}\right||c_2|}{1-\left(\frac{3^n}{2}\left|\frac{c}{c+2}\right||c_1|\right)^2} = \frac{4^{n+1}|c||c+2|^2|c_2|}{|c+3|(4|c+2|^2-3^{2n}|c|^2|c_1|^2)},$$

wherefrom

$$\frac{2}{1+\frac{4^{n+1}|c||c+2|^2|c_2|}{|c+3|(4|c+2|^2-3^{2n}|c|^2|c_1|^2)}} \leqslant \frac{2|c+2|+3^n|c||c_1|}{2|c+2|-3^n|c||c_1|}\{2|(I^nH(z))''_{z=b}|-2\},$$

$$\frac{2|c+3|(4|c+2|^2-3^{2n}|c|^2|c_1|^2)}{|c+3|(4|c+2|^2-3^{2n}|c|^2|c_1|^2)-4^{n+1}|c||c+2|^2|c_2|}\,\frac{2|c+2|-3^n|c||c_1|}{2|c+2|+3^n|c||c_1|} \leqslant 2|(I^nH(z))''_{z=b}|-2,$$

$$\frac{2|c+3|(2|c+2|-3^n|c||c_1|)^2}{|c+3|(4|c+2|^2-3^{2n}|c|^2|c_1|^2)-4^{n+1}|c||c+2|^2|c_2|}+2\leqslant 2|(I^nH(z))''_{z=b}|$$

Therefore, we obtain

$$|(I^nH(z))''_{z=b}|\geqslant 1+\frac{|c+3|(2|c+2|-3^n|c||c_1|)^2}{|c+3|(4|c+2|^2-3^{2n}|c|^2|c_1|^2)-4^{n+1}|c||c+2|^2|c_2|}.$$

Also, since $b(I^nH(z))''_{z=b}=c(I^nf(z))'_{z=b}$, we have

$$|(I^nf(z))'_{z=b}|\geqslant \frac{1}{|c|}\left(1+\frac{|c+3|(2|c+2|-3^n|c||c_1|)^2}{|c+3|(4|c+2|^2-3^{2n}|c|^2|c_1|^2)-4^{n+1}|c||c+2|^2|c_2|}\right).$$

Now, we shall show that the inequality (2.3) is sharp. Consider the function $(I^nH(z))'=\frac{z^2-1}{z^2(1+z^2)}$ with the condition $\lim_{z\to 0}z^2H(z)=0$ at $z=0$. We have

$$(I^nH(z))''=\frac{2z(z^4+z^2)-(4z^3+2z)(z^2-1)}{(z^4+z^2)^2},$$

$$|(I^nH(z))''_{z=1}|=1,\quad |(I^nf(z))'_{z=1}|=1/|c|.$$

Since $|c_1|=\frac{2}{3^n}|\frac{c+2}{c}|$, (2.3) is satisfied with equality. That is,

$$\begin{aligned}&\frac{1}{|c|}\left(1+\frac{|c+3|(2|c+2|-3^n|c||c_1|)^2}{|c+3|(4|c+2|^2-3^{2n}|c|^2|c_1|^2)-4^{n+1}|c||c+2|^2|c_2|}\right)\\&=\frac{1}{|c|}\left(1+\frac{|c+3|\big(2|c+2|-3^n|c|\frac{2}{3^n}\big|\frac{c+2}{c}\big|\big)^2}{|c+3|\big(4|c+2|^2-3^{2n}|c|^2\big(\frac{2}{3^n}\big|\frac{c+2}{c}\big|\big)^2\big)-4^{n+1}|c||c+2|^2|c_2|}\right)=\frac{1}{|c|}.\end{aligned}$$

Thus, we obtain $|(I^nf(z))'_{z=1}|=1/|c|$. □

If $z^2(I^nf(z))'$ has no zeros different from $z=0$ in Theorem 6, the inequality (2.3) can be further strengthened. This is given by the following Theorem.

THEOREM 2.4. *Let $f(z)\in\mathcal{W}(c>0)$ and $z^2(I^nf(z))'$ has no zeros in $U$ except $z=0$ and $c_1>0$. Assume that, for some $b\in\partial U$, $(I^nf(z))'$ has angular limit $(I^nf(z))'_{z=b}$ at $b$ and $(I^nf(z))_{z=b}=\frac{c+1}{c}\big(I^n\big(\frac{c}{z^{c+1}}\int_0^z t^cf(t)dt\big)\big)_{z=b}$. Then we have the inequality*

$$|(I^nf(z))'_{z=b}|\geqslant\frac{1}{c}\left(1-\frac{1}{2}\frac{3^n\frac{c}{c+2}c_1\ln^2\big(\frac{3^n}{2}\frac{c}{c+2}c_1\big)}{3^n\frac{c}{c+2}c_1\ln\big(\frac{3^n}{2}\frac{c}{c+2}c_1\big)-4^n\frac{c}{c+3}|c_2|}\right).\tag{2.5}$$

*The inequality* (2.5) *is sharp.*

PROOF. Let $c_1>0$ and $c>0$ in the expression of the function $f(z)$. Having in mind the inequality (2.4) and the function $z^2(I^nf(z))'$ has no zeros in $U$ except $U-\{0\}$, we denote by $\ln\varphi(z)$ the holomorphic branch of the logarithm normed by the condition

$$\ln\varphi(0)=\ln\left(\frac{3^n}{2}\frac{c}{c+2}c_1\right)<0.$$

That is, since $z^2(I^nf(z))'$ has no zeros in $U$ except $U-\{0\}$, we have that $z^2(I^nH(z))'$ has no zeros in $U$ except $z=0$

The auxiliary function

$$F(z) = \frac{\ln\varphi(z) - \ln\varphi(0)}{\ln\varphi(z) + \ln\varphi(0)}$$

is holomorphic in the unit disc $U$, $|F(z)| < 1$, $F(0) = 0$ and $|F(b)| = 1$ for $b \in \partial U$.

For $p = 1$, from (1.7), we obtain

$$\begin{aligned}\frac{2}{1+|F'(0)|} \leqslant |F'(b)| &= \frac{|2\ln\varphi(0)|}{|\ln\varphi(b) + \ln\varphi(0)|^2}\left|\frac{\varphi'(b)}{\varphi(b)}\right| \\ &= \frac{-2\ln\varphi(0)}{\ln^2\varphi(0) + \arg^2\varphi(b)}\{|\psi'(b)| - 2\}.\end{aligned}$$

Replacing $\arg^2\omega(b)$ by zero, we get

$$\begin{aligned}&\frac{1}{1 - \frac{4^n\frac{c}{c+3}|c_2|}{\frac{3^n}{2}\frac{c}{c+2}c_1 2\ln\left(\frac{3^n}{2}\frac{c}{c+2}c_1\right)}} \leqslant \frac{-1}{\ln\left(\frac{3^n}{2}\frac{c}{c+2}c_1\right)}\{2|(I^nH(z))''_{z=b}| - 2\}, \\ &-\frac{3^n\frac{c}{c+2}c_1\ln^2\left(\frac{3^n}{2}\frac{c}{c+2}c_1\right)}{3^n\frac{c}{c+2}c_1\ln\left(\frac{3^n}{2}\frac{c}{c+2}c_1\right) - 4^n\frac{c}{c+3}|c_2|} \leqslant 2|(I^nH(z))''_{z=b}| - 2, \\ &-\frac{1}{2}\frac{3^n\frac{c}{c+2}c_1\ln^2\left(\frac{3^n}{2}\frac{c}{c+2}c_1\right)}{3^n\frac{c}{c+2}c_1\ln\left(\frac{3^n}{2}\frac{c}{c+2}c_1\right) - 4^n\frac{c}{c+3}|c_2|} \leqslant |(I^nH(z))''_{z=b}| - 1, \\ &|(I^nH(z))''_{z=b}| \geqslant \left(1 - \frac{1}{2}\frac{3^n\frac{c}{c+2}c_1\ln^2\left(\frac{3^n}{2}\frac{c}{c+2}c_1\right)}{3^n\frac{c}{c+2}c_1\ln\left(\frac{3^n}{2}\frac{c}{c+2}c_1\right) - 4^n\frac{c}{c+3}|c_2|}\right).\end{aligned}$$

Also, since $(I^nH(z))''_{z=b} = c(I^nf(z))'_{z=b}$, we have

$$|(I^nf(z))'_{z=b}| \geqslant \frac{1}{c}\left(1 - \frac{1}{2}\frac{\frac{3^n}{2}\frac{c}{c+2}c_1\ln^2\left(\frac{3^n}{2}\frac{c}{c+2}c_1\right)}{3^n\frac{c}{c+2}c_1\ln\left(\frac{3^n}{2}\frac{c}{c+2}c_1\right) - 4^n\frac{c}{c+3}|c_2|}\right).$$

Equality in (2.5) occurs for the solution of the equation

$$(I^nH(z))' = \frac{z^2-1}{z^2(1+z^2)},$$

with the condition $\lim_{z\to 0} z^2H(z) = 0$ at $z = 0$. □

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