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THE SPACE OF OPERATOR VALUED FUNCTIONS SEEN AS HILBERT H^*-MODULE

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Abstract. Let M be a space of weakly *-measurable functions $\mathcal{F}\colon \Omega \to B(H)$ on measure space (Ω, Σ, μ), for which the function $\mathcal{F}^*\mathcal{F}$ is Gel'fand integrable and Gel'fand integral $\int_\Omega \mathcal{F}^*\mathcal{F}\, d\mu$ is a nuclear operator on Hilbert space H. We show that M is a Hilbert H^*-module, which contains an orthonormal basis.

1. Introduction

A Hilbert H^*-module W over an H^*-algebra Λ is a right Λ-module which possesses a $\tau(\Lambda)$-valued product, where $\tau(\Lambda) = \{ab \mid a, b \in \Lambda\}$ is the trace-class. At the same time, W is a Hilbert space with the inner product given by the action of the trace on the $\tau(\Lambda)$-valued product.

The notion of H^*-module is introduced by Saworotnow in [**7**] under the name of generalized Hilbert space. It has been studied by Smith [**9**], Giellis [**4**] Molnar [**6**], Cabrera et al. [**3**], Bakić and Guljaš [**2**] and others.

Unlike Hilbert C^*-modules, it is well known that each Hilbert H^*-module contains basic elements, orthonormal systems and orthonormal bases (see [**3**] and [**6**]). Moreover, all orthonormal bases for W have the same cardinal number.

In the present paper, we construct an example of right Hilbert H^*-module over the algebra of Hilbert–Schmidt operators and find basic elements, orthonormal system and orthonormal basis.

2. Basic notations and preliminary results

We recall that an H^*-algebra is a complex associative Banach algebra Λ with an inner product $\langle\cdot,\cdot\rangle$ such that $\langle a, a\rangle = \|a\|^2$ for all $a \in \Lambda$ and for each $a \in \Lambda$, there exists some $a^* \in \Lambda$ such that $\langle ab, c\rangle = \langle b, a^*c\rangle$ and $\langle ba, c\rangle = \langle b, ca^*\rangle$ for all $b, c \in \Lambda$. The adjoint a^* of a need not be unique (see [**1**]). Throughout this paper, Λ will always denote a proper H^*-algebra, i.e., an H^*-algebra where each element has a unique adjoint.

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An idempotent in an H^*-algebra is an element e such that $e^2 = e \neq 0$. A projection e is a selfadjoint idempotent in Λ. A projection e is minimal if $e \neq 0$ and $e\Lambda e = \mathbb{C}e$.

The trace-class in a H^*-algebra Λ is defined as the set $\tau(\Lambda) = \{ab \mid a, b \in \Lambda\}$. The trace-class is selfadjoint ideal of Λ and it is dense in Λ, with respect to norm $\tau(\cdot)$. The norm τ is related to the given norm $\|\cdot\|$ on Λ by $\tau(a^*a) = \|a\|^2$ for all $a \in \Lambda$. There exists a continuous linear form sp on $\tau(\Lambda)$ (trace) satisfying $\mathrm{sp}(ab) = \mathrm{sp}(ba) = \langle a^*, b\rangle$. In particular, $\mathrm{sp}(a^*a) = \mathrm{sp}(aa^*) = \langle a, a\rangle = \|a\|^2 = \tau(a^*a)$.

Let $C_\infty(H)$ be the space of all compact and $B(H)$ the space of all bounded linear operators acting on a separable, infinite-dimensional and complex Hilbert space H. In addition, let $s_j(A)$ be the sequence of singular values of the operator A. The algebra $C_2 = \{A \in C_\infty(H) \mid \|A\|_2^2 = \sum_{j=1}^{+\infty} s_j^2(A) < +\infty\}$ is H^*-algebra with minimal projections of rank one $\Theta_{e,f}$, given by $\Theta_{e,f}(g) = e\,\langle f, g\rangle$, for $e, f, g \in H$; and with inner product $\langle A, B\rangle = \mathrm{sp}(A^*B)$ which satisfies $\langle AB, C\rangle = \mathrm{sp}(B^*A^*C) = \langle B, A^*C\rangle$ and $\langle BA, C\rangle = \mathrm{sp}(A^*B^*C) = \mathrm{sp}(B^*CA^*) = \langle B, CA^*\rangle$ for all $A, B, C \in C_2$.

A Hilbert Λ-module is a right module W over a H^*-algebra Λ provided with a mapping $[\cdot,\cdot]\colon W \times W \to \tau(\Lambda)$ which satisfies the following conditions: $[x, \alpha y] = \alpha[x, y]$, $[x, y+z] = [x, y] + [x, z]$, $[x, ya] = [x, y]\,a$, $[x, y]^* = [y, x]$; W is Hilbert space with the inner product $\langle x, y\rangle = \mathrm{sp}\,([x, y])$ for all $\alpha \in \mathbb{C}$, $x, y, z \in W$, $a \in \Lambda$ and for all $x \in W$, $x \neq 0$ there is $a \in \Lambda$, $a \neq 0$ such that $[x, x] = a^*a$. Since M is a Hilbert space, it is complete in the derived scalar-valued inner product $\mathrm{sp}\,([x, y])$.

An element u in a Hilbert H^*-module W is said to be basic if there exists a minimal projection $e \in \Lambda$ such that $[u, u] = e$. An orthonormal system in W is a family of basic elements $(u_\lambda), \lambda \in \Upsilon$, satisfying $[u_\lambda, u_\mu] = 0$, for all $\lambda, \mu \in \Upsilon$, $\lambda \neq \mu$. An orthonormal basis in W is an orthonormal system generating a dense submodule of W. It is well known that each Hilbert H^*-module contains basic elements, orthonormal systems and orthonormal bases (see [**3**] and [**6**]).

The following theorems are very important for Hilbert H^*-module.

Theorem 2.1. [**3**, Remark 1] *Let W be a Hilbert H^*-module over an algebra Λ. Then for all $x, y \in W$ and $a \in \Lambda$*

$$\|x\|^2 = \mathrm{sp}([x, x]) = \tau([x, x]);$$
$$\|[x, y]\| \leqslant \tau([x, y]) \leqslant \|x\| \cdot \|y\|; \quad \|xa\| \leqslant \|a\| \cdot \|x\|.$$

Theorem 2.2. [**3**, Theorem 1.6] *If $(u_\lambda), \lambda \in \Upsilon$ is orthonormal basis for a Hilbert H^*-module W over an algebra Λ, then for all $x \in W$*

$$x = \sum_\lambda u_\lambda [u_\lambda, x]; \text{ (Fourier expansion);}$$
$$[x, x] = \sum_\lambda [x, u_\lambda][u_\lambda, x]; \text{ (Parseval's identity);}$$
$$\|x\|^2 = \sum_\lambda \|[u_\lambda, x]\|^2.$$

For more details, we refer to Saworotnow [**7**], Smith [**9**], Giellis [**4**], Molnar [**6**], Cabrera et al. [**3**], Bakić and Guljaš [**2**] and others.

Next, we introduce weak *-integrals of operator valued functions and state some preliminary results. Let (Ω, Σ, μ) be a measure space. A mapping $\mathcal{A}\colon \Omega \to B(H)$ is called weakly*-measurable if the scalar function $t \mapsto \langle \mathcal{A}_t f, f\rangle$ is measurable for any $f \in H$. A mapping $\mathcal{A}$ is weak*-integrable if the function $t \mapsto \langle \mathcal{A}_t f, f\rangle$ is integrable for any $f \in H$. Let $C_p = C_p(H)$ $(1 \leqslant p < +\infty)$ be the space of all compact linear operators acting on H with norm $\|A\|_p = \big(\sum_{i=1}^{+\infty} s_i^p(A)\big)^{1/p} < +\infty$ where s_i are s-numbers of the operator A, and let C_∞ be the space of all compact operators with norm $\|A\|_\infty = \|A\| = s_1(A)$. If $\mathcal{A}\colon \Omega \to B(H)$ is weak*-integrable, then the sesquilinear form $\sigma\colon H \times H \to \mathbb{C}$, defined by $\sigma(f,f) = \int_\Omega \langle \mathcal{A}_t f, f\rangle \,\mathrm{d}\mu(t)$, is bounded, so there exists unique bounded operator A (or $\int_\Omega \mathcal{A}\,\mathrm{d}\mu$) which satisfies

$$\langle A f, f\rangle = \int_\Omega \langle \mathcal{A}_t f, f\rangle \,\mathrm{d}\mu(t) \quad \text{for all } f \in H.$$

We formalize this in the following definition.

Definition 2.1. Let $\mathcal{A}\colon \Omega \to B(H)$ be a weak*-integrable function. The bounded operator $\int_\Omega \mathcal{A}\,\mathrm{d}\mu$ is unique operator for which

$$\Big\langle \Big(\int_\Omega \mathcal{A}\,\mathrm{d}\mu\Big) f, f \Big\rangle = \int_\Omega \langle \mathcal{A}_t f, f\rangle \,\mathrm{d}\mu(t)$$

holds for all $f \in H$.

For $p \geqslant 1$, denote by $l_G^2(\Omega, \mathrm{d}\mu, C_p)$ the set

$$\Big\{\mathcal{F}\colon \Omega \to B(H) \mid \mathcal{F}^*\mathcal{F} \text{ is weak}^*\text{-integrable}\ , \int_\Omega \mathcal{F}^*\mathcal{F}\,\mathrm{d}\mu \in C_p\Big\}.$$

On this set introduce the following equivalence relation $\mathcal{F} \sim \mathcal{G}$ iff $(\mathcal{F}_t - \mathcal{G}_t)f = 0$ for all $f \in H$, except on a set of zero measure. The quotient space denote by M_p for $p > 1$, and by M for $p = 1$.

The following theorem will be necessary for the proof of the main results.

Theorem 2.3. [**5**, Theorem 2.1] *The space* $(M, \|\cdot\|)$ *is a Banach space with norm* $\|\cdot\|\colon M \to [0,+\infty)$*, defined by* $\|\mathcal{F}\|_M = \big\|\int_\Omega \mathcal{F}^*\mathcal{F}\,\mathrm{d}\mu\big\|_1^{1/2}$ *for all* $\mathcal{F} \in M$.

3. Main result

Our aim in this section is to study an example of H^*-module.

Theorem 3.1. *The space* M *is a right Hilbert* H^**-module over* H^**-algebra* C_2*, with the inner product* $[\cdot,\cdot]\colon M \times M \to C_1$ *defined by*

$$[\mathcal{F}, \mathcal{G}] = \int_\Omega \mathcal{F}^*\mathcal{G}\,\mathrm{d}\mu \quad \textit{for all } \mathcal{F}, \mathcal{G} \in M.$$

Proof. We shall prove that it satisfies the conditions of Hilbert H^*-module. For $\mathcal{F} \in M$, we have $\big\langle \int_\Omega \mathcal{F}^*\mathcal{F}\,\mathrm{d}\mu\, f, f\big\rangle = \int_\Omega \|\mathcal{F}_t f\|^2 \,\mathrm{d}\mu(t) \geqslant 0$, so $[\mathcal{F}, \mathcal{F}] = \int_\Omega \mathcal{F}^*\mathcal{F}\,\mathrm{d}\mu \geqslant 0$.

If $[\mathcal{F},\mathcal{F}]=0$, then $0=\left\langle\int_\Omega \mathcal{F}^*\mathcal{F}\,\mathrm{d}\mu\, f,f\right\rangle=\int_\Omega\|\mathcal{F}_t f\|^2\,\mathrm{d}\mu(t)$ for all $f\in H$, so $\|\mathcal{F}_t f\|=0$ for all $f\in H$, except on a set of zero measure. Therefore, $\mathcal{F}=0$.

We define the norm in the space M by $\|\mathcal{F}\|=\|[\mathcal{F},\mathcal{F}]\|_1^{1/2}$. We have $\langle[\mathcal{F},\alpha\mathcal{G}]f,f\rangle$ $=\left\langle\int_\Omega \mathcal{F}^*\alpha\mathcal{G}\,\mathrm{d}\mu f,f\right\rangle=\left\langle\alpha\int_\Omega \mathcal{F}^*\mathcal{G}\,\mathrm{d}\mu f,f\right\rangle=\langle\alpha[\mathcal{F},\mathcal{G}]f,f\rangle$, for all $\mathcal{F},\mathcal{G}\in M,\alpha\in\mathbb{C}$, hence $[\mathcal{F},\alpha\mathcal{G}]=\alpha[\mathcal{F},\mathcal{G}]$.

For $\mathcal{F},\mathcal{G},\mathcal{H}\in M$, we have

$$
\begin{aligned}
\langle[\mathcal{F},\mathcal{G}+\mathcal{H}]f,f\rangle &= \int_\Omega \langle\mathcal{F}_t^*(\mathcal{G}+\mathcal{H})_t f,f\rangle\,\mathrm{d}\mu(t)\\
&=\int_\Omega \langle\mathcal{F}_t^*\mathcal{G}_t f,f\rangle\,\mathrm{d}\mu(t)+\int_\Omega \langle\mathcal{F}_t^*\mathcal{H}_t f,f\rangle\,\mathrm{d}\mu(t)=\langle\langle\mathcal{F},\mathcal{G}\rangle f,f\rangle+\langle[\mathcal{F},\mathcal{H}]f,f\rangle\,.
\end{aligned}
$$

Hence $[\mathcal{F},\mathcal{G}+\mathcal{H}]=[\mathcal{F},\mathcal{G}]+[\mathcal{F},\mathcal{H}]$.

Next, we have $\langle[\mathcal{F},\mathcal{G}C]f,f\rangle=\int_\Omega\langle\mathcal{F}_t^*\mathcal{G}_t Cf,f\rangle\,\mathrm{d}\mu(t)=\langle[\mathcal{F},\mathcal{G}]Cf,f\rangle$ for $\mathcal{F},\mathcal{G}\in M$, $C\in C_2$. Thus $[\mathcal{F},\mathcal{G}C]=[\mathcal{F},\mathcal{G}]C$.

Let $\mathcal{F},\mathcal{G}\in M$. The function $t\mapsto\langle\mathcal{F}_t^*\mathcal{G}_t f,f\rangle$ is measurable for each $f\in H$. Indeed, it follows from the Parseval identity that $\langle\mathcal{F}_t^*\mathcal{G}_t f,f\rangle=\sum_{n=1}^\infty\langle\mathcal{G}_t f,e_n\rangle\langle e_n,\mathcal{F}_t f\rangle$ for an orthonormal basis $\{e_n\}$ of H, and thus the pointwise limit of measurable functions is also a measurable one. Moreover, for each $f\in H$ the function above is integrable since

$$
|\langle\mathcal{F}_t^*G_t f,f\rangle|\leqslant\langle\mathcal{F}_t^*\mathcal{F}_t f,f\rangle^{1/2}\langle\mathcal{G}_t^*\mathcal{G}_t f,f\rangle^{1/2}\leqslant\frac{1}{2}\left(\langle\mathcal{F}_t^*\mathcal{F}_t f,f\rangle+\langle\mathcal{G}_t^*\mathcal{G}_t f,f\rangle\right).
$$

For each orthonormal basis $\{e_n\}$ of H it holds

$$
\begin{aligned}
&\sum_{n=1}^\infty\left|\left\langle\left(\int_\Omega\mathcal{F}^*\mathcal{G}d\mu\right)e_n,e_n\right\rangle\right|=\sum_{n=1}^\infty\left|\int_\Omega\langle\mathcal{F}_t^*\mathcal{G}_t e_n,e_n\rangle\,d\mu\right|\\
&\leqslant\sum_{n=1}^\infty\int_\Omega|\langle\mathcal{F}_t^*\mathcal{G}_t e_n,e_n\rangle|\,d\mu\leqslant\sum_{n=1}^\infty\int_\Omega\langle\mathcal{F}_t^*\mathcal{F}_t e_n,e_n\rangle^{1/2}\langle\mathcal{G}_t^*\mathcal{G}_t e_n,e_n\rangle^{1/2}\,d\mu\\
&\leqslant\sum_{n=1}^\infty\left(\int_\Omega\langle\mathcal{F}_t^*\mathcal{F}_t e_n,e_n\rangle\,d\mu\right)^{1/2}\left(\int_\Omega\langle\mathcal{G}_t^*\mathcal{G}_t e_n,e_n\rangle\,d\mu\right)^{1/2}\\
&=\left(\sum_{n=1}^\infty\left\langle\left(\int_\Omega\mathcal{F}^*\mathcal{F}d\mu\right)e_n,e_n\right\rangle\right)^{1/2}\left(\sum_{n=1}^\infty\left\langle\left(\int_\Omega\mathcal{G}^*\mathcal{G}d\mu\right)e_n,e_n\right\rangle\right)^{1/2}\\
&=\left\|\int_\Omega\mathcal{F}^*\mathcal{F}d\mu\right\|_1^{1/2}\cdot\left\|\int_\Omega\mathcal{G}^*\mathcal{G}d\mu\right\|_1^{1/2},
\end{aligned}
$$

hence $\int_\Omega\mathcal{F}^*\mathcal{G}d\mu,\int_\Omega\mathcal{G}^*\mathcal{F}d\mu\in C_1$ and

$$
\left\|\int_\Omega\mathcal{F}^*\mathcal{G}d\mu\right\|_1\leqslant\left\|\int_\Omega\mathcal{F}^*\mathcal{F}d\mu\right\|_1^{1/2}\cdot\left\|\int_\Omega\mathcal{G}^*\mathcal{G}d\mu\right\|_1^{1/2}.
$$

Next, $\left\langle\left(\int_\Omega\mathcal{F}^*\mathcal{G}\,\mathrm{d}\mu\right)^*f,g\right\rangle=\int_\Omega\overline{\langle\mathcal{F}_t^*\mathcal{G}_t\,g,f\rangle}\,\mathrm{d}\mu(t)=\left\langle\int_\Omega\mathcal{G}^*\mathcal{F}\,\mathrm{d}\mu\,f,g\right\rangle$. We have proved $[\mathcal{F},\mathcal{G}]^*=[\mathcal{G},\mathcal{F}]$.

The space M is a Hilbert space with the scalar product $\langle \mathcal{F}, \mathcal{G}\rangle = \mathrm{sp}([\mathcal{F}, \mathcal{G}]) = \mathrm{sp}\left(\int_\Omega \mathcal{F}^* \mathcal{G}\, \mathrm{d}\mu\right)$. Indeed, since

$$\sum_k \langle [\mathcal{F}, \mathcal{G}] e_k, e_k\rangle = \overline{\sum_k \langle [\mathcal{G}, \mathcal{F}] e_k, e_k\rangle}, \quad \sum_k \langle [\alpha\mathcal{F}, \mathcal{G}] e_k, e_k\rangle = \alpha \sum_k \langle [\mathcal{F}, \mathcal{G}] e_k, e_k\rangle,$$
$$\sum_k \langle [\mathcal{F} + \mathcal{H}, \mathcal{G}] e_k, e_k\rangle = \sum_k \langle [\mathcal{F}, \mathcal{G}] e_k, e_k\rangle + \sum_k \langle [\mathcal{H}, \mathcal{G}] e_k, e_k\rangle$$

for some orthonormal basis $\{e_k\}$ of H, we have $\langle \mathcal{F}, \mathcal{G}\rangle = \overline{\langle \mathcal{G}, \mathcal{F}\rangle}$, $\langle \alpha\mathcal{F}, \mathcal{G}\rangle = \alpha \langle \mathcal{F}, \mathcal{G}\rangle$ and $\langle \mathcal{F} + \mathcal{H}, \mathcal{G}\rangle = \langle \mathcal{F}, \mathcal{G}\rangle + \langle \mathcal{H}, \mathcal{G}\rangle$, for all $\alpha \in \mathbb{C}$, $\mathcal{F}, \mathcal{G} \in M$. We proved that if $\langle \mathcal{F}, \mathcal{F}\rangle = 0$, then $\mathcal{F} = 0$. The completeness of the space M follows from Theorem 2.3. □

4. Applications

In this section we will show how the structure theorems for Hilbert H*-modules can be applied to our case.

THEOREM 4.1. *Let $\mathcal{F}, \mathcal{G} \in M$ and let $X \in C_2$. Then*

1) $\left\|\int_\Omega \mathcal{F}^* \mathcal{G}\, \mathrm{d}\mu\right\|_1 \leqslant \left\|\int_\Omega \mathcal{F}^* \mathcal{F}\, \mathrm{d}\mu\right\|_1^{1/2} \cdot \left\|\int_\Omega \mathcal{G}^* \mathcal{G}\, \mathrm{d}\mu\right\|_1^{1/2}$;
2) $\left\|\int_\Omega X^* \mathcal{F}^* \mathcal{F} X\, \mathrm{d}\mu\right\|_1 \leqslant \left\|\int_\Omega \mathcal{F}^* \mathcal{F}\, \mathrm{d}\mu\right\|_1 \cdot \|X\|_2^2$;
3) $\int_\Omega \mathcal{F}^* \mathcal{G}\, \mathrm{d}\mu \int_\Omega \mathcal{G}^* \mathcal{F}\, \mathrm{d}\mu \leqslant \left\|\int_\Omega \mathcal{G}^* \mathcal{G}\, \mathrm{d}\mu\right\|_{B(H)} \int_\Omega \mathcal{F}^* \mathcal{F}\, \mathrm{d}\mu$.

PROOF. Properties 1) and 2) follow directly from Theorem 2.1. To prove 3), let $\mathcal{F}, \mathcal{G} \in M$ and let φ be a positive linear functional on $B(H)$. Applying the Cauchy–Bunyakovskii inequality for the degenerate inner product $\varphi([\cdot, \cdot])$ on M, we obtain

$$\begin{aligned}\varphi([\mathcal{F}, \mathcal{G}][\mathcal{G}, \mathcal{F}]) &= \varphi([\mathcal{F}, \mathcal{G}[\mathcal{G}, \mathcal{F}]]) \leqslant \varphi([\mathcal{F}, \mathcal{F}])^{1/2} \varphi([\mathcal{G}[\mathcal{G}, \mathcal{F}], \mathcal{G}[\mathcal{G}, \mathcal{F}]])^{1/2} \\ &= \varphi([\mathcal{F}, \mathcal{F}])^{1/2} \varphi([\mathcal{F}, \mathcal{G}][\mathcal{G}, \mathcal{G}][\mathcal{G}, \mathcal{F}]) \\ &\leqslant \varphi([\mathcal{F}, \mathcal{F}])^{1/2} \|[\mathcal{G}, \mathcal{G}]\|_{B(H)}^{1/2} \varphi([\mathcal{F}, \mathcal{G}][\mathcal{G}, \mathcal{F}])^{1/2}.\end{aligned}$$

Therefore, we have the inequality $\varphi([\mathcal{F}, \mathcal{G}][\mathcal{G}, \mathcal{F}]) \leqslant \|[\mathcal{G}, \mathcal{G}]\|_{B(H)} \varphi([\mathcal{F}, \mathcal{F}])$ for any positive linear functional φ, hence statement 3) is proved. □

In the following proposition we apply some properties of the Hilbert H-module to the particular module M.

PROPOSITION 4.1. (a) *The space M has orthonormal basis $\mathcal{U}_\lambda$ which for all $\mathcal{F} \in M$ satisfies*

$$\text{(i)}\ \ \mathcal{F} = \sum_\lambda \mathcal{U}_\lambda \left([\mathcal{U}_\lambda, \mathcal{F}]\right); \quad \text{(ii)}\ \ [\mathcal{F}, \mathcal{F}] = \sum_\lambda [\mathcal{F}, \mathcal{U}_\lambda][\mathcal{U}_\lambda, \mathcal{F}];$$
$$\text{(iii)}\ \ \|[\mathcal{F}, \mathcal{F}]\|_1 = \sum_\lambda \|[\mathcal{U}_\lambda, \mathcal{F}]\|_1^2.$$

(b) *Let* $\mathcal{F}_n, \mathcal{F}, \mathcal{G}_n, \mathcal{G}, \mathcal{H} \in M$. *If* (1) $\lim_{n\to\infty} \operatorname{sp}([\mathcal{F}_n - \mathcal{F}, \mathcal{H}]) = 0$ *holds for each* $\mathcal{H} \in M$, (2) $\lim_{n\to\infty} \operatorname{sp}([\mathcal{G}_n - \mathcal{G}, \mathcal{G}_n - \mathcal{G}]) = 0$, *then*

$$\lim_{n\to\infty} \operatorname{sp}([\mathcal{F}_n, \mathcal{G}_n] - [\mathcal{F}, \mathcal{G}]) = 0.$$

(c) *Let* $\mathcal{F}_n, \mathcal{F}, \mathcal{G}_n, \mathcal{G}, \mathcal{H} \in M$. *If* (1′) $\lim_{n\to\infty} \|[\mathcal{F}_n - \mathcal{F}, \mathcal{H}]\|_1 = 0$ *holds for each* $\mathcal{H} \in M$, (2′) $\lim_{n\to\infty} \|[\mathcal{G}_n - \mathcal{G}, \mathcal{G}_n - \mathcal{G}]\|_1 = 0$, *then*

$$\lim_{n\to\infty} \|[\mathcal{F}_n, \mathcal{G}_n] - [\mathcal{F}, \mathcal{G}]\|_1 = 0.$$

(d) *Let* $[\mathcal{F}, \mathcal{F}]$ *be a projection in* C_2 *(not necessarily minimal) for some* $\mathcal{F} \in M$. *Then* $\mathcal{F}[\mathcal{F}, \mathcal{F}] = \mathcal{F}$.

(e) *Let* $\Theta_{f,g} \in C_2$ *be a minimal projection for some* $f, g \in H$. *Then there exists an orthonormal basis* $(\mathcal{U}_\lambda) \in M$ *such that* $[\mathcal{U}_\lambda, \mathcal{U}_\lambda] = \Theta_{f,g}$.

(f) *If there exists* $N > 0$ *such that* $\|[\mathcal{U}_\lambda, \mathcal{U}_\lambda]\|_1 \leqslant N$ *for some mutually orthogonal elements* $(\mathcal{U}_\lambda)$ *in* M, *then* $\|[\mathcal{U}_\lambda, \mathcal{F}]\|_1$ *converges to 0, for all* $\mathcal{F} \in M$.

Proof. The property (a) follows directly from Theorem 2.2.

Since M is a Hilbert space with inner product $\langle \mathcal{F}, \mathcal{G} \rangle = \operatorname{sp}([\mathcal{F}, \mathcal{G}])$, it satisfies property (b).

The inequality $\|([\mathcal{F}_n, \mathcal{G}_n] - [\mathcal{F}, \mathcal{G}])\|_1 \leqslant \|\mathcal{F}_n\|_M \cdot \|\mathcal{G}_n - \mathcal{G}\|_M + \|[\mathcal{F}_n - \mathcal{F}, \mathcal{G}]\|_1$ holds, as in the case of Hilbert spaces. From the uniform boundedness principle, we have $\sup\limits_n \|\widetilde{\mathcal{F}_n}\| < \infty$, hence property (c) follows.

Properties (d), (e) and (f) follow from [**2**, Lemma 1.4 or Propositions 1.5,1.9] applied to Hilbert H^*-module M. □

Remark 4.1. Properties (b) and (c) hold for any Hilbert H^*-module with the trace replaced by the scalar product and the norm with the appropriate one.

Remark 4.2. The special case of [**5**, Theorem 3.4 a)], for $p = 1$, is a corollary of Theorems 3.1 and 4.1.

Define the set $M_\mathcal{F} = \{\mathcal{F}X \mid X \in C_2\}$, for some $\mathcal{F} \in M$. The hilbertian dimension C_2-dim $M_\mathcal{F}$, generated by an element $\mathcal{F}$, is equal to the cardinal number of the set I of indices such that $\int_\Omega \mathcal{F}^*\mathcal{F}\, \mathrm{d}\mu = \sum_{\lambda\in I} \alpha_\lambda \Theta_{e_\lambda, e_\lambda}$ where $(\Theta_{e_\lambda, e_\lambda})$ are orthogonal minimal projections in C_2 and $\alpha_\lambda > 0$. The hilbertian dimension of a submodule $M_\mathcal{F}$ can be greater than 1. Hence $\mathcal{F} = \sum_{\lambda\in I} \sqrt{\alpha_\lambda}\mathcal{F}_\lambda$ for $\mathcal{F}_\lambda = (\sqrt{\alpha_\lambda})^{-1}\mathcal{F}\Theta_{e_\lambda, e_\lambda}$, and $(\mathcal{F}_\lambda)$ is orthonormal basis in $M_\mathcal{F}$ (see [**2**]).

An operator $A\colon M \to M$ is called C_2-linear if it is linear and satisfies $A(\mathcal{F}X) = A(\mathcal{F})X$, for all $\mathcal{F} \in M$, $X \in C_2$. The set of all bounded C_2-linear operators on M is denoted by $B_{C_2}(M)$.

Theorem 4.2. *Let* $X \in B(H)$ *and* $\mathcal{X} \in M$, *where* $\sup_{t\in\Omega} \|\mathcal{X}_t\| = N < \infty$. *The operators* $L_X, L_\mathcal{X}\colon M \to M$ *defined by*

$$L_X(\mathcal{F}) = X\mathcal{F}, \quad L_\mathcal{X}(\mathcal{F}) = \mathcal{X}\mathcal{F},$$

belong to $B_{C_2}(M)$ *and the inequalities* $\|L_X\| \leqslant \|X\|$, $\|L_\mathcal{X}\| \leqslant N$ *hold.*

PROOF. The operators are well defined, because $X\mathcal{F}, \mathcal{X}\mathcal{F} \in M$ when $\mathcal{F} \in M$. Indeed, $t \to \mathcal{F}_t^* X^* X \mathcal{F}_t$ is weak*-integrable since

$$\left\langle \int_\Omega \mathcal{F}^* X^* X \mathcal{F} \, \mathrm{d}\mu \, f, f \right\rangle = \int_\Omega \|X\mathcal{F}_t f\|^2 \, \mathrm{d}\mu(t) \leqslant \|X\|^2 \int_\Omega \|\mathcal{F}_t f\|^2 \, \mathrm{d}\mu(t) < +\infty.$$

From the inequality $\mathcal{F}_t^* X^* X \mathcal{F} \leqslant \|X\|^2 \mathcal{F}_t^* \mathcal{F}_t$ for all $t \in \Omega$, we have $\int_\Omega \mathcal{F}^* \mathcal{F} \, \mathrm{d}\mu \in C_1$ and

$$\|L_X(\mathcal{F})\|_M = \|X\mathcal{F}\|_M = \left\| \int_\Omega \mathcal{F}^* X^* X \mathcal{F} \, \mathrm{d}\mu \right\|_1^{1/2} \leqslant \left\| \int_\Omega \mathcal{F}^* \mathcal{F} \, \mathrm{d}\mu \right\|_1^{1/2} \cdot \|X\|.$$

Hence $\|L_X\| \leqslant \|X\|$.

Next, $\mathcal{F}^* \mathcal{X}^* \mathcal{X} \mathcal{F}$ is weak*-integrable since

$$\begin{aligned}
\left\langle \int_\Omega \mathcal{F}^* \mathcal{X}^* \mathcal{X} \mathcal{F} \, \mathrm{d}\mu \, f, f \right\rangle &= \int_\Omega \|\mathcal{X}_t \mathcal{F}_t f\|^2 \, \mathrm{d}\mu(t) \leqslant \int_\Omega \|\mathcal{X}_t\| \cdot \|\mathcal{F}_t f\|^2 \, \mathrm{d}\mu(t) \\
&\leqslant N \int_\Omega \|\mathcal{F}_t f\|^2 \, \mathrm{d}\mu(t) = N \int_\Omega \langle \mathcal{F}_t^* \mathcal{F}_t f, f \rangle \, \mathrm{d}\mu(t) \\
&= N \left\langle \int_\Omega \mathcal{F}_t^* \mathcal{F}_t \, \mathrm{d}\mu(t) f, f \right\rangle \\
&\leqslant N \left\| \int_\Omega \mathcal{F}_t^* \mathcal{F}_t \, \mathrm{d}\mu(t) \right\|_{B(H)} \cdot \|f\|^2.
\end{aligned}$$

Hence $\int_\Omega \mathcal{F}^* \mathcal{X}^* \mathcal{X} \mathcal{F} \, \mathrm{d}\mu \in B(H)$.

We will prove that $\int_\Omega \mathcal{F}^* \mathcal{X}^* \mathcal{X} \mathcal{F} \, \mathrm{d}\mu \in C_1$. We have

$$\begin{aligned}
\|L_\mathcal{X}(\mathcal{F})\|_M^2 = \|\mathcal{X}\mathcal{F}\|_M^2 &= \left\| \int_\Omega \mathcal{F}^* \mathcal{X}^* \mathcal{X} \mathcal{F} \, \mathrm{d}\mu \right\|_1 = \sum_{j=1}^{+\infty} s_j \left(\int_\Omega \mathcal{F}^* \mathcal{X}^* \mathcal{X} \mathcal{F} \, \mathrm{d}\mu \right) \\
&\leqslant \sum_{j=1}^{+\infty} s_j \left(N^2 \int_\Omega \mathcal{F}^* \mathcal{F} \, \mathrm{d}\mu \right) = N^2 \sum_{j=1}^{+\infty} s_j \left(\int_\Omega \mathcal{F}^* \mathcal{F} \, \mathrm{d}\mu \right) = N^2 \|\mathcal{F}\|^2.
\end{aligned}$$

Therefore, $\|L_\mathcal{X}\| \leqslant N$. □

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