PUBLICATIONS DE L'INSTITUT MATHÉMATIQUE
Nouvelle série, tome 109(123) (2021), 109–123 DOI: https://doi.org/10.2298/PIM2123109L

# ON ESTIMATES OF DEVIATION OF CONJUGATE FUNCTIONS FROM MATRIX OPERATORS OF THEIR FOURIER SERIES BY SOME EXPRESSIONS WITH *R*-DIFFERENCES OF THE ENTRIES

**Włodzimierz Łenski and Bogdan Szal**

Abstract. We extend the results of the authors from [Abstract and Applied Analysis, Volume 2016, Article ID 9712878] to the case conjugate Fourier series.

## 1. Introduction

Let $X = L^p$ or $X = C$, where $L^p$ $(1 \leqslant p \leqslant \infty)$ or $C$ be the class of all $2\pi$-periodic real-valued functions, integrable in the Lebesgue sense with the $p$-th power when $p \geqslant 1$ and essentially bounded when $p = \infty$ or continuous over $Q = [-\pi, \pi]$ with the norms

$$\|f\|_{L^p} := \|f(\bullet)\|_{L^p} = \begin{cases} \left(\int_Q |f(t)|^p dt\right)^{1/p} & \text{when } 1 \leqslant p < \infty, \\ \text{ess}\sup_{t\in Q} |f(t)| & \text{when } p = \infty, \end{cases}$$

$$\|f\|_C := \|f(\bullet)\|_C = \sup_{t\in Q} |f(t)|$$

and consider the trigonometric Fourier series

$$Sf(x) := \frac{a_0(f)}{2} + \sum_{\nu=1}^{\infty} (a_\nu(f) \cos \nu x + b_\nu(f) \sin \nu x)$$

with the partial sums $S_k f$ and the conjugate one

$$\widetilde{S}f(x) := \sum_{\nu=1}^{\infty} (a_\nu(f) \sin \nu x - b_\nu(f) \cos \nu x)$$

with the partial sums $\widetilde{S}_k f$. We know that if $f \in L^1$, then

$$\widetilde{f}(x) := -\frac{1}{\pi} \int_0^\pi \psi_x(t) \frac{1}{2} \cot \frac{t}{2} dt = \lim_{\epsilon\to 0^+} \widetilde{f}(x, \epsilon),$$

2010 *Mathematics Subject Classification:* 42A24.
*Key words and phrases:* rate of approximation, summability of Fourier series.
Communicated by Stevan Pilipović.

where

$$\widetilde{f}(x,\epsilon) := -\frac{1}{\pi}\int_{\epsilon}^{\pi} \psi_x(t)\frac{1}{2}\cot\frac{t}{2}dt,$$

with $\psi_x(t) := f(x+t) - f(x-t)$, exists for almost all $x$ [**6**, Th.(3.1)IV].

Let $A := (a_{n,k})$ be an infinite matrix of real numbers such that $a_{n,k} \geqslant 0$ when $k,n = 0,1,2,\dots$, $\lim_{n\to\infty} a_{n,k} = 0$ and $\sum_{k=0}^{\infty} a_{n,k} = 1$. We will use the notation $A_{n,r} = \sum_{k=0}^{\infty} |a_{n,k} - a_{n,k+r}|$, for $r \in \mathbb{N}$.

The $A$-transformation of $S_k f$ and of $\widetilde{S}_k f$ be given, by a matrix convention, as follows

$$\begin{pmatrix} T_{n,A}f(x) \\ \widetilde{T}_{n,A}f(x) \end{pmatrix} := \sum_{k=0}^{\infty} a_{n,k} \begin{pmatrix} S_k f(x) \\ \widetilde{S}_k f(x) \end{pmatrix} \quad (n = 0,1,2,\dots)$$

provided the series are convergent. In this paper, we study the upper bounds of $\big\|\widetilde{T}_{n,A}f - \widetilde{f}\big\|_X$ and $\big\|\widetilde{T}_{n,A}f(\bullet) - \widetilde{f}(\bullet,\epsilon)\big\|_X$ by the modulus of continuity of $f$ in the space $X$ defined by the formula

$$\widetilde{\omega}(f,\delta)_X = \sup_{0<t\leqslant\delta} \|\psi_\bullet(t)\|_X.$$

We will also use the modulus of continuity of $f$ in the space $X$ defined by $\omega(f,\delta)_X := \sup_{0\leqslant|t|\leqslant\delta} \|\varphi_\bullet(t)\|_X$, where $\varphi_x(t) := f(x+t) + f(x-t) - 2f(x)$.

We will consider a function $\omega$ of modulus of continuity type on the interval $[0,2\pi]$, i.e., a nondecreasing continuous function having the properties: $\omega(0) = 0$, $\omega(\delta_1+\delta_2) \leqslant \omega(\delta_1) + \omega(\delta_2)$ for any $0 \leqslant \delta_1 \leqslant \delta_2 \leqslant \delta_1+\delta_2 \leqslant 2\pi$.

The deviation $T_{n,A}f - f$ was estimated in [**2**] (see also [**1**, Theorems 3.4, p. 290] and [**5**]) as follows:

Theorem A. *Let* $f \in \big\{f \in X : \omega(f,\delta)_X = O(\omega(\delta))\,when\ \delta \in [0,2\pi]\big\}$ *and* $r \in \mathbb{N}$. *Then*

$$\|T_{n,A}f - f\|_X = O_r\Big(H\Big(\frac{\pi}{n+1}\Big)\Big(\frac{\pi}{n+1} + A_{n,r}\Big)\Big),$$

*where a function of modulus of continuity type* $\omega$ *satisfies the condition*

$$\int_u^{\pi} t^{-2}\omega(t)dt = O(H(u)) \ \text{when}\ u \in [0,\pi], \tag{1.1}$$

*with* $H(u) \geqslant 0$, *such that*

$$\int_0^u H(t)\,dt = O(uH(u)) \ \text{when}\ u \in [0,\pi]. \tag{1.2}$$

*Additionally, if*

$$\left[\sum_{l=0}^{n}\sum_{k=l}^{r+l-1} a_{n,k}\right]^{-1} = O_r(1), \tag{1.3}$$

*then*

$$\|T_{n,A}f - f\|_X = O_r\Big(H\Big(\frac{\pi}{n+1}\Big)A_{n,r}\Big)$$

*but if*

$$\sum_{k=0}^{\infty}(k+1)a_{n,k}=O(n+1), \tag{1.4}$$

*then*

$$\|T_{n,A}f-f\|_X=O_r\Big(\omega\Big(\frac{\pi}{n+1}\Big)+H\Big(\frac{\pi}{n+1}\Big)A_{n,r}\Big).$$

THEOREM B. *If $f\in X$ and a matrix $A$ is such that* (1.4) *holds, then for $r\in\mathbb{N}$*

$$\|T_{n,A}f-f\|_X=O_r\Bigg(\omega\Big(f,\frac{\pi}{n+1}\Big)_X+\sum_{\mu=1}^{n}\frac{\omega(f,\pi/\mu)_X}{\mu}\sum_{k=0}^{\mu+1}a_{n,k}$$
$$+\sum_{\mu=1}^{n}\omega(f,\pi/\mu)_X\sum_{k=\mu}^{\infty}|a_{n,k}-a_{n,k+r}|\Bigg).$$

From our theorems we also derived a corollary for a matrix $A$ satisfying the condition $\sum_{k=m}^{\infty}|a_{n,k}-a_{n,k+r}|=O_r(1)\sum_{k=[m/c]}^{\infty}\frac{a_{n,k}}{k+1}$ with some $c>1$ and $r\in\mathbb{N}$.

## 2. Statement of the results

Let $X_\omega=\big\{f\in X:\widetilde{\omega}(f,\delta)_X=O(\omega(\delta))$ when $\delta\in[0,2\pi]\big\}$. We present the estimates of the quantities $\big\|\widetilde{T}_{n,A}f(\bullet)-\widetilde{f}(\bullet)\big\|_X$ and $\big\|\widetilde{T}_{n,A}f(\bullet)-\widetilde{f}(\bullet,\epsilon)\big\|_X$ simultaneously. Finally, we give a corollary and a remark.

THEOREM 2.1. *If $f\in X_\omega$, where $\omega$ satisfies condition* (1.1) *such that* (1.2) *holds and $r\in\mathbb{N}$, then*

$$\left.\begin{array}{r}\big\|\widetilde{T}_{n,A}f(\bullet)-\widetilde{f}(\bullet)\big\|_X\\ \big\|\widetilde{T}_{n,A}f(\bullet)-\widetilde{f}\big(\bullet,\frac{\pi}{r(n+1)}\big)\big\|_X\end{array}\right\}=O_r\Big(H\Big(\frac{\pi}{n+1}\Big)\Big(\frac{\pi}{n+1}+A_{n,r}\Big)\Big).$$

*Additionally, if a matrix $A$ is such that* (1.3) *is true, then*

$$\left.\begin{array}{r}\big\|\widetilde{T}_{n,A}f(\bullet)-\widetilde{f}(\bullet)\big\|_X\\ \big\|\widetilde{T}_{n,A}f(\bullet)-\widetilde{f}\big(\bullet,\frac{\pi}{r(n+1)}\big)\big\|_X\end{array}\right\}=O_r\Big(H\Big(\frac{\pi}{n+1}\Big)A_{n,r}\Big).$$

THEOREM 2.2. *If $f\in X_\omega$, where $\omega$ satisfies condition* (1.1) *such that* (1.2) *holds, $r\in\mathbb{N}$ and a matrix $A$ is such that* (1.3) *is true, then*

$$\left.\begin{array}{r}\big\|\widetilde{T}_{n,A}f(\bullet)-\widetilde{f}(\bullet)\big\|_X\\ \big\|\widetilde{T}_{n,A}f(\bullet)-\widetilde{f}\big(\bullet,\frac{1}{r}A_{n,r}\big)\big\|_X\end{array}\right\}=O_r(H(A_{n,r})A_{n,r}).$$

THEOREM 2.3. *If $f\in X_\omega$, where $\omega$ satisfies condition* (1.1) *such that* (1.2) *holds and $r\in\mathbb{N}$, then*

$$\left.\begin{array}{r}\big\|\widetilde{T}_{n,A}f(\bullet)-\widetilde{f}(\bullet)\big\|_X\\ \big\|\widetilde{T}_{n,A}f(\bullet)-\widetilde{f}\big(\bullet,\frac{\pi}{r(n+1)}\big)\big\|_X\end{array}\right\}=O_r\Big(\omega\Big(\frac{\pi}{n+1}\Big)+H\Big(\frac{\pi}{n+1}\Big)A_{n,r}\Big),$$

*where in the case of the first estimate $\omega$ satisfies the extra condition*

$$\int_0^u t^{-1}\omega(t)dt=O(\omega(u))\ \text{when}\ u\in[0,2\pi], \tag{2.1}$$

*but in the case of the second estimate a matrix $A$ is such that* (1.4) *is true.*

THEOREM 2.4. *If $f \in X$ and $r \in \mathbb{N}$, then*

$$\left.\begin{array}{r}\left\|\widetilde{T}_{n,A}f(\bullet)-\widetilde{f}(\bullet)\right\|_X\\ \left\|\widetilde{T}_{n,A}f(\bullet)-\widetilde{f}\left(\bullet,\frac{\pi}{r(n+1)}\right)\right\|_X\end{array}\right\}=O_r\left(\widetilde{\omega}\left(f,\frac{\pi}{n+1}\right)_X+\sum_{\mu=1}^{n}\frac{\widetilde{\omega}\left(f,\frac{\pi}{\mu}\right)_X}{\mu}\sum_{k=0}^{\mu+1}a_{n,k}\right.$$
$$\left.+\sum_{\mu=1}^{n}\widetilde{\omega}\left(f,\frac{\pi}{\mu}\right)_X\sum_{k=\mu}^{\infty}\left|a_{n,k}-a_{n,k+r}\right|\right),$$

*were in the case of the first estimate $\widetilde{\omega}$ instead of $\omega$ satisfies the extra condition* (2.1)*, but in the case of the second estimate a matrix $A$ is such that* (1.4) *is true.*

COROLLARY 2.1. *If $f \in X_\omega$, where $\omega$ satisfies the condition* (1.1) *such that* (1.2) *is true and*

$$\sum_{k=m}^{\infty}\left|a_{n,k}-a_{n,k+r}\right|=O_r(1)\sum_{k=[m/c]}^{\infty}\frac{a_{n,k}}{k+1}, \tag{2.2}$$

*with some $c > 1$ and $r \in \mathbb{N}$ holds, then*

$$\left.\begin{array}{r}\left\|\widetilde{T}_{n,A}f(\bullet)-\widetilde{f}(\bullet)\right\|_X\\ \left\|\widetilde{T}_{n,A}f(\bullet)-\widetilde{f}\left(\bullet,\frac{\pi}{r(n+1)}\right)\right\|_X\end{array}\right\}=O_r\left(\frac{H\left(\frac{\pi}{n+1}\right)}{n+1}+\sum_{k=0}^{n}a_{n,k}\frac{H\left(\frac{\pi}{k+1}\right)}{k+1}\right),$$

*were in the case of the first estimate $\widetilde{\omega}$ instead of $\omega$ satisfies extra condition* (2.1)*, but in the case of the second estimate a matrix $A$ is such that* (1.4) *is true.*

REMARK 2.1. We note that our extra conditions (1.3) and (1.4) for a lower triangular infinite matrix $A$ always hold.

## 3. Auxiliary results

We begin this section by some notations from [**4**] and [**6**, Section 5 of Chapter II]. Let for $r = \pm1, \pm2, \dots$

$$D^{\circ}_{k,r}(t)=\frac{\sin\frac{(2k+r)t}{2}}{2\sin\frac{rt}{2}},\quad \widetilde{D^{\circ}}_{k,r}(t)=\frac{\cos\frac{(2k+r)t}{2}}{2\sin\frac{rt}{2}},\quad \widetilde{D}_{k,r}(t)=\frac{\cos\frac{t}{2}-\cos\frac{(2k+r)t}{2}}{2\sin\frac{rt}{2}}.$$

It is clear by [**6**] that $\widetilde{S}_k f(x) = -\frac{1}{\pi}\int_{-\pi}^{\pi} f(x+t)\widetilde{D}_{k,1}(t)dt$, whence

$$\widetilde{T}_{n,A}f(x)-\widetilde{f}(x)=\frac{1}{\pi}\int_0^{\pi}\psi_x(t)\sum_{k=0}^{\infty}a_{n,k}\widetilde{D^{\circ}}_{k,1}(t)dt$$

and

$$\widetilde{T}_{n,A}f(x)-\widetilde{f}\left(x,\frac{\pi}{r(n+1)}\right)=-\frac{1}{\pi}\int_0^{\frac{\pi}{r(n+1)}}\psi_x(t)\sum_{k=0}^{\infty}a_{n,k}\widetilde{D}_{k,1}(t)dt$$
$$+\frac{1}{\pi}\int_{\frac{\pi}{r(n+1)}}^{\pi}\psi_x(t)\sum_{k=0}^{\infty}a_{n,k}\widetilde{D^{\circ}}_{k,1}(t)dt.$$

Next, we present the known estimates and relations.

LEMMA 3.1. [6] *If* $0 < |t| \leqslant \pi$, *then* $|\widetilde{D^{\circ}}_{k,1}(t)| \leqslant \frac{\pi}{2|t|}$, $|\widetilde{D}_{k,1}(t)| \leqslant \frac{\pi}{|t|}$ *and, for any real* $t$, *we have* $|\widetilde{D}_{k,1}(t)| \leqslant \frac{1}{2}k(k+1)|t|$ *and* $|\widetilde{D}_{k,1}(t)| \leqslant k < k+1$.

LEMMA 3.2. [4] *Let* $r \in N$, $l \in Z$ *and* $a := (a_n) \subset \mathbb{C}$. *If* $x \neq \frac{2l\pi}{r}$, *then for every* $m \geqslant n$

$$\sum_{k=n}^{m} a_k \sin kx = -\sum_{k=n}^{m} (a_k - a_{k+r})\, \widetilde{D^{\circ}}_{k,r}(t) + \sum_{k=m+1}^{m+r} a_k \widetilde{D^{\circ}}_{k,-r}(t) - \sum_{k=n}^{n+r-1} a_k \widetilde{D^{\circ}}_{k,-r}(t),$$

$$\sum_{k=n}^{m} a_k \cos kx = \sum_{k=n}^{m} (a_k - a_{k+r}) D^{\circ}_{k,r}(t) - \sum_{k=m+1}^{m+r} a_k D^{\circ}_{k,-r}(t) + \sum_{k=n}^{n+r-1} a_k D^{\circ}_{k,-r}(t).$$

We additionally need two estimates with a function of modulus of continuity type $\omega$.

LEMMA 3.3. [2] *If* (1.1) *and* (1.2) *hold, then for* $c \geqslant 1$ *and* $\beta > \alpha > 0$

$$\int_{\alpha}^{\beta} t^{-1}\omega(t)dt = O((\beta-\alpha)H(c(\beta-\alpha))) \text{ when } (\beta-\alpha) \in [0, 2\pi].$$

LEMMA 3.4. [2] *If* (1.1) *and* (1.2) *hold, then for* $b \geqslant 1$,

$$\int_{u}^{\pi} t^{-2}\omega(t)dt = O(H(bu)) \text{ when } u \in [0, \pi].$$

Finally, we present a very useful trivial property of a function of modulus of continuity type $\omega$.

LEMMA 3.5. *A function* $\omega$ *of modulus of continuity type on the interval* $[0, 2\pi]$ *satisfies the following conditions* $\delta_2^{-1}\omega(\delta_2) \leqslant 2\delta_1^{-1}\omega(\delta_1)$ *for* $\delta_2 \geqslant \delta_1 > 0$ *and* $\omega(n\delta) \leqslant n\omega(\delta)$ *for* $\delta > 0$, $n \in N$.

## 4. Proofs of the results

PROOF OF THEOREM 2.1. It is clear that for an odd $r$

$$\begin{pmatrix} \widetilde{T}_{n,A}f(x) - \widetilde{f}(x) \\ \widetilde{T}_{n,A}f(x) - \widetilde{f}\big(x, \frac{\pi}{r(n+1)}\big) \end{pmatrix}$$

$$= \begin{pmatrix} + \\ - \end{pmatrix} \frac{1}{\pi} \int_{0}^{\frac{\pi}{r(n+1)}} \psi_x(t) \sum_{k=0}^{\infty} a_{n,k} \begin{pmatrix} \widetilde{D^{\circ}}_{k,1}(t) \\ \widetilde{D}_{k,1}(t) \end{pmatrix} dt + \frac{1}{\pi} \int_{\frac{\pi}{r(n+1)}}^{\frac{\pi}{r}} \psi_x(t) \sum_{k=0}^{\infty} a_{n,k} \widetilde{D^{\circ}}_{k,1}(t)dt$$

$$+ \frac{1}{\pi} \sum_{m=1}^{[r/2]} \int_{\frac{2m\pi}{r}}^{\frac{2m\pi}{r}+\frac{\pi}{r}} \psi_x(t) \sum_{k=0}^{\infty} a_{n,k} \widetilde{D^{\circ}}_{k,1}(t)dt + \frac{1}{\pi} \sum_{m=0}^{[r/2]-1} \int_{\frac{2m\pi}{r}+\frac{\pi}{r}}^{\frac{2(m+1)\pi}{r}} \psi_x(t) \sum_{k=0}^{\infty} a_{n,k} \widetilde{D^{\circ}}_{k,1}(t)dt$$

$$= \begin{pmatrix} J_1(x) \\ J_1'(x) \end{pmatrix} + J_2(x) + I_1''(x) + I_2(x)$$

and for an even $r$

$$\begin{pmatrix} \widetilde{T}_{n,A}f(x)-\widetilde{f}(x) \\ \widetilde{T}_{n,A}f(x)-\widetilde{f}\big(x,\frac{\pi}{r(n+1)}\big) \end{pmatrix}$$

$$= \begin{pmatrix} + \\ - \end{pmatrix} \frac{1}{\pi} \int\limits_{0}^{\frac{\pi}{r(n+1)}} \psi_x(t) \sum_{k=0}^{\infty} a_{n,k} \begin{pmatrix} \widetilde{D^{\circ}}_{k,1}(t) \\ \widetilde{D}_{k,1}(t) \end{pmatrix} dt + \frac{1}{\pi} \int\limits_{\frac{\pi}{r(n+1)}}^{\frac{\pi}{r}} \psi_x(t) \sum_{k=0}^{\infty} a_{n,k} \widetilde{D^{\circ}}_{k,1}(t) dt$$

$$+ \frac{1}{\pi} \sum_{m=1}^{[r/2]-1} \int\limits_{\frac{2m\pi}{r}}^{\frac{2m\pi}{r}+\frac{\pi}{r}} \psi_x(t) \sum_{k=0}^{\infty} a_{n,k} \widetilde{D^{\circ}}_{k,1}(t) dt + \frac{1}{\pi} \sum_{m=0}^{[r/2]-1} \int\limits_{\frac{2m\pi}{r}+\frac{\pi}{r}}^{\frac{2(m+1)\pi}{r}} \psi_x(t) \sum_{k=0}^{\infty} a_{n,k} \widetilde{D^{\circ}}_{k,1}(t) dt$$

$$= \begin{pmatrix} J_1(x) \\ J_1'(x) \end{pmatrix} + J_2(x) + I_1'''(x) + I_2(x).$$

Then

$$\begin{pmatrix} \big\|\widetilde{T}_{n,A}f(\bullet)-\widetilde{f}(\bullet)\big\|_X \\ \big\|\widetilde{T}_{n,A}f(\bullet)-\widetilde{f}\big(\bullet,\frac{\pi}{r(n+1)}\big)\big\|_X \end{pmatrix} \leqslant \begin{pmatrix} \|J_1+J_2+I_1''\|_X + \|J_1+J_2+I_1'''\|_X \\ \|J_1'\|_X + \|J_2+I_1''\|_X + \|J_2+I_1'''\|_X \end{pmatrix} + \|I_2\|_X.$$

By Lemma 3.1

$$\|J_1'\|_X \leqslant \frac{1}{\pi} \int_0^{\frac{\pi}{r(n+1)}} \|\psi_{\bullet}(t)\|_X \left| \sum_{k=0}^{\infty} a_{n,k} \widetilde{D}_{k,1}(t) \right| dt$$

$$\leqslant \frac{1}{2\pi} \int_0^{\frac{\pi}{r(n+1)}} \|\psi_{\bullet}(t)\|_X \sum_{k=0}^{\infty} a_{n,k} \frac{\pi}{t} dt \leqslant \frac{1}{2} \int_0^{\frac{\pi}{r(n+1)}} \frac{O(\omega(t))}{t} dt.$$

Since, by Lemma 3.2,

$$\sum_{k=0}^{\infty} a_{n,k} \widetilde{D^{\circ}}_{k,1}(t) = \sum_{k=0}^{\infty} a_{n,k} \frac{\cos\frac{(2k+1)t}{2}}{2\sin\frac{t}{2}}$$

$$= \frac{1}{2\sin\frac{t}{2}} \left( \sum_{k=0}^{\infty} a_{n,k} \cos kt \cos\frac{t}{2} - \sum_{k=0}^{\infty} a_{n,k} \sin kt \sin\frac{t}{2} \right)$$

$$= \frac{\cos\frac{t}{2}}{2\sin\frac{t}{2}} \left( \sum_{k=0}^{\infty} \Big( a_{n,k} - a_{n,k+r} \Big) D^{\circ}_{k,r}(t) + \sum_{k=0}^{r-1} a_{n,k} D^{\circ}_{k,-r}(t) \right)$$

$$- \frac{1}{2} \left( -\sum_{k=0}^{\infty} \Big( a_{n,k} - a_{n,k+r} \Big) \widetilde{D^{\circ}}_{k,r}(t) - \sum_{k=0}^{r-1} a_{n,k} \widetilde{D^{\circ}}_{k,-r}(t) \right),$$

whence

$$\left| \sum_{k=0}^{\infty} a_{n,k} \widetilde{D^{\circ}}_{k,1}(t) \right| \leqslant \frac{1}{2\left|\sin\frac{t}{2}\sin\frac{rt}{2}\right|} \left( A_{n,r} + \sum_{k=0}^{r-1} a_{n,k} \right) \leqslant \frac{1}{\left|\sin\frac{t}{2}\sin\frac{rt}{2}\right|} A_{n,r}.$$

Hence and by Lemma 3.1,

$$\begin{pmatrix} \|J_1+J_2+I_1''\|_X + \|J_1+J_2+I_1'''\|_X \\ \|J_2+I_1''\|_X + \|J_2+I_1'''\|_X \end{pmatrix}$$

$$\leqslant \begin{pmatrix} \frac{2}{\pi}\sum\limits_{m=0}^{[r/2]-\kappa}\left(\int\limits_{\frac{2m\pi}{r}}^{\frac{2m\pi}{r}+\frac{\pi}{r(n+1)}} + \int\limits_{\frac{2m\pi}{r}+\frac{\pi}{r(n+1)}}^{\frac{2m\pi}{r}+\frac{\pi}{r}}\right)\|\psi_\bullet(t)\|_X\left|\sum\limits_{k=0}^{\infty}a_{n,k}\widetilde{D^\circ}_{k,1}(t)\right|dt \\ \frac{2}{\pi}\left(\sum\limits_{m=1}^{[r/2]-\kappa}\int\limits_{\frac{2m\pi}{r}}^{\frac{2m\pi}{r}+\frac{\pi}{r(n+1)}} + \sum\limits_{m=0}^{[r/2]-\kappa}\int\limits_{\frac{2m\pi}{r}+\frac{\pi}{r(n+1)}}^{\frac{2m\pi}{r}+\frac{\pi}{r}}\right)\|\psi_\bullet(t)\|_X\left|\sum\limits_{k=0}^{\infty}a_{n,k}\widetilde{D^\circ}_{k,1}(t)\right|dt \end{pmatrix}$$

and therefore

$$\left.\begin{matrix}\|J_1+J_2+I_1''\|_X+\|J_1+J_2+I_1'''\|_X\\ \|J_1'\|_X+\|J_2+I_1''\|_X+\|J_2+I_1'''\|_X\end{matrix}\right\}$$

$$\leqslant \sum_{m=0}^{[r/2]-\kappa}\int\limits_{\frac{2m\pi}{r}}^{\frac{2m\pi}{r}+\frac{\pi}{r(n+1)}}\frac{O(\omega(t))}{t}dt+\frac{2}{\pi}\sum_{m=0}^{[r/2]-\kappa}\int\limits_{\frac{2m\pi}{r}+\frac{\pi}{r(n+1)}}^{\frac{2m\pi}{r}+\frac{\pi}{r}}\frac{O(\omega(t))}{\left|\sin\frac{t}{2}\sin\frac{rt}{2}\right|}A_{n,r}dt,$$

where $\kappa=1$ when $r$ is even, and $\kappa=0$ when $r$ is odd.

Using Lemmas 3.3, 3.4, with $c=b=r$, and the estimates $\left|\sin\frac{t}{2}\right|\geqslant\frac{|t|}{\pi}$, $\left|\sin\frac{rt}{2}\right|\geqslant\frac{rt}{\pi}-2m$ for $t\in\left[\frac{2m\pi}{r},\frac{2m\pi}{r}+\frac{\pi}{r}\right]\subset[0,\pi]$, where $m\in\left\{0,\dots,[r/2]-\kappa\right\}$, we obtain

$$\left.\begin{matrix}\|J_1+J_2+I_1''\|_X+\|J_1+J_2+I_1'''\|_X\\ \|J_1'\|_X+\|J_2+I_1''\|_X+\|J_2+I_1'''\|_X\end{matrix}\right\}$$

$$\leqslant O(1)([r/2]+1)\frac{\pi}{r(n+1)}H\Big(\frac{\pi}{n+1}\Big)+2A_{n,r}\sum_{m=0}^{[r/2]-\kappa}\int\limits_{\frac{2m\pi}{r}+\frac{\pi}{r(n+1)}}^{\frac{2m\pi}{r}+\frac{\pi}{r}}\frac{O(\omega(t))}{t\left(\frac{rt}{\pi}-2m\right)}dt$$

$$=O(1)\frac{\pi}{n+1}H\Big(\frac{\pi}{n+1}\Big)+2A_{n,r}\sum_{m=0}^{[r/2]-\kappa}\int\limits_{\frac{2m\pi}{r}+\frac{\pi}{r(n+1)}}^{\frac{2m\pi}{r}+\frac{\pi}{r}}\frac{O(\omega(t))}{\frac{rt}{\pi}\left(t-\frac{2m\pi}{r}\right)}dt$$

$$=O(1)\frac{\pi}{n+1}H\Big(\frac{\pi}{n+1}\Big)+2A_{n,r}\sum_{m=0}^{[r/2]-\kappa}\int\limits_{\frac{\pi}{r(n+1)}}^{\frac{\pi}{r}}\frac{O\left(\omega\left(t+\frac{2m\pi}{r}\right)\right)}{\frac{rt}{\pi}\left(t+\frac{2m\pi}{r}\right)}dt$$

$$\leqslant O(1)\left[\frac{\pi}{n+1}H\Big(\frac{\pi}{n+1}\Big)+([r/2]+1)\frac{4\pi}{r}A_{n,r}\int\limits_{\frac{\pi}{r(n+1)}}^{\frac{\pi}{r}}\frac{\omega(t)}{t^2}dt\right]$$

$$=O(1)\Big[\frac{\pi}{n+1}H\Big(\frac{\pi}{n+1}\Big)+A_{n,r}H\Big(\frac{\pi}{n+1}\Big)\Big].$$

Similarly, by Lemma 3.1, Lemmas 3.3, 3.4, with $c=b=r$ and the estimates $\left|\sin\frac{t}{2}\right|\geqslant\frac{|t|}{\pi}$, $\left|\sin\frac{rt}{2}\right|\geqslant 2(m+1)-\frac{rt}{\pi}$ for $t\in\left[\frac{2(m+1)\pi}{r}-\frac{\pi}{r},\frac{2(m+1)\pi}{r}-\frac{\pi}{r(n+1)}\right]\subset[0,\pi]$, where $m\in\{0,\dots,[r/2]-1\}$, we get

$$\|I_2\|_X\leqslant\frac{1}{\pi}\sum_{m=0}^{[r/2]-1}\int_{\frac{2m\pi}{r}+\frac{\pi}{r}}^{\frac{2(m+1)\pi}{r}}\|\psi_\bullet(t)\|_X\left|\sum_{k=0}^{\infty}a_{n,k}\widetilde{D^\circ}_{k,1}(t)\right|dt$$

$$= \frac{1}{\pi}\sum_{m=0}^{[r/2]-1}\left(\int_{\frac{2(m+1)\pi}{r}-\frac{\pi}{r}}^{\frac{2(m+1)\pi}{r}-\frac{\pi}{r(n+1)}}+\int_{\frac{2(m+1)\pi}{r}-\frac{\pi}{r(n+1)}}^{\frac{2(m+1)\pi}{r}}\right)\|\psi_{\bullet}(t)\|_X\left|\sum_{k=0}^{\infty}a_{n,k}\widetilde{D^{\circ}}_{k,1}(t)\right|dt$$

$$\leqslant \frac{1}{2}\sum_{m=0}^{[r/2]-1}\int\limits_{\frac{2(m+1)\pi}{r}-\frac{\pi}{r(n+1)}}^{\frac{2(m+1)\pi}{r}}\frac{O(\omega(t))}{t}dt+\frac{1}{\pi}A_{n,r}\sum_{m=0}^{[r/2]-1}\int\limits_{\frac{2(m+1)\pi}{r}-\frac{\pi}{r}}^{\frac{2(m+1)\pi}{r}-\frac{\pi}{r(n+1)}}\frac{O(\omega(t))}{\left|\sin\frac{t}{2}\sin\frac{rt}{2}\right|}dt$$

$$\leqslant \frac{1}{2}\sum_{m=0}^{[r/2]-1}\int\limits_{\frac{2(m+1)\pi}{r}-\frac{\pi}{r(n+1)}}^{\frac{2(m+1)\pi}{r}}\frac{O(\omega(t))}{t}dt+A_{n,r}\sum_{m=0}^{[r/2]-1}\int\limits_{\frac{2(m+1)\pi}{r}-\frac{\pi}{r}}^{\frac{2(m+1)\pi}{r}-\frac{\pi}{r(n+1)}}\frac{O(\omega(t))}{\frac{rt}{\pi}\left[\frac{2(m+1)\pi}{r}-t\right]}dt$$

$$= \frac{1}{2}\sum_{m=0}^{[r/2]-1}\int\limits_{\frac{2(m+1)\pi}{r}-\frac{\pi}{r(n+1)}}^{\frac{2(m+1)\pi}{r}}\frac{O(\omega(t))}{t}dt+A_{n,r}\sum_{m=0}^{[r/2]-1}\int\limits_{\frac{\pi}{r(n+1)}}^{\frac{\pi}{r}}\frac{O\left(\omega\left(-t+\frac{2(m+1)\pi}{r}\right)\right)}{\frac{r}{\pi}t\left(-t+\frac{2(m+1)\pi}{r}\right)}dt$$

$$\leqslant \frac{1}{2}\sum_{m=0}^{[r/2]-1}\int_{\frac{2(m+1)\pi}{r}-\frac{\pi}{r(n+1)}}^{\frac{2(m+1)\pi}{r}}\frac{O(\omega(t))}{t}dt+A_{n,r}\left[r/2\right]\frac{2\pi}{r}\int_{\frac{\pi}{r(n+1)}}^{\frac{\pi}{r}}\frac{O(\omega(t))}{t^2}dt.$$

Thus

$$\|I_2\|_X=O(1)\left[\frac{\pi}{n+1}H\left(\frac{\pi}{n+1}\right)+A_{n,r}H\left(\frac{\pi}{n+1}\right)\right].$$

Collecting these estimates we obtain the first result.

Applying condition (1.3) we have

$$\left[(n+1)\sum_{k=0}^{\infty}\|a_{n,k}-a_{n,k+r}\|\right]^{-1}=\left[\sum_{l=0}^{n}\sum_{k=0}^{\infty}\|a_{n,k}-a_{n,k+r}\|\right]^{-1}$$
$$\leqslant\left[\sum_{l=0}^{n}\sum_{k=l}^{\infty}\|a_{n,k}-a_{n,k+r}\|\right]^{-1}\leqslant\left[\sum_{l=0}^{n}\left|\sum_{k=l}^{\infty}(a_{n,k}-a_{n,k+r})\right|\right]^{-1}$$
$$=\left[\sum_{l=0}^{n}\sum_{k=l}^{r+l-1}a_{n,k}\right]^{-1}=O_r(1)$$

and the second result also follows. □

Proof of Theorem 2.2. Analogously, as in the proof of Theorem 2.1, we consider an odd $r$ and an even $r$. Then,

$$\begin{pmatrix}\widetilde{T}_{n,A}f(x)-\widetilde{f}(x)\\ \widetilde{T}_{n,A}f(x)-\widetilde{f}\left(x,\frac{1}{r}A_{n,r}\right)\end{pmatrix}$$
$$=\binom{+}{-}\frac{1}{\pi}\int\limits_{0}^{\frac{1}{r}A_{n,r}}\psi_x(t)\sum_{k=0}^{\infty}a_{n,k}\begin{pmatrix}\widetilde{D^{\circ}}_{k,1}(t)\\ \widetilde{D}_{k,1}(t)\end{pmatrix}dt+\frac{1}{\pi}\int\limits_{\frac{1}{r}A_{n,r}}^{\frac{\pi}{r}}\psi_x(t)\sum_{k=0}^{\infty}a_{n,k}\widetilde{D^{\circ}}_{k,1}(t)dt$$

$$+\frac{1}{\pi}\sum_{m=1}^{[r/2]}\int_{\frac{2m\pi}{r}}^{\frac{2m\pi}{r}+\frac{\pi}{r}}\psi_x(t)\sum_{k=0}^{\infty}a_{n,k}\widetilde{D^{\circ}}_{k,1}(t)dt+\frac{1}{\pi}\sum_{m=0}^{[r/2]-1}\int_{\frac{2m\pi}{r}+\frac{\pi}{r}}^{\frac{2(m+1)\pi}{r}}\psi_x(t)\sum_{k=0}^{\infty}a_{n,k}\widetilde{D^{\circ}}_{k,1}(t)dt$$

or

$$\begin{pmatrix}\widetilde{T}_{n,A}f(x)-\widetilde{f}(x)\\ \widetilde{T}_{n,A}f(x)-\widetilde{f}\left(x,\frac{1}{r}A_{n,r}\right)\end{pmatrix}$$

$$=\begin{pmatrix}+\\-\end{pmatrix}\frac{1}{\pi}\int_{0}^{\frac{1}{r}A_{n,r}}\psi_x(t)\sum_{k=0}^{\infty}a_{n,k}\begin{pmatrix}\widetilde{D^{\circ}}_{k,1}(t)\\ \widetilde{D}_{k,1}(t)\end{pmatrix}dt+\frac{1}{\pi}\int_{\frac{1}{r}A_{n,r}}^{\frac{\pi}{r}}\psi_x(t)\sum_{k=0}^{\infty}a_{n,k}\widetilde{D^{\circ}}_{k,1}(t)dt$$

$$+\frac{1}{\pi}\sum_{m=1}^{[r/2]-1}\int_{\frac{2m\pi}{r}}^{\frac{2m\pi}{r}+\frac{\pi}{r}}\psi_x(t)\sum_{k=0}^{\infty}a_{n,k}\widetilde{D^{\circ}}_{k,1}(t)dt+\frac{1}{\pi}\sum_{m=0}^{[r/2]-1}\int_{\frac{2m\pi}{r}+\frac{\pi}{r}}^{\frac{2(m+1)\pi}{r}}\psi_x(t)\sum_{k=0}^{\infty}a_{n,k}\widetilde{D^{\circ}}_{k,1}(t)dt,$$

respectively. Since $A_{n,r}\leqslant 2$, we can estimate our terms analogously as in the proof of Theorem 2.1 with $A_{n,r}$ instead of $\frac{\pi}{n+1}$ and thus we obtain the desired estimate. □

Proof of Theorem 2.3. Similarly, as in the proof of Theorem 2.1

$$\begin{pmatrix}\left\|\widetilde{T}_{n,A}f(\bullet)-\widetilde{f}(\bullet)\right\|_X\\ \left\|\widetilde{T}_{n,A}f(\bullet)-\widetilde{f}\left(\bullet,\frac{\pi}{r(n+1)}\right)\right\|_X\end{pmatrix}\leqslant\begin{pmatrix}\|J_1\|_X\\ \|J_1'\|_X\end{pmatrix}+\|J_2+I_1''\|_X+\|J_2+I_1'''\|_X+\|I_2\|_X.$$

By Lemma 3.1 and (1.4)

$$\|J_1'\|_X\leqslant\frac{1}{\pi}\int_{0}^{\frac{\pi}{r(n+1)}}\|\psi_\bullet(t)\|_X\left|\sum_{k=0}^{\infty}a_{n,k}\widetilde{D}_{k,1}(t)\right|dt\leqslant\frac{1}{\pi}\sum_{k=0}^{\infty}(k+1)a_{n,k}\int_{0}^{\frac{\pi}{r(n+1)}}\omega(t)dt$$

$$=O(n+1)\int_0^{\frac{\pi}{r(n+1)}}\omega(t)dt=O(1)\omega\Big(\frac{\pi}{r(n+1)}\Big)=O\Big(\omega\Big(\frac{\pi}{n+1}\Big)\Big)$$

and by Lemma 3.1 and (2.1)

$$\|J_1\|_X\leqslant\frac{1}{\pi}\int_{0}^{\frac{\pi}{r(n+1)}}\|\psi_\bullet(t)\|_X\left|\sum_{k=0}^{\infty}a_{n,k}\widetilde{D^{\circ}}_{k,1}(t)\right|dt\leqslant\frac{1}{2\pi}\int_{0}^{\frac{\pi}{r(n+1)}}\|\psi_\bullet(t)\|_X\sum_{k=0}^{\infty}a_{n,k}\frac{\pi}{t}dt$$

$$\leqslant\frac{1}{2}\int_0^{\frac{\pi}{r(n+1)}}\frac{\omega(t)}{t}dt=O\Big(\omega\Big(\frac{\pi}{r(n+1)}\Big)\Big)=O\Big(\omega\Big(\frac{\pi}{n+1}\Big)\Big).$$

Further, by the same lemmas and conditions as in the above proofs and Lemma 3.5, we obtain with $\kappa=1$ when $r$ is even, and $\kappa=0$ when $r$ is odd, that

$$\|J_2+I_1''\|_X+\|J_2+I_1'''\|_X$$

$$\begin{aligned}
&\leqslant \frac{2}{\pi}\left(\sum_{m=1}^{[r/2]-\kappa}\int_{\frac{2m\pi}{r}}^{\frac{2m\pi}{r}+\frac{\pi}{r}}+\int_{\frac{\pi}{r(n+1)}}^{\frac{\pi}{r}}\right)\|\psi_{\bullet}(t)\|_X\left|\sum_{k=0}^{\infty}a_{n,k}\widetilde{D^{\circ}}_{k,1}(t)\right|dt\\
&=\frac{2}{\pi}\left(\sum_{m=1}^{[r/2]-\kappa}\int_{\frac{2m\pi}{r}}^{\frac{2m\pi}{r}+\frac{\pi}{r(n+1)}}+\sum_{m=0}^{[r/2]-\kappa}\int_{\frac{2m\pi}{r}+\frac{\pi}{r(n+1)}}^{\frac{2m\pi}{r}+\frac{\pi}{r}}\right)\|\psi_{\bullet}(t)\|_X\left|\sum_{k=0}^{\infty}a_{n,k}\widetilde{D^{\circ}}_{k,1}(t)\right|dt\\
&\leqslant \frac{2}{\pi}\sum_{m=1}^{[r/2]-\kappa}\int_{\frac{2m\pi}{r}}^{\frac{2m\pi}{r}+\frac{\pi}{r(n+1)}}\frac{O(\omega(t))}{2\left|\sin\frac{t}{2}\right|}dt+\frac{2}{\pi}\sum_{m=0}^{[r/2]-\kappa}\int_{\frac{2m\pi}{r}+\frac{\pi}{r(n+1)}}^{\frac{2m\pi}{r}+\frac{\pi}{r}}\frac{O(\omega(t))}{\left|\sin\frac{t}{2}\sin\frac{rt}{2}\right|}A_{n,r}dt\\
&\leqslant \sum_{m=1}^{[r/2]-\kappa}\int_{\frac{2m\pi}{r}}^{\frac{2m\pi}{r}+\frac{\pi}{r(n+1)}}\frac{O(\omega(t))}{t}dt+2A_{n,r}\sum_{m=0}^{[r/2]-\kappa}\int_{\frac{2m\pi}{r}+\frac{\pi}{r(n+1)}}^{\frac{2m\pi}{r}+\frac{\pi}{r}}\frac{O(\omega(t))}{t\left(\frac{rt}{\pi}-2m\right)}dt\\
&\leqslant \sum_{m=1}^{[r/2]-\kappa}\frac{O\left(\omega\left(\frac{2m\pi}{r}\right)\right)}{\frac{2m\pi}{r}}\int_{\frac{2m\pi}{r}}^{\frac{2m\pi}{r}+\frac{\pi}{r(n+1)}}dt+2A_{n,r}\sum_{m=0}^{[r/2]-\kappa}\int_{\frac{\pi}{r(n+1)}}^{\frac{\pi}{r}}\frac{O\left(\omega\left(t+\frac{2m\pi}{r}\right)\right)}{\frac{rt}{\pi}\left(t+\frac{2m\pi}{r}\right)}dt\\
&\leqslant 4\sum_{m=1}^{[r/2]-\kappa}\frac{O\left(\omega\left(\frac{2\pi}{r}\right)\right)}{\frac{2\pi}{r}}\frac{\pi}{r(n+1)}+\frac{4\pi}{r}\left([r/2]+1\right)A_{n,r}\int_{\frac{\pi}{r(n+1)}}^{\frac{\pi}{r}}\frac{O(\omega(t))}{t^2}dt\\
&\leqslant 4\sum_{m=1}^{[r/2]-\kappa}\frac{O\big(\omega\big(2(n+1)\frac{\pi}{r(n+1)}\big)\big)}{\frac{2\pi}{r}}\frac{\pi}{r(n+1)}+O_r(1)A_{n,r}H\Big(\frac{\pi}{n+1}\Big)\\
&\leqslant 4\sum_{m=1}^{[r/2]-\kappa}\frac{O\big(2(n+1)\omega\big(\frac{\pi}{r(n+1)}\big)\big)}{\frac{2\pi}{r}}\frac{\pi}{r(n+1)}+O_r(1)A_{n,r}H\Big(\frac{\pi}{n+1}\Big)\\
&=O_r(1)\Big[\omega\Big(\frac{\pi}{n+1}\Big)+A_{n,r}H\Big(\frac{\pi}{n+1}\Big)\Big]
\end{aligned}$$

and

$$\begin{aligned}
\|I_2\|_X &\leqslant \frac{1}{\pi}\sum_{m=0}^{[r/2]-1}\left(\int_{\frac{2(m+1)\pi}{r}-\frac{\pi}{r}}^{\frac{2(m+1)\pi}{r}-\frac{\pi}{r(n+1)}}+\int_{\frac{2(m+1)\pi}{r}-\frac{\pi}{r(n+1)}}^{\frac{2(m+1)\pi}{r}}\right)\|\psi_{\bullet}(t)\|_X\left|\sum_{k=0}^{\infty}a_{n,k}\widetilde{D^{\circ}}_{k,1}(t)\right|dt\\
&\leqslant \frac{1}{\pi}\sum_{m=0}^{[r/2]-1}\int_{\frac{2(m+1)\pi}{r}-\frac{\pi}{r}}^{\frac{2(m+1)\pi}{r}-\frac{\pi}{r(n+1)}}\frac{O(\omega(t))}{\left|\sin\frac{t}{2}\sin\frac{rt}{2}\right|}A_{n,r}dt+\frac{1}{2}\sum_{m=0}^{[r/2]-1}\int_{\frac{2(m+1)\pi}{r}-\frac{\pi}{r(n+1)}}^{\frac{2(m+1)\pi}{r}}\frac{O(\omega(t))}{t}dt
\end{aligned}$$

$$\leqslant \sum_{m=0}^{[r/2]-1} A_{n,r} \int_{\frac{2(m+1)\pi}{r}-\frac{\pi}{r}}^{\frac{2(m+1)\pi}{r}-\frac{\pi}{r(n+1)}} \frac{O(\omega(t))}{\frac{rt}{\pi}\left[\frac{2(m+1)\pi}{r}-t\right]}dt + \sum_{m=0}^{[r/2]-1} \frac{O\left(\omega\left(\frac{2(m+1)\pi}{r}-\frac{\pi}{r(n+1)}\right)\right)}{r(n+1)\left(\frac{2(m+1)\pi}{r}-\frac{\pi}{r(n+1)}\right)}$$

$$\leqslant [r/2]\left[\frac{2\pi}{r}A_{n,r}\int_{\frac{\pi}{r(n+1)}}^{\frac{\pi}{r}} \frac{O(\omega(t))}{t^2}dt + 2\frac{O\left(\omega\left(\frac{\pi}{r}\right)\right)}{\frac{\pi}{r}}\frac{\pi}{r(n+1)}\right]$$

$$= O_r(1)\Big[A_{n,r}H\Big(\frac{\pi}{n+1}\Big)+\omega\Big(\frac{\pi}{n+1}\Big)\Big].$$

Thus our proof is complete. □

PROOF OF THEOREM 2.4. Let as above

$$\begin{pmatrix} \big\|\widetilde{T}_{n,A}f(\bullet)-\widetilde{f}(\bullet)\big\|_X \\ \big\|\widetilde{T}_{n,A}f(\bullet)-\widetilde{f}\big(\bullet,\frac{\pi}{r(n+1)}\big)\big\|_X \end{pmatrix} \leqslant \begin{pmatrix} \|J_1\|_X \\ \|J_1'\|_X \end{pmatrix} + \|J_2+I_1''\|_X + \|J_2+I_1'''\|_X + \|I_2\|_X,$$

$$\|J_1\|_X \leqslant \frac{1}{\pi}\int_0^{\frac{\pi}{r(n+1)}} \|\psi_\bullet(t)\|_X \Big|\sum_{k=0}^{\infty} a_{n,k}\widetilde{D^\circ}_{k,1}(t)\Big|dt \leqslant \frac{1}{2}\sum_{k=0}^{\infty} a_{n,k}\int_0^{\frac{\pi}{r(n+1)}} \frac{\widetilde{\omega}(f,t)_X}{t}dt$$
$$= O(1)\widetilde{\omega}\Big(f,\frac{\pi}{r(n+1)}\Big)_X = O\Big(\widetilde{\omega}\Big(f,\frac{\pi}{n+1}\Big)_X\Big), \text{ by (2.1)},$$

and

$$\|J_1'\|_X \leqslant \frac{1}{\pi}\int_0^{\frac{\pi}{r(n+1)}} \|\psi_\bullet(t)\|_X \Big|\sum_{k=0}^{\infty} a_{n,k}\widetilde{D}_{k,1}(t)\Big|dt \leqslant \frac{1}{2\pi}\int_0^{\frac{\pi}{r(n+1)}} \|\psi_\bullet(t)\|_X \sum_{k=0}^{\infty} a_{n,k}(k+1)dt$$
$$\leqslant O(n+1)\int_0^{\frac{\pi}{r(n+1)}} \widetilde{\omega}(f,t)_X dt = O\Big(\widetilde{\omega}\Big(f,\frac{\pi}{n+1}\Big)_X\Big), \text{ by (1.4)}.$$

Further, taking $\tau_m^1 = \left[\frac{\pi}{rt-2m\pi}\right]$ and $\tau = \left[\frac{\pi}{rt}\right]$, using Lemma 3.5, and with $\kappa = 1$ when $r$ is even, and $\kappa = 0$ when $r$ is odd, we obtain

$$\|J_2+I_1''\|_X + \|J_2+I_1'''\|_X$$
$$\leqslant \frac{2}{\pi}\left(\sum_{m=1}^{[r/2]-\kappa}\int_{\frac{2m\pi}{r}}^{\frac{2m\pi}{r}+\frac{\pi}{r}} + \int_{\frac{\pi}{r(n+1)}}^{\frac{\pi}{r}}\right)\|\psi_\bullet(t)\|_X\Big|\sum_{k=0}^{\infty} a_{n,k}\widetilde{D^\circ}_{k,1}(t)\Big|dt$$
$$= \frac{2}{\pi}\left(\sum_{m=1}^{[r/2]-\kappa}\int_{\frac{2m\pi}{r}}^{\frac{2m\pi}{r}+\frac{\pi}{r(n+1)}} + \sum_{m=0}^{[r/2]-\kappa}\int_{\frac{2m\pi}{r}+\frac{\pi}{r(n+1)}}^{\frac{2m\pi}{r}+\frac{\pi}{r}}\right)\|\psi_\bullet(t)\|_X\Big|\sum_{k=0}^{\infty} a_{n,k}\widetilde{D^\circ}_{k,1}(t)\Big|dt$$

$$\leqslant \frac{2}{\pi}\sum_{m=1}^{[r/2]-\kappa}\int\limits_{\frac{2m\pi}{r}}^{\frac{2m\pi}{r}+\frac{\pi}{r(n+1)}}\frac{\widetilde{\omega}(f,t)_X}{2|\sin\frac{t}{2}|}\sum_{k=0}^{\infty}a_{n,k}dt$$

$$+\frac{2}{\pi}\sum_{m=0}^{[r/2]-\kappa}\int\limits_{\frac{2m\pi}{r}+\frac{\pi}{r(n+1)}}^{\frac{2m\pi}{r}+\frac{\pi}{r}}\bigg(\frac{\widetilde{\omega}(f,t)_X}{2|\sin\frac{t}{2}|}\sum_{k=0}^{\tau_m^1}a_{n,k}+\frac{\widetilde{\omega}(f,t)_X}{|\sin\frac{t}{2}\sin\frac{rt}{2}|}\sum_{k=\tau_m^1}^{\infty}|a_{n,k}-a_{n,k+r}|\bigg)dt$$

$$\leqslant \sum_{m=1}^{[r/2]-\kappa}\int\limits_{\frac{2m\pi}{r}}^{\frac{2m\pi}{r}+\frac{\pi}{r(n+1)}}\frac{\widetilde{\omega}(f,t)_X}{t}dt+2([r/2]+1)\int\limits_{\frac{\pi}{r(n+1)}}^{\frac{\pi}{r}}\frac{\widetilde{\omega}(f,t)_X}{t}\sum_{k=0}^{\tau}a_{n,k}dt$$

$$+\frac{4\pi}{r}([r/2]+1)\int\limits_{\frac{\pi}{r(n+1)}}^{\frac{\pi}{r}}\frac{\widetilde{\omega}(f,t)_X}{t^2}\sum_{k=\tau}^{\infty}|a_{n,k}-a_{n,k+r}|dt$$

$$\leqslant 2\sum_{m=1}^{[r/2]-\kappa}\frac{\widetilde{\omega}\big(f,\frac{2m\pi}{r}\big)_X}{\frac{2m\pi}{r}}\int\limits_{\frac{2m\pi}{r}}^{\frac{2m\pi}{r}+\frac{\pi}{r(n+1)}}dt+2([r/2]+1)\sum_{\mu=1}^{n}\int\limits_{\mu}^{\mu+1}\frac{\widetilde{\omega}\big(f,\frac{\pi}{rt}\big)_X}{\frac{\pi}{rt}}\sum_{k=0}^{[t]}a_{n,k}\frac{\pi dt}{rt^2}$$

$$+\frac{4\pi}{r}([r/2]+1)\sum_{\mu=1}^{n}\int\limits_{\mu}^{\mu+1}\frac{\widetilde{\omega}\big(f,\frac{\pi}{rt}\big)_X}{\big(\frac{\pi}{rt}\big)^2}\sum_{k=[t]}^{\infty}|a_{n,k}-a_{n,k+r}|\frac{\pi dt}{rt^2}$$

$$\leqslant O_r(1)\widetilde{\omega}\Big(f,\frac{\pi}{n+1}\Big)_X+O_r(1)\sum_{\mu=1}^{n}\frac{\widetilde{\omega}\Big(f,\frac{\pi}{\mu}\Big)_X}{\mu}\sum_{k=0}^{\mu+1}a_{n,k}$$

$$+O_r(1)\sum_{\mu=1}^{n}\widetilde{\omega}\Big(f,\frac{\pi}{\mu}\Big)_X\sum_{k=\mu}^{\infty}|a_{n,k}-a_{n,k+r}|.$$

Next, taking $\tau_m^2=\big[\frac{\pi}{-rt+2(m+1)\pi}\big]$, we obtain

$$\|I_2\|_X\leqslant\frac{1}{\pi}\sum_{m=0}^{[r/2]-1}\int\limits_{\frac{2m\pi}{r}+\frac{\pi}{r}}^{\frac{2(m+1)\pi}{r}}\|\psi_\bullet(t)|_X\Big|\sum_{k=0}^{\infty}a_{n,k}\widetilde{D^\circ}_{k,1}(t)\Big|dt$$

$$\leqslant\frac{1}{\pi}\sum_{m=0}^{[r/2]-1}\Bigg(\int\limits_{\frac{2(m+1)\pi}{r}-\frac{\pi}{r}}^{\frac{2(m+1)\pi}{r}-\frac{\pi}{r(n+1)}}+\int\limits_{\frac{2(m+1)\pi}{r}-\frac{\pi}{r(n+1)}}^{\frac{2(m+1)\pi}{r}}\Bigg)\|\psi_\bullet(t)\|_X\Big|\sum_{k=0}^{\infty}a_{n,k}\widetilde{D^\circ}_{k,1}(t)\Big|dt$$

$$\leqslant\frac{1}{\pi}\sum_{m=0}^{[r/2]-1}\int\limits_{\frac{2(m+1)\pi}{r}-\frac{\pi}{r}}^{\frac{2(m+1)\pi}{r}-\frac{\pi}{r(n+1)}}\bigg(\frac{\widetilde{\omega}(f,t)_X}{2|\sin\frac{t}{2}|}\sum_{k=0}^{\tau_m^2}a_{n,k}+\frac{\widetilde{\omega}(f,t)_X}{|\sin\frac{t}{2}\sin\frac{rt}{2}|}\sum_{k=\tau_m^2}^{\infty}|a_{n,k}-a_{n,k+r}|dt\bigg)$$

$$+\frac{1}{\pi}\sum_{m=0}^{[r/2]-1}\int_{\frac{2(m+1)\pi}{r}-\frac{\pi}{r(n+1)}}^{\frac{2(m+1)\pi}{r}}\frac{\widetilde{\omega}(f,t)_X}{2|\sin\frac{t}{2}|}\sum_{k=0}^{\infty}a_{n,k}dt$$

$$\leqslant\frac{1}{2}\sum_{m=0}^{[r/2]-1}\int_{\frac{\pi}{r(n+1)}}^{\frac{\pi}{r}}\frac{\widetilde{\omega}\big(f,-t+\frac{2(m+1)\pi}{r}\big)_X}{-t+\frac{2(m+1)\pi}{r}}\sum_{k=0}^{\tau}a_{n,k}dt$$

$$+\sum_{m=0}^{[r/2]-1}\int_{\frac{\pi}{r(n+1)}}^{\frac{\pi}{r}}\frac{\widetilde{\omega}\big(f,-t+\frac{2(m+1)\pi}{r}\big)_X}{\frac{r}{\pi}t\big(-t+\frac{2(m+1)\pi}{r}\big)}\sum_{k=\tau}^{\infty}|a_{n,k}-a_{n,k+r}|dt$$

$$+\frac{1}{2}\sum_{m=0}^{[r/2]-1}\int_{\frac{2(m+1)\pi}{r}-\frac{\pi}{r(n+1)}}^{\frac{2(m+1)\pi}{r}}\frac{\widetilde{\omega}(f,t)_X}{t}dt$$

$$\leqslant[r/2]\int_{\frac{\pi}{r(n+1)}}^{\frac{\pi}{r}}\frac{\widetilde{\omega}(f,t)_X}{t}\sum_{k=0}^{\tau}a_{n,k}+\frac{2\pi}{r}[r/2]\int_{\frac{\pi}{r(n+1)}}^{\frac{\pi}{r}}\frac{\widetilde{\omega}(f,t)_X}{t^2}\sum_{k=\tau}^{\infty}|a_{n,k}-a_{n,k+r}|dt$$

$$+\sum_{m=0}^{[r/2]-1}\frac{\widetilde{\omega}\big(f,\frac{2(m+1)\pi}{r}-\frac{\pi}{r(n+1)}\big)_X}{\frac{2(m+1)\pi}{r}-\frac{\pi}{r(n+1)}}\int_{\frac{2(m+1)\pi}{r}-\frac{\pi}{r(n+1)}}^{\frac{2(m+1)\pi}{r}}dt$$

$$\leqslant O_r(1)\sum_{\mu=1}^{n}\frac{\widetilde{\omega}\big(f,\frac{\pi}{\mu}\big)_X}{\mu}\sum_{k=0}^{\mu+1}a_{n,k}+O_r(1)\sum_{\mu=1}^{n}\widetilde{\omega}\Big(f,\frac{\pi}{\mu}\Big)_X\sum_{k=\mu}^{\infty}|a_{n,k}-a_{n,k+r}|$$

$$+O_r(1)\widetilde{\omega}\Big(f,\frac{\pi}{n+1}\Big)_X.$$

Thus the result follows. □

Proof of Corollary 2.1. Theorem 2.3 implies that

$$\left.\begin{matrix}\big\|\widetilde{T}_{n,A}f(\bullet)-\widetilde{f}(\bullet)\big\|_X\\ \big\|\widetilde{T}_{n,A}f(\bullet)-\widetilde{f}\big(\bullet,\frac{\pi}{r(n+1)}\big)\big\|_X\end{matrix}\right\}=O_r\Bigg(\widetilde{\omega}\Big(f,\frac{\pi}{n+1}\Big)_X+\sum_{\mu=1}^{n}\frac{\widetilde{\omega}\big(f,\frac{\pi}{\mu}\big)_X}{\mu}\sum_{k=0}^{\mu+1}a_{n,k}$$

$$+\sum_{\mu=1}^{n}\widetilde{\omega}\Big(f,\frac{\pi}{\mu}\Big)_X\sum_{k=\mu}^{\infty}|a_{n,k}-a_{n,k+r}|\Bigg).$$

Since (2.2)

$$\sum_{\mu=1}^{n}\widetilde{\omega}\Big(f,\frac{\pi}{\mu}\Big)_X\sum_{k=\mu}^{\infty}|a_{n,k}-a_{n,k+r}|=O_r(1)\sum_{\mu=1}^{n}\widetilde{\omega}\Big(f,\frac{\pi}{\mu}\Big)_X\Bigg(\sum_{k=[\mu/c]}^{\mu}+\sum_{k=\mu}^{n}+\sum_{k=n+1}^{\infty}\Bigg)\frac{a_{n,k}}{k+1}$$

$$\leqslant O_r(1)\sum_{\mu=1}^{n}\widetilde{\omega}\Big(f,\frac{\pi}{\mu}\Big)_X\Big(\sum_{k=[\mu/c]}^{\mu}\frac{a_{n,k}}{k+1}\Big)+O_r(1)\sum_{\mu=1}^{n}\widetilde{\omega}\Big(f,\frac{\pi}{\mu}\Big)_X\Big(\sum_{k=\mu}^{n}+\sum_{k=n+1}^{\infty}\Big)\frac{a_{n,k}}{k+1}$$

$$\leqslant O_r(1)c\sum_{\mu=1}^{n}\frac{\widetilde{\omega}\big(f,\frac{\pi}{\mu}\big)_X}{\mu}\sum_{k=0}^{\mu+1}a_{n,k}+O_r(1)\sum_{\mu=1}^{n}\widetilde{\omega}\Big(f,\frac{\pi}{\mu}\Big)_X\Big(\sum_{k=\mu}^{n}+\sum_{k=n+1}^{\infty}\Big)\frac{a_{n,k}}{k+1}$$

one has

$$\left.\begin{array}{c}\big\|\widetilde{T}_{n,A}f(\bullet)-\widetilde{f}(\bullet)\big\|_X\\ \big\|\widetilde{T}_{n,A}f(\bullet)-\widetilde{f}\big(\bullet,\frac{\pi}{r(n+1)}\big)\big\|_X\end{array}\right\}$$

$$=O_r(1)\widetilde{\omega}\Big(f,\frac{\pi}{n+1}\Big)_X+O_r(1)(1+c)\sum_{\mu=1}^{n}\frac{\widetilde{\omega}\big(f,\frac{\pi}{\mu}\big)_X}{\mu}\sum_{k=0}^{\mu+1}a_{n,k}$$

$$+O_r(1)\sum_{\mu=1}^{n}\widetilde{\omega}\Big(f,\frac{\pi}{\mu}\Big)_X\sum_{k=\mu}^{n}\frac{a_{n,k}}{k+1}+O_r(1)\sum_{k=n+1}^{\infty}\frac{a_{n,k}}{k+1}\sum_{\mu=1}^{n}\widetilde{\omega}\Big(f,\frac{\pi}{\mu}\Big)_X$$

$$\leqslant O_r(1)\widetilde{\omega}\Big(f,\frac{\pi}{n+1}\Big)_X+O_r(1)(1+c)\Big\{\sum_{\mu=1}^{n}\frac{\widetilde{\omega}\big(f,\frac{\pi}{\mu}\big)_X}{\mu}\sum_{k=0}^{\mu+1}a_{n,k}$$

$$+2\sum_{\mu=1}^{n}\frac{\widetilde{\omega}\big(f,\frac{\pi}{\mu}\big)_X}{\mu+1}a_{n,\mu+1}+\sum_{\mu=1}^{n}\frac{\widetilde{\omega}\big(f,\frac{\pi}{\mu}\big)_X}{\mu}a_{n,\mu}\Big\}$$

$$+O_r(1)\sum_{\mu=1}^{n}\widetilde{\omega}\Big(f,\frac{\pi}{\mu}\Big)_X\sum_{k=\mu}^{n}\frac{a_{n,k}}{k+1}+O_r(1)\sum_{k=n+1}^{\infty}\frac{a_{n,k}}{k+1}\sum_{\mu=1}^{n}\widetilde{\omega}\Big(f,\frac{\pi}{\mu}\Big)_X$$

$$\leqslant O_r(1)\widetilde{\omega}\Big(f,\frac{\pi}{n+1}\Big)_X+O_r(1)(1+c)\sum_{\mu=0}^{n}\frac{\widetilde{\omega}\big(f,\frac{\pi}{\mu+1}\big)_X}{\mu+1}\sum_{k=0}^{\mu}a_{n,k}$$

$$+O_r(1)[3(1+c)+1]\sum_{\mu=1}^{n}\widetilde{\omega}\Big(f,\frac{\pi}{\mu}\Big)_X\sum_{k=\mu}^{n}\frac{a_{n,k}}{k}$$

$$+O_r(1)\sum_{k=n+1}^{\infty}\frac{a_{n,k}}{k}\sum_{\mu=1}^{n}\widetilde{\omega}\Big(f,\frac{\pi}{\mu}\Big)_X$$

$$=O_r(1)\widetilde{\omega}\Big(f,\frac{\pi}{n+1}\Big)_X+O_r(1)\sum_{k=0}^{n}a_{n,k}\sum_{\mu=k}^{n}\frac{\widetilde{\omega}\big(f,\frac{\pi}{\mu+1}\big)_X}{\mu+1}$$

$$+O_r(1)\sum_{k=1}^{n}\frac{a_{n,k}}{k+1}\sum_{\mu=1}^{k}\widetilde{\omega}\Big(f,\frac{\pi}{\mu}\Big)_X+O_r(1)\sum_{k=n+1}^{\infty}\frac{a_{n,k}}{k+1}\sum_{\mu=1}^{n}\widetilde{\omega}\Big(f,\frac{\pi}{\mu}\Big)_X.$$

If (1.1) and (1.2) hold, then

$$\widetilde{\omega}\Big(f,\frac{\pi}{n+1}\Big)_X\leqslant\frac{1}{n+1}\sum_{\mu=0}^{n}\widetilde{\omega}\Big(f,\frac{\pi}{\mu+1}\Big)_X=O(1)\frac{H\big(\frac{\pi}{n+1}\big)}{n+1},$$

$$\sum_{\mu=k}^{n} \frac{\widetilde{\omega}\left(f, \frac{\pi}{\mu+1}\right)_X}{\mu+1} = O(1) \int_{\frac{\pi}{n+2}}^{\frac{\pi}{k+1}} \frac{\widetilde{\omega}(f,t)_X}{t} dt = O(1) \frac{H\left(\frac{\pi}{k+1}\right)}{k+1}$$

and therefore

$$\left.\begin{array}{c} \left\| \widetilde{T}_{n,A} f(\bullet) - \widetilde{f}(\bullet) \right\|_X \\ \left\| \widetilde{T}_{n,A} f(\bullet) - \widetilde{f}\left(\bullet, \frac{\pi}{r(n+1)}\right) \right\|_X \end{array}\right\}$$

$$= O_r\left(\frac{H\left(\frac{\pi}{n+1}\right)}{n+1}\right) + O_r\left(\sum_{k=0}^{n} a_{n,k} \frac{H\left(\frac{\pi}{k+1}\right)}{k+1}\right) + O_r\left(H\left(\frac{\pi}{n+1}\right) \sum_{k=n+1}^{\infty} \frac{a_{n,k}}{k+1}\right).$$

Since

$$\sum_{k=n+1}^{\infty} \frac{a_{n,k}}{k+1} \leqslant \frac{1}{n+1} \sum_{k=n+1}^{\infty} a_{n,k} \leqslant \frac{1}{n+1}$$

the result follows. □

**Acknowledgement.** We are very grateful for essential and constructive comments and suggestions of the anonymous referee that improved the paper.

## References


[1] Xh. Z. Krasniqi, *Some further results on the degree of approximation of continuous functions*, Annales Univ. Sci. Budapest., Sect. Comp., **38** (2012), 279–294.

[2] W. Łenski, B. Szal, *On estimates of deviation of functions from matrix operators of their Fourier series by some expressions with r-differences of the entries*, Abstr. Appl. Anal. 2016, Article ID 9712878, 10 pages, doi:10.1155/2016/9712878.

[3] ———, *A new class of numerical sequences and its applications to uniform convergence of sine series*, Math. Nachr. **284**(14–15) (2011), 1985–2002.

[4] B. Szal, *On L-convergence of trigonometric series*, J. Math. Anal. Appl. **373** (2011), 449–463.

[5] B. Wei, D. Yu, *On the degree of approximation of continuous functions by means of Fourier series*, Math. Commun. **17** (2012), 211–219.

[6] A. Zygmund, *Trigonometric series*, Cambridge, 2002.


University of Zielona Góra
Faculty of Mathematics, Computer Science and Econometrics
Zielona Góra, Poland
W.Lenski@wmie.uz.zgora.pl
B.Szal@wmie.uz.zgora.pl

(Received 08 04 2017)
(Revised 15 06 2020)