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CONTINUED FRACTIONS AND DIOPHANTINE EQUATIONS IN POSITIVE CHARACTERISTIC

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Abstract. We exhibit explicitly the continued fraction expansion of some algebraic power series over a finite field. We also discuss some Diophantine equations on the ring of polynomials, which are intimately related to these power series.

1. Introduction

Formal power series over a given field have been studied for a long time in Number Theory. The case of a finite base field is particularly important and the analogy between these power series and the real numbers is striking. Nevertheless the positive characteristic, inducing the existence of the Frobenius isomorphism, makes rational approximation to algebraic elements very different from the case of real numbers and somehow more complex. Baum Sweet's article [**3**] was the starting point of the diophantine approximation in positive characteristic through the continued fraction expansion and opened several questions and research axis in this area. The continued fraction expansion and the irrationality measure of the solution of many algebraic equation was computed. We recall for example the case of the equation algebraic irreducible $x^n = R(*)$, where n is a positive integer, not divisible by p, and $R \in \mathbb{F}(T)$. Such an equation has a root in $\mathbb{F}((T^{-1}))$ if (and only if) $\deg R$ is a multiple of n and the first coefficient of R belong to $\mathbb{F}^n$. Osgood [**11**], Voloch [**14**], de Mathan [**8**] and Lasjaunias [**4**] have studied the rational approximation of the solution of the equation $(*)$. For instance, we know that it is well approximable by rationals for suitable R. Furthermore, it is an element of a particular subset of algebraic elements called hyperquadratic. Let $r = p^t$ with $t \geqslant 0$; we say that α belonging to $\mathbb{F}((T^{-1}))$ is hyperquadratic if α is irrational and satisfies an algebraic equation of the particular form $A\alpha^{r+1} + B\alpha^r + C\alpha + D = 0$, where A, B, C and D belong to $\mathbb{F}[T]$. Note that the quadratic elements are hyperquadratic. Many explicit continued fractions are known for nonquadratic but hyperquadratic elements; see for example [**2**], [**5**] and [**12**]. However, the explicit continued fraction

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expansion of the solution of $(*)$ is not yet completely described. In [**3**], Baum and Sweet have given the continued fraction expansion of the irrational solution of the equation $x^{2^n-1} = \frac{A}{A+1}$ in $\mathbb{F}((T^{-1}))$, where $\mathbb{F} = \mathbb{F}_2$ and A is nonconstant polynomial. This has allowed them to describe the solutions of the Diophantine equation $(A+1)P^{2^n-1} - AQ^{2^n-1} = 1$. They have also computed the all partial quotients of other algebraic elements and discussed the solutions of some other Diophantine equation.

The remainder of the paper is organised in the following way. We gather some definitions and theorems in Section 2. We then consider a nonconstant polynomial A and r is a power of p. In Section 3, we will give explicitly the continued fraction expansion of the solution of the equation $x^{r+1} = A^{r+1} - 1$ in $\mathbb{F}((T^{-1}))$, Theorem 3.1. We will compute, in Theorem 3.2, the solutions of some Diophantine equations related to it. Note that the solution of this equation for $r = 2$ and $A = T$ was described in [**1**]. The Section 4 is devoted to describe all partial quotients of the solution of the equation $A^{r-1}x^{r+1} - (A^r + 1)x^r + A^{r-1}x - A^r = 0$, Theorem 4.1. Further, in Theorem 4.2, we will also discuss a Diophantine equation related to it.

So by this work we add other examples to the explicitly known hyperquadratic continued fractions. Furthermore, for these examples, Liouville's theorem is sharp, and thus a Thue–Siegel–Roth theorem cannot hold for such examples. Indeed, we will improve the following result of Baum ans Sweet [**3**], Theorems 3.3, 4.3.

THEOREM 1.1. *Let $d, n \in \mathbb{N} \setminus \{0\}$. Then there exist an algebraic formal power series $\theta \in \mathbb{F}_2((T^{-1}))$ of degree $2^n + 1$ such that the equation*

$$\left|\theta - \frac{P}{Q}\right| = \frac{2^{-d}}{|Q|^{2^n+1}}$$

has infinitely many solutions $(P, Q) \in \mathbb{F}_2[T] \times \mathbb{F}_2[T]$.

We finally note that in [**10**], this result with some Diophantine equations studied in [**3**] was improved.

2. Preliminaries

Let p be a prime number and let $\mathbb{F}$ be a finite field of characteristic p. For a formal indeterminate T let $\mathbb{F}[T]$, $\mathbb{F}(T)$ and $\mathbb{F}((T^{-1}))$, respectively, denote the ring of polynomials, the field of rational functions and the field of power series in $1/T$ over $\mathbb{F}$. These fields are valuated by the ultrametric absolute value introduced on $\mathbb{F}(T)$ by $|P/Q| = e^{\deg(P)-\deg(Q)}$, where e is Euler's number. Hence a nonzero element of $\mathbb{F}((T^{-1}))$ is written as $\alpha = \sum_{k \leqslant k_0} a_k T^k$ with $k_0 \in \mathbb{Z}$, $a_k \in \mathbb{F}$, and $a_{k_0} \neq 0$ and we have $|\alpha| = e^{k_0}$. This field $\mathbb{F}((T^{-1}))$ is the completion of the field $\mathbb{F}(T)$ for this absolute value. We recall that each irrational (rational) element α of $\mathbb{F}((T^{-1}))$ can be expanded as an infinite (finite) continued fraction. This will be denoted $\alpha = [a_0, a_1, \ldots, a_n, \ldots]$ where the $a_i \in \mathbb{F}[T]$, with $\deg(a_i) > 0$ for $i \geqslant 1$, are the partial quotients and the tail $\alpha_i = [a_i, a_{i+1}, \ldots] \in \mathbb{F}((T^{-1}))$ is the complete quotient. As in the classical theory, we define recursively the two sequences of polynomials $(P_n)_{n\geqslant 0}$ and $(Q_n)_{n\geqslant 0}$ by $P_n = a_n P_{n-1} + P_{n-2}$ and $Q_n = a_n Q_{n-1} + Q_{n-2}$, with the initial conditions $P_0 = a_0$, $P_1 = a_0 a_1 + 1$, $Q_0 = 1$ and

$Q_1 = a_1$. We have $P_{n+1}Q_n - Q_{n+1}P_n = (-1)^n$, whence P_n and Q_n are coprime polynomials. The rational function P_n/Q_n is called a convergent to α and we have $P_n/Q_n = [a_0, a_1, \dots, a_n]$. It is easy to see that $\deg Q_{n+1} = \deg a_{n+1} + \deg Q_n$, thus $\deg Q_n = \sum_{j=1}^{n} \deg a_j$. Moreover we have for $n \geqslant 1$ the equality:

$$\alpha = [a_0, a_1, \dots, a_n, \alpha_{n+1}] = \frac{P_n\alpha_{n+1} + P_{n-1}}{Q_n\alpha_{n+1} + Q_{n-1}} \tag{2.1}$$

where $\alpha_{n+1} = [a_{n+1}, a_{n+2}, \dots]$ is called a complete quotient of α.

We have to introduce the following lemma and theorem. Mahler [**7**] proved the equivalence of Liouville's theorem [**6**] in the formal series case:

THEOREM 2.1. *Let $\alpha \in \mathbb{F}((T^{-1}))$ algebraic of degree $d \geqslant 2$; then there exists $c > 0$ such that for all $P, Q \in \mathbb{F}[T]$ one has $\left|\alpha - \frac{P}{Q}\right| \geqslant \frac{c}{|Q|^d}$.*

We have to introduce the following result of Mkaouar.

THEOREM 2.2. *Let $P(x) = \sum_{0\leqslant i\leqslant n} A_i x^i$ with $A_i \in \mathbb{F}[T]$ and $n \geqslant 1$. Suppose that $\deg A_i < \deg A_{n-1}$ for all $0 \leqslant i \leqslant n$ and $i \neq n-1$. Then there exists a unique power series γ with positive degree satisfying $P(\gamma) = 0$. Moreover $[\gamma] = -[A_{n-1}/A_n]$.*

The proof of this theorem can be found in Mkaouar's paper [**9**]. The reader who is interested in a survey on the different contributions to this topic and for full references can consult for example [**4**], [**12**] and [[**13**], Chap 9].

3. Continued fraction expansion of the solution of the equation $x^{r+1} = A^{r+1} - 1$

THEOREM 3.1. *Let r be a power of p. Let $\alpha \in \mathbb{F}((T^{-1}))$ be the irrational solution of the equation*

$$x^{r+1} = A^{r+1} - 1. \tag{3.1}$$

Then, the continued fraction expansion of α is $[a_0, \dots, a_n, \dots]$, where $a_0 = A$, $a_1 = -A^r$ and for $n \geqslant 0$

$$a_{n+2} = -A^{r^n}\left(1 + A^{r+1} + A^{2(r+1)} + \cdots + A^{(r-2)(r+1)}\right)^{r^n}\left(A^{r+1} - 1\right)^{\frac{r^n + (-1)^{n+1}}{r+1}}$$

PROOF. Let γ the irrational solution of the equation

$$\gamma^{r+1} + A^r\gamma^r + A\gamma + 1 = 0. \tag{3.2}$$

So from Theorem 2.2 we have $[\gamma] = -A^r$. Let $\alpha = [a_0, \dots, a_n, \dots]$ be such that $\alpha = A + \gamma^{-1}$. By a simple calculation we check that α is the solution of equation (3.1) namely $\alpha^{r+1} = A^{r+1} - 1$. As $|\gamma| > 1$ then $[\alpha] = a_0 = A$ and $\gamma = \alpha_1$. Moreover we can write (3.2) as

$$\alpha_1^r = \frac{-A\alpha_1 - 1}{\alpha_1 + A^r}. \tag{3.3}$$

We know that $\alpha_1 = -A^r + \alpha_2^{-1}$. So equation (3.3) becomes

$$\alpha_1^r = \frac{-A(-A^r + \alpha_2^{-1}) - 1}{\alpha_2^{-1}} = (A^{r+1} - 1)\alpha_2 - A. \tag{3.4}$$

Equation (3.4) gives that $a_1^r + \alpha_2^{-r} = (A^{r+1} - 1)\alpha_2 - A$. So

$$\alpha_2 = \frac{a_1^r + A}{A^{r+1} - 1} + (A^{r+1} - 1)^{-1}\alpha_2^{-r} = \frac{-A^{r^2} + A}{A^{r+1} - 1} + (A^{r+1} - 1)^{-1}\alpha_2^{-r}.$$

Since

$$-A^{r^2} + A = -A(A^{r+1} - 1)\big(1 + A^{r+1} + A^{2(r+1)} + \cdots + A^{(r-2)(r+1)}\big),$$

then $a_2 = -A(1 + A^{r+1} + A^{2(r+1)} + \cdots + A^{(r-2)(r+1)})$ and

$$\alpha_3 = (A^{r+1} - 1)\alpha_2^r. \tag{3.5}$$

Again, we know that $\alpha_2 = a_2 + \alpha_3^{-1}$. So we obtain from equation (3.5) that $\alpha_3 = (A^{r+1} - 1)a_2^r + (A^{r+1} - 1)\alpha_3^{-r}$. Then

$$a_3 = (A^{r+1} - 1)b_2^r = -A^r(A^{r+1} - 1)\big(1 + A^{r+1} + A^{2(r+1)} + \cdots + A^{(r-2)(r+1)}\big)^r,$$

and $\alpha_4 = (A^{r+1} - 1)^{-1}\alpha_3^r$. The last equality gives that

$$\alpha_4 = a_4 + \alpha_5^{-1} = (A^{r+1} - 1)^{-1}a_3^r + (A^{r+1} - 1)^{-1}\alpha_4^{-r}.$$

Then

$$a_4 = -A^{r^2}(A^{r+1} - 1)^{r-1}\big(1 + A^{r+1} + A^{2(r+1)} + \cdots + A^{(r-2)(r+1)}\big)^{r^2},$$

and $\alpha_5 = (A^{r+1} - 1)\alpha_4^r$. So by a simple recursion we prove that for all $k \geqslant 1$:

$$\alpha_{2k+1} = (A^{r+1} - 1)\alpha_{2k}^r \quad \text{and} \quad \alpha_{2k+2} = (A^{r+1} - 1)^{-1}\alpha_{2k+1}^r.$$

Hence for all $k \geqslant 1$:

$$a_{2k+1} = (A^{r+1} - 1)a_{2k}^r \quad \text{and} \quad a_{2k+2} = (A^{r+1} - 1)^{-1}a_{2k+1}^r.$$

Thus for all $k \geqslant 1$:

$$a_{2k+1} = -A^{r^{2k-1}}\big(1 + A^{r+1} + A^{2(r+1)} + \cdots + A^{(r-2)(r+1)}\big)^{r^{2k-1}}\big(A^{r+1} - 1\big)^{\frac{r^{2k-1}+1}{r+1}},$$

$$a_{2k+2} = -A^{r^{2k}}\big(1 + A^{r+1} + A^{2(r+1)} + \cdots + A^{(r-2)(r+1)}\big)^{r^{2k}}\big(A^{r+1} - 1\big)^{\frac{r^{2k}-1}{r+1}}.$$

So we obtain the desired result. □

THEOREM 3.2. *Let A be a nonconstant polynomial of $\mathbb{F}[T]$ and r a power of p. The Diophantine equations*

$$P^{r+1} - (A^{r+1} - 1)Q^{r+1} = 1,$$
$$P^{r+1} - (A^{r+1} - 1)Q^{r+1} = -(A^{r+1} - 1)$$

have infinitely many solutions $(P, Q) \in \mathbb{F}[T] \times \mathbb{F}[T]$, which are respectively the even and the odd convergents of the solution α of equation (3.1).

For the proof of this theorem, we need the following lemma.

LEMMA 3.1. *Let r be a power of p. Let α be the irrational solution of equation (3.1) and $(P_n/Q_n)_{n\geqslant 0}$ the sequence of convergent of α. Then $A^{r+1}-1$ divides P_{2s+1} for all $s \geqslant 0$.*

PROOF. If $(P_n/Q_n)_{n\geqslant 0}$ the sequence of convergent of α, then $P_0 = A$ and $P_1 = A^{r+1}-1$. Then $A^{r+1}-1$ divides P_1. Suppose that $A^{r+1}-1$ divides P_{2s-1} for all $s \geqslant 1$. We have $P_{2s+1} = a_{2s+1}P_{2s} + P_{2s-1}$ and since

$$a_{2s+1} = -A^{r^{2s-1}}\big(1 + A^{r+1} + A^{2(r+1)} + \cdots + A^{(r-2)(r+1)}\big)^{r^{2s-1}}\big(A^{r+1}-1\big)^{\frac{r^{2s-1}+1}{r+1}}$$

is divisible by $A^{r+1}-1$ for all $s \geqslant 1$, then P_{2s+1} is divisible by $A^{r+1}-1$ for all $s \geqslant 1$. □

PROOF OF THEOREM 3.2. Let $H(Y,Z) = Y^{r+1} - (A^{r+1}-1)Z^{r+1}$ and α the unique root of $L(Y) = H(Y,1)$ satisfying $|\alpha| = |A|$, then by writing $H(Y,1)$ in the form $H(Y,1) = \alpha^r(Y-\alpha) + Y(Y-\alpha)^r$, we can conclude, for all integer $s \geqslant 0$, that

$$\Big|H\Big(\frac{P_{2s}}{Q_{2s}},1\Big)\Big| = \Big|A|^r|\alpha - \frac{P_{2s}}{Q_{2s}}\Big|, \tag{3.6}$$

$$\Big|H\Big(\frac{P_{2s+1}}{Q_{2s+1}},1\Big)\Big| = |A|^r\Big|\alpha - \frac{P_{2s+1}}{Q_{2s+1}}\Big|. \tag{3.7}$$

On the other hand, a simple calculation gives that

$$\begin{aligned}
|Q_{2s}| = \prod_{i=1}^{2s}|a_i| &= |A|^r|A|^{2\frac{r^{2s-1}-r-1}{r-1}}|A|^{(r-2)(r+1)\frac{r^{2s-1}-1}{r-1}},\\
|Q_{2s+1}| = \prod_{i=1}^{2s+1}|a_i| &= |A|^r|A|^{2\frac{r^{2s}-1}{r-1}}|A|^{(r-2)(r+1)\frac{r^{2s}-1}{r-1}},\\
|a_{2s+2}| &= |A|^{2r^{2s}-1}|A|^{(r-2)(r+1)r^{2s}},\\
|a_{2s+1}| &= |A|^{2r^{2s-1}+1}|A|^{(r-2)(r+1)r^{2s-1}}.
\end{aligned}$$

We can easily check that $|a_{2s+1}| = |A|^r|Q_{2s}|^{r-1}$ and $|a_{2s+2}| = |A|^{-1}|Q_{2s+1}|^{r-1}$. This gives that

$$\begin{aligned}
\Big|\alpha - \frac{P_{2s}}{Q_{2s}}\Big| &= \frac{1}{|a_{2s+1}||Q_{2s}|^2} = \frac{1}{|A|^r|Q_{2s}|^{r+1}},\\
\Big|\alpha - \frac{P_{2s+1}}{Q_{2s+1}}\Big| &= \frac{1}{|a_{2s+2}||Q_{2s+1}|^2} = \frac{1}{|A|^{-1}|Q_{2s+1}|^{r+1}},
\end{aligned} \tag{3.8}$$

So equation (3.6) becomes

$$\Big|H\Big(\frac{P_{2s}}{Q_{2s}},1\Big)\Big| = |A|^r\frac{1}{|A|^r|Q_{2s}|^{r+1}} = \frac{1}{|Q_{2s}|^{r+1}}.$$

and equation (3.7) becomes

$$\Big|H\Big(\frac{P_{2s+1}}{Q_{2s+1}},1\Big)\Big| = |A|^r\frac{1}{|A|^{-1}|Q_{2s}|^{r+1}} = \frac{|A|^{r+1}}{|Q_{2s}|^{r+1}}.$$

Since $H(P,Q) = Q^{r+1}H\big(\frac{P}{Q},1\big)$ we have $|H(P_{2s},Q_{2s})| = 1$ and $|H(P_{2s+1},Q_{2s+1})| = |A|^{r+1}$. From $|H(P_{2s},Q_{2s})| = 1$ we get $H(P_{2s},Q_{2s}) \in \mathbb{F}^*$. Since $P_0 = A$ and $Q_0 = 1$ then $H(P_0,Q_0) = 1$. This gives that $H(P_{2s},Q_{2s}) = 1$ for all $s \geqslant 0$.

On the other hand, we have $H(P_{2s+1},Q_{2s+1}) = P_{2s+1}^{r+1} - (A^{r+1}-1)Q_{2s+1}^{r+1}$ then $H(P_{2s+1},Q_{2s+1})$ is divisible by $A^{r+1}-1$ for all $s \geqslant 0$. As $|H(P_{2s+1},Q_{2s+1})| = |A|^{r+1}$ and since $H(P_1,Q_1) = -A^{r+1}+1$ then $H(P_{2s+1},Q_{2s+1}) = -A^{r+1}+1$ for all $s \geqslant 0$. □

The following theorem improves Baum and Sweet's theorem stated in Theorem 1.1.

THEOREM 3.3. *Let $d \in \mathbb{N} \setminus \{0\}$ and r a power of p. Then there exist an algebraic formal power series $\alpha \in \mathbb{F}_q((T^{-1}))$ of degree $r+1$ such that the equation*

$$\Big|\alpha - \frac{P}{Q}\Big| = \frac{e^{-d}}{|Q|^{r+1}}$$

has infinitely many solutions $(P,Q) \in \mathbb{F}[T] \times \mathbb{F}[T]$.

PROOF. The proof is directly deduced from equality (3.8). □

4. Continued fraction expansion of the solution of the equation $A^{r-1}x^{r+1} - (A^r+1)x^r + A^{r-1}x - A^r = 0$

THEOREM 4.1. *Let r be a power of a prime p. Let A be a nonzero polynomial of $\mathbb{F}_q[T]$. Let $\beta \in \mathbb{F}_q((T^{-1}))$ be the irrational solution of strictly positive degree of the equation*

$$A^{r-1}\beta^{r+1} - (A^r+1)\beta^r + A^{r-1}\beta - A^r = 0. \tag{4.1}$$

Then $\beta = [b_0,\dots,b_n,\dots]$ where $b_0 = A$ and for all $n \geqslant 1$:

$$b_n = \begin{cases} A^{r\frac{n+1}{2}-1} & \text{if } n \text{ is odd} \\ A & \text{if } n \text{ is even.} \end{cases}$$

PROOF. First, we have to check the value of $|\beta|$. Let $F(x) = A^{r-1}x^{r+1} - (A^r+1)x^r + A^{r-1}x - A^r$. We consider the Newton polygon of $F(x)$, which is denoted by $\mathrm{N}(F)$. There are four points $(0, r\deg A)$, $(1,(r-1)\deg A)$, $(r, r\deg A)$ and $(r+1,(r-1)\deg A)$ on the xy-plane, and hence $\mathrm{N}(F)$ consists of two line segments. The first one has slope 0 and the second one has slope $\deg A$. This gives that $\deg\beta = \deg A$.

We have $\frac{A^r+1}{A^{r-1}}$ is a convergent to β. In fact, as $\beta = \frac{(A^r+1)\beta^r + A^r}{A^{r-1}\beta^r + A^{r-1}}$ then

$$\Big|\beta - \frac{A^r+1}{A^{r-1}}\Big| = \Big|\frac{(A^r+1)\beta^r - A^r}{A^{r-1}\beta^r + A^{r-1}} - \frac{A^r+1}{A^{r-1}}\Big| = \frac{1}{|A|^{2(r-1)}|A|}$$

Since $\frac{A^r+1}{A^{r-1}} = [A, A^{r-1}]$, then $b_0 = A$ and $b_1 = A^{r-1}$.

On the other hand, (4.1) can be written as

$$\beta^r = \frac{-A^{r-1}\beta + A^r}{A^{r-1}\beta - (A^r+1)} \tag{4.2}$$

From equality (2.1) we have

$$\beta = \frac{(A^r+1)\beta_2 + A}{A^{r-1}\beta_2 + 1} \tag{4.3}$$

Combining (4.2) and (4.3) we obtain that

$$\beta^r = A^{r-1}\beta_2 \tag{4.4}$$

We have $\beta = A + \frac{1}{\beta_1}$ so (4.4) gives that

$$\beta_2 = \frac{\beta^r}{A^{r-1}} = \frac{A^r}{A^{r-1}} + \frac{1}{A^{r-1}\beta_1^r} = A + \frac{1}{A^{r-1}\beta_1^r}$$

This gives that $b_2 = A$ and $\beta_3 = A^{r-1}\beta_1^r$. Again this equation gives that

$$\beta_3 = A^{r-1}b_1^r + \frac{A^{r-1}}{\beta_2^r} = A^{r^2-1} + \frac{A^{r-1}}{\beta_2^r}.$$

Since $|\beta_2^r| > |A^{r-1}|$ then $b_3 = A^{r^2-1}$ and

$$\beta_4 = \frac{\beta_2^r}{A^{r-1}}. \tag{4.5}$$

We see that (4.5) and (4.4) are of the same shape. We now claim that for all $k \geqslant 1$,

$$b_{2k} = A, b_{2k-1} = A^{r^k-1}, \beta_{2k+1} = A^{r-1}\beta_{2k-1}^r, \beta_{2k+2} = \beta_{2k}^r/A^{r-1}. \tag{4.6}$$

Clearly (4.6) is true for $k = 1$. So we assume (4.6) for $k = l \geqslant 1$. Then

$$\beta_{2l+2} = A^{r-1}((A^{r^l-1})^r + \frac{1}{\beta_{2l+1}^r}) = A^{r^{l+1}-1} + \frac{A^{r-1}}{\beta_{2l+1}^r},$$

which implies $b_{2l+2} == A^{r^{l+1}-1}$ and $\beta_{2l+3} = \beta_{2l+1}^r/A^{r-1}$. Then

$$\beta_{2l+3} = \frac{A^r}{A^{r-1}} + \frac{1}{A^{r-1}\beta_{2l+2}^r} = A + \frac{1}{A^{r-1}\beta_{2l+2}^r},$$

which implies $b_{2l+3} = A$ and $\beta_{2l+4} = A^{r-1}\beta_{2l+2}^r$. Thus (4.6) is also true for $k = l+1$. By induction, we see that (4.6) holds for all $k \geqslant 1$. □

THEOREM 4.2. *Let A be a nonconstant polynomial of $\mathbb{F}[T]$ and q a power of p. The Diophantine equations*

$$A^{r-1}P^{r+1} - (A^r+1)P^rQ + A^{r-1}PQ^r - A^rQ^{r+1} = -A^r$$

have infinitely many solutions $(P, Q) \in \mathbb{F}[T] \times \mathbb{F}[T]$, which are the even convergents of the solution β of equation (4.1).

LEMMA 4.1. *Let θ be the irrational solution of the equation* (4.1) *and $\left(\frac{P_n}{Q_n}\right)_{n\geqslant 0}$ the sequence of convergents of θ. Then A divides P_{2s} for all $s \geqslant 0$.*

PROOF. If $(P_n/Q_n)_{n\geqslant 0}$ the sequence of convergent of θ, then $P_0 = A$. Then A divides P_0. Suppose that A divides P_{2s} for all $s \geqslant 1$. We have $P_{2s+2} = b_{2s+2}P_{2s+1} + P_{2s}$ and since $b_{2s+2} = A$ is divisible by A then P_{2s+2} is divisible by A for all $s \geqslant 1$. □

PROOF OF THEOREM 4.2. Let $(P_n/Q_n)_{n\geqslant 0}$ be the sequence of convergent of β. Let $H(Y,Z) = A^{r-1}Y^{r+1} - (A^r+1)Y^rZ + A^{r-1}YZ^r - A^rZ^{r+1}$ and β be the unique root of $L(Y) = H(Y,1)$ satisfying $|\beta| = A$; then by writing $H(Y,1)$ in the form

$$H(Y,1) = A^{r-1}(\beta^r+1)(Y-\beta) + (A^{r-1}(Y-A)-1)(Y-\beta)^r,$$

we can conclude, for every integer $s \geqslant 0$, that

$$\Big|H\Big(\frac{P_{2s}}{Q_{2s}},1\Big)\Big| = |A^{2r-1}|\Big|\beta - \frac{P_{2s}}{Q_{2s}}\Big|, \tag{4.7}$$

On the other hand, a simple calculation gives that $|Q_{2s}| = \prod_{i=1}^{2s}|b_i| = |A|^{r(\frac{r^s-1}{r-1})}$, and $|b_{2s+1}| = |A|^{r^{s+1}-1}$. We can easily check that $|b_{2s+1}| = |A^{r-1}||Q_{2s}|^{r-1}$. This gives that

$$\Big|\beta - \frac{P_{2s}}{Q_{2s}}\Big| = \frac{1}{|b_{2s+1}||Q_{2s}|^2} = \frac{1}{|A|^{r-1}|Q_{2s}|^{r+1}}. \tag{4.8}$$

So equation (4.7) becomes

$$\Big|H\Big(\frac{P_{2s}}{Q_{2s}},1\Big)\Big| = |A^{2r-1}|\frac{1}{|A^{r-1}||Q_{2s}|^{r+1}} = \frac{|A|^r}{|Q_{2s}|^{r+1}}.$$

Since $H(P,Q) = Q^{r+1}H\big(\frac{P}{Q},1\big)$ we obtain that $|H(P_{2s},Q_{2s})| = |A|^r$. Since $P_0 = A$, $Q_0 = 1$ and $H(P_{2s},Q_{2s}) = A^{r-1}P_{2s}^{r+1} - (A^r+1)P_{2s}^rQ_{2s} + A^{r-1}P_{2s}Q_{2s}^r - A^rQ_{2s}^{r+1}$ then $H(P_0,Q_0) = -A^r$. Further, from Lemma 4.1, we have A divides P_{2s} for all $s \geqslant 0$. Then clearly A^r divides $H(P_{2s},Q_{2s})$ for all $s \geqslant 0$. This gives that $H(P_{2s},Q_{2s}) = -A^r$ and we obtain the desired result. □

We next prove that the algebraic power series α is of degree $r+1$.

LEMMA 4.2. *Let $A \in \mathbb{F}[T] \setminus \{0\}$. Then the polynomial*

$$L(Y) = A^{r-1}Y^{r+1} - (A^r+1)Y^r + A^{r-1}Y - A^r$$

is irreducible over $\mathbb{F}(T)$.

PROOF. Let $\beta \in \mathbb{F}((T^{-1}))$ the unique root of L such that $|\beta| > 1$. If β has degree $d < r+1$ and P/Q is a convergent of β, then by (4.8) and Theorem 2.1, there are two constants c and $c' > 0$ such that

$$\frac{c}{|Q|^d} \leqslant \Big|\beta - \frac{P}{Q}\Big| = \frac{c'}{|Q|^{r+1}},$$

for arbitrarily large $|Q|$, which is a contradiction. □

So we can introduce the following result.

THEOREM 4.3. *Let $d \in \mathbb{N} \setminus \{0\}$ and r a power of p. Then there exist an algebraic formal power series $\beta \in \mathbb{F}_q((T^{-1}))$ of degree $r+1$ such that the equation*

$$\Big|\beta - \frac{P}{Q}\Big| = \frac{e^{-(r-1)d}}{|Q|^{r+1}}$$

has infinitely many solutions $(P,Q) \in \mathbb{F}[T] \times \mathbb{F}[T]$.

PROOF. The proof is directly deduced from equality (4.8). □

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