PUBLICATIONS DE L'INSTITUT MATHÉMATIQUE
Nouvelle série, tome 108(122) (2020), 91–102 DOI: https://doi.org/10.2298/PIM2022091H

$*$-CONFORMAL η-RICCI SOLITONS IN ϵ-KENMOTSU MANIFOLDS

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Abstract. We characterize ϵ-Kenmotsu manifolds admitting $*$-conformal η-Ricci solitons. At last, an example of 7-dimension ϵ-Kenmotsu manifold is given.

1. Introduction

In 1993, Bejancu and Duggal [**2**] introduced the concept of ϵ-Sasakian manifolds. Later, it was shown by Xufeng and Xiaoli [**19**] that every ϵ-Sasakian manifolds are real hypersurfaces of indefinite Kahlerian manifolds. In 1972, Kenmotsu studied a class of contact Riemannian manifolds satisfying some special conditions [**14**]. We call it Kenmotsu manifold. The concept of an ϵ-Kenmotsu manifold was introduced by De and Sarkar [**5**] who showed that the existence of new structure on an indefinite metric influences the curvatures. ϵ-Kenmotsu manifolds have also been studied by various authors in several ways to a different extent such as [**10, 11, 12, 18**] and many others.

In 1982, Hamilton [**9**] introduced the notion of Ricci flow to find a canonical metric on a smooth manifold. The Ricci flow is an evolution equation for metrics on a Riemannian manifold defined by

$$\frac{\partial}{\partial t}g_{ij}(t) = -2R_{ij}.$$

A Ricci soliton emerges as the limit of the solutions of the Ricci flow. A solution to the Ricci flow is called Ricci soliton if it moves only by a one parameter group of diffeomorphism and scaling. A Ricci soliton (g, V, λ) on a Riemannian manifold (M, g) is a generalization of an Einstein metric such that $\pounds_V g + 2S + 2\lambda g = 0$, where S is the Ricci tensor, $\pounds_V$ is the Lie derivative operator along the vector field V on M and λ is a real number. The Ricci soliton is said to be shrinking, steady or expanding according to λ being negative, zero or positive, respectively.

2010 *Mathematics Subject Classification:* 53C15, 53C25.

Key words and phrases: $*$-conformal η-Ricci solitons, ϵ-Kenmotsu manifolds, concircular curvature tensor, η-Einstein manifolds, Einstein manifolds.

As a generalization of the Ricci soliton, the notion of η-Ricci soliton was introduced by Cho and Kimura [**4**] and is given by $\pounds_V g + 2S + 2\lambda g + 2\mu\eta \otimes \eta = 0$, where λ and μ are real numbers.

In 2004, the concept of conformal Ricci flow which is a variation of the classical Ricci flow equation was introduced by Fischer [**6**]. In the classical Ricci flow equation the unit volume constraint plays an important role, but in the conformal Ricci flow equation, the scalar curvature r is considered as a constraint. The conformal Ricci flow on M is defined by the equation [**6**]

$$\frac{\partial g}{\partial t} + 2\Big(S + \frac{g}{n}\Big) = -pg$$

and $r = -1$, where p is a scalar non-dynamical field(time dependent scalar field), r is the scalar curvature of the manifold and n is the dimension of manifold.

The conformal Ricci soliton equation and conformal η-Ricci soliton equation are given by [**1**]

$$\pounds_V g + 2S = \Big(2\lambda - \Big(p + \frac{2}{n}\Big)\Big)g,$$

$$\pounds_V g + 2S + (2\lambda - \Big(p + \frac{2}{n}\Big)\Big)g + 2\mu\eta \otimes \eta = 0,$$

respectively, where λ and μ are constants.

The notion of $*$-Ricci tensor on almost Hermitian manifolds was introduced by Tachibana [**17**]. Later, Hamada [**8**] studied $*$-Ricci flat real hypersurfaces of complex space forms and Blair [**3**] defined $*$-Ricci tensor in contact metric manifolds given by

$$S^*(X,Y) = g(Q^*X,Y) = \text{Trace}\,\{\phi \circ R(X,\phi Y)\}\,, \tag{1.1}$$

where Q^* is the $*$-Ricci operator and S^* is a tensor field of type $(0,2)$.

Definition 1.1. [**13**] A Riemannian metric g on M is called a $*$-Ricci soliton, if

$$(\pounds_V g)(X,Y) + 2S^*(X,Y) + 2\lambda g(X,Y) = 0$$

for all vector fields X, Y on M and λ is a constant.

If $S^*(X,Y) = \lambda g(X,Y) + \mu\eta(X)\eta(Y)$ for all vector fields X, Y and λ, μ are smooth functions, then the manifold is called $*$-η-Einstein manifold. Further if $\mu = 0$, that is, $S^*(X,Y) = \lambda g(X,Y)$ for all vector fields X, Y, then the manifold becomes $*$-Einstein.

Recently, the $*$-Ricci solitons on almost contact metric manifolds have been studied by various authors such as [**7, 13, 15, 16**] and many others.

The notion of $*$-conformal η-Ricci soliton is defined as follows:

$$\pounds_V g + 2S^* + \Big(2\lambda - \Big(p + \frac{2}{n}\Big)\Big)g + 2\mu\eta \otimes \eta = 0, \tag{1.2}$$

where $\pounds_V$ is the Lie derivative along the vector field V, S^* is the $*$-Ricci tensor and λ, μ are constants.

In the present paper we study $*$-conformal η-Ricci solitons in an ϵ-Kenmotsu manifold satisfying certain curvature conditions.

2. Preliminaries

An n-dimensional smooth manifold (M, g) is said to be an ϵ-almost contact metric manifold [**2**], if it admits a $(1,1)$ tensor field ϕ, a structure vector field ξ, a 1-form η and an indefinite metric g such that

$$\phi^2 X = -X + \eta(X)\xi, \quad \eta(\xi) = 1, \tag{2.1}$$

$$g(\xi, \xi) = \epsilon, \quad \eta(X) = \epsilon g(X, \xi), \tag{2.2}$$

$$g(\phi X, \phi Y) = g(X, Y) - \epsilon\eta(X)\eta(Y) \tag{2.3}$$

for all vector fields X, Y on M, where ϵ is 1 or-1 according as ξ is spacelike or timelike vector fields and rank ϕ is $(n-1)$. If $d\eta(X, Y) = g(X, \phi Y)$ for every $X, Y \in \chi(M)$, then we say that M is an ϵ-contact metric manifold. Also, we have $\phi\xi = 0$, $\eta(\phi X) = 0$. If an ϵ-contact metric manifold satisfies

$$(\nabla_X \phi)(Y) = -g(X, \phi Y)\xi - \epsilon\eta(Y)\phi X,$$

where ∇ denotes the Levi-Civita connection with respect to g, then M is called an ϵ-Kenmotsu manifold [**5**].

An ϵ-almost contact metric manifold is an ϵ-Kenmotsu, if and only if

$$\nabla_X \xi = \epsilon(X - \eta(X)\xi). \tag{2.4}$$

Moreover, the curvature tensor R, the Ricci tensor S and the Ricci operator Q in an ϵ-Kenmotsu manifold M with respect to the Levi-Civita connection satisfies

$$(\nabla_X \eta)Y = g(X, Y) - \epsilon\eta(X)\eta(Y), \tag{2.5}$$

$$R(X, Y)\xi = \eta(X)Y - \eta(Y)X, \tag{2.6}$$

$$R(\xi, X)Y = \eta(Y)X - \epsilon g(X, Y)\xi, \tag{2.7}$$

$$R(\xi, X)\xi = -R(X, \xi)\xi = X - \eta(X)\xi, \tag{2.8}$$

$$\eta(R(X, Y)Z) = \epsilon(g(X, Z)\eta(Y) - g(Y, Z)\eta(X)), \tag{2.9}$$

$$S(X, \xi) = -(n-1)\eta(X), \quad S(\xi, \xi) = -(n-1), \tag{2.10}$$

$$Q\xi = -\epsilon(n-1)\xi \tag{2.11}$$

for any X, Y, Z on M, where $g(QX, Y) = S(X, Y)$. We note that if $\epsilon = 1$ and the structure vector field ξ is spacelike, then an ϵ-Kenmotsu manifold is usual Kenmotsu manifold.

Definition 2.1. An ϵ-Kenmotsu manifold M is said to be η-Einstein manifold if its Ricci tensor S is of the form $S(X, Y) = ag(X, Y) + b\eta(X)\eta(Y)$, where a and b are smooth functions on M. If $b = 0$ (resp., $a = 0$), then the manifold is called Einstein (resp., special type of an η-Einstein) manifold.

Definition 2.2. The concircular curvature tensor C in an n-dimensional ϵ-Kenmotsu manifold M is defined by [**20**]

$$C(X, Y)Z = R(X, Y)Z - \frac{r}{n(n-1)}[g(Y, Z)X - g(X, Z)Y], \tag{2.12}$$

where R is the Riemannian curvature tensor and r is the scalar curvature of the manifold.

Lemma 2.1. *In an n-dimensional ϵ-Kenmotsu manifold, we have*

$$\begin{aligned}\bar{R}(X,Y,\phi Z,\phi W) = \bar{R}(X,Y,Z,W) \\ + \epsilon\Phi(X,Z)\Phi(Y,W) - \epsilon\Phi(Y,Z)\Phi(X,W) \\ + \epsilon g(Y,Z)g(X,W) - \epsilon g(X,Z)g(Y,W)\end{aligned} \tag{2.13}$$

for any X, Y, Z, W on M, where $\bar{R}(X,Y,Z,W) = g(R(X,Y)Z,W)$ and Φ is the fundamental 2-form of M defined by $\Phi(X,Y) = g(X,\phi Y)$.

Proof. By using equations (2.3)-(2.4), (2.6) and the expression of the curvature tensor $R(X,Y)Z = \nabla_X\nabla_Y Z - \nabla_Y\nabla_X Z - \nabla_{[X,Y]}Z$ in $\bar{R}(X,Y,\phi Z,\phi W) = g(\bar{R}(X,Y)\phi Z,\phi W)$, after straightforward calculations (2.13) follows. □

Lemma 2.2. *In an n-dimensional ϵ-Kenmotsu manifold the $*$-Ricci tensor is given by*

$$S^*(Y,Z) = S(Y,Z) + \epsilon(n-2)g(Y,Z) + \eta(Y)\eta(Z) \tag{2.14}$$

for any Y, Z on M.

Proof. Let $\{e_i\}$, $i = 1, 2, \ldots, n$ be an orthonormal basis of the tangent space at each point of the manifold. Therefore from (2.13) and (1.1), we have

$$\begin{aligned}S^*(Y,Z) &= \sum_{i=1}^{n} \bar{R}(e_i,Y,\phi Z,\phi e_i) \\ &= \sum_{i=1}^{n} [\bar{R}(e_i,Y,Z,e_i) + \epsilon\Phi(e_i,Z)\Phi(Y,e_i) - \epsilon\Phi(Y,Z)\Phi(e_i,e_i) \\ &\qquad + \epsilon g(Y,Z)g(e_i,e_i) - \epsilon g(e_i,Z)g(Y,e_i)].\end{aligned}$$

By using (2.3) and $\Phi(X,Y) = g(X,\phi Y)$ in the above equation, (2.14) follows. □

3. $*$-conformal η-Ricci solitons in ϵ-Kenmotsu manifolds

Let an n-dimensional ϵ-Kenmotsu manifold admits $*$-conformal η-Ricci soliton. Then (1.2) holds and thus we have

$$(\pounds_\xi g)(Y,Z) + 2S^*(Y,Z) + \Big[2\lambda - \Big(p + \frac{2}{n}\Big)\Big]g(Y,Z) + 2\mu\eta(Y)\eta(Z) = 0. \tag{3.1}$$

In an ϵ-Kenmotsu manifold, we have [**11**]

$$(\pounds_\xi g)(Y,Z) = 2\epsilon(g(Y,Z) - \epsilon\eta(Y)\eta(Z)). \tag{3.2}$$

Therefore, from (3.1) and (3.2), we find

$$S^*(Y,Z) = -\Big[\epsilon + \lambda - \frac{1}{2}\Big(p + \frac{2}{n}\Big)\Big]g(Y,Z) + (1-\mu)\eta(Y)\eta(Z). \tag{3.3}$$

By using (3.3), (2.14) takes the form

$$S(Y,Z) = -\Big[n\epsilon - \epsilon + \lambda - \frac{1}{2}\Big(p + \frac{2}{n}\Big)\Big]g(Y,Z) - \mu\eta(Y)\eta(Z) \tag{3.4}$$

which is of the form $S(Y,Z) = Ag(Y,Z) + B\eta(Y)\eta(Z)$, where $A = -[n\epsilon - \epsilon + \lambda - \frac{1}{2}(p + \frac{2}{n})]$ and $B = -\mu$. Taking $Z = \xi$ in (3.4), we find

$$S(Y,\xi) = -\Big[n - 1 + \epsilon\lambda + \mu - \frac{\epsilon}{2}\Big(p + \frac{2}{n}\Big)\Big]\eta(Y). \tag{3.5}$$

From equations (2.10) and (3.5), we obtain

$$\lambda + \epsilon\mu = \frac{1}{2}\Big(p + \frac{2}{n}\Big). \tag{3.6}$$

Thus we have the following:

Theorem 3.1. *If an n-dimensional ϵ-Kenmotsu manifold admits $*$-conformal η-Ricci soliton, then the manifold is an η-Einstein manifold of the form* (3.4) *and the scalars λ and μ are related by $\lambda + \epsilon\mu = \frac{1}{2}(p + \frac{2}{n})$.*

Now we consider an ϵ-Kenmotsu manifold admitting $*$-conformal η-Ricci soliton and have Codazzi type of Ricci tensor and cyclic parallel Ricci tensor.

Definition 3.1. An ϵ-Kenmotsu manifold is said to have Codazzi type of Ricci tensor if its Ricci tensor S of type $(0,2)$ is non-zero and satisfies the following condition

$$(\nabla_X S)(Y,Z) = (\nabla_Y S)(X,Z) \tag{3.7}$$

for all $X, Y, Z \in \chi(M)$,

Taking covariant derivative of (3.4) and making use of (2.5), we find

$$(\nabla_X S)(Y,Z) = -\mu[g(X,Y)\eta(Z) + g(X,Z)\eta(Y) - 2\epsilon\eta(X)\eta(Y)\eta(Z)]. \tag{3.8}$$

If the Ricci tensor S is of Codazzi type, then we have from (3.7) and (3.8) that

$$\mu[g(X,Z)\eta(Y) - g(Y,Z)\eta(X)] = 0$$

from which it follows that either $\mu = 0$ or $g(X,Z)\eta(Y) - g(Y,Z)\eta(X) = 0$. Therefore,

(i) if $\mu = 0$, then the $*$-conformal η-Ricci soliton becomes $*$-conformal Ricci soliton. Hence we state the following:

Theorem 3.2. *A $*$-conformal η-Ricci soliton in an ϵ-Kenmotsu manifold whose Ricci tensor is of Codazzi type becomes a $*$-conformal Ricci soliton.*

Again for $\mu = 0$, (3.4) becomes $S(Y,Z) = -\epsilon(n-1)g(Y,Z)$. Therefore the manifold becomes an Einstein manifold. Also it is known that a 3-dimensional Einstein manifold is a manifold of constant curvature [**20**]. Thus we have:

Corollary 3.1. *An ϵ-Kenmotsu manifold whose Ricci tensor is of Codazzi-type admitting $*$-conformal Ricci solitons is a manifold of constant curvature.*

(ii) If $g(X,Z)\eta(Y) - g(Y,Z)\eta(X) = 0$, then we replace $Y = \xi$ in the foregoing equation, we obtain $g(X,Z) = \epsilon\eta(X)\eta(Z)$ which by substituting X by QX turns to $S(X,Z) = -(n-1)\eta(X)\eta(Z)$. Thus we have the following:

Theorem 3.3. *If an n-dimensional ϵ-Kenmotsu manifold admits $*$-conformal η-Ricci soliton and the manifold has Ricci tensor of Codazzi type, then the manifold is a special type of an η-Einstein manifold.*

Definition 3.2. An ϵ-Kenmotsu manifold is said to have cyclic parallel Ricci tensor if its Ricci tensor S of type $(0, 2)$ is non-zero and satisfies the following condition

$$(\nabla_X S)(Y, Z) + (\nabla_Y S)(Z, X) + (\nabla_Z S)(X, Y) = 0 \tag{3.9}$$

for all $X, Y, Z \in \chi(M)$.

Let an n-dimensional ϵ-Kenmotsu manifold admitting $*$-conformal η-Ricci soliton and the manifold has cyclic parallel Ricci tensor, then (3.9) holds. By virtue of (3.8), we have

$$(\nabla_Y S)(Z, X) = -\mu[g(Y, Z)\eta(X) + g(Y, X)\eta(Z) - 2\epsilon\eta(X)\eta(Y)\eta(Z)], \tag{3.10}$$

$$(\nabla_Z S)(X, Y) = -\mu[g(Z, X)\eta(Y) + g(Z, Y)\eta(X) - 2\epsilon\eta(X)\eta(Y)\eta(Z)]. \tag{3.11}$$

By making use of (3.8), (3.10) and (3.11) in (3.9), we have

$$\mu[g(X, Y)\eta(Z) + g(Y, Z)\eta(X) + g(Z, X)\eta(Y) - 6\epsilon\eta(X)\eta(Y)\eta(Z)] = 0$$

which by putting $Z = \xi$ reduces to

$$\mu[g(X, Y) - \epsilon\eta(X)\eta(Y)] = 0 \implies \mu g(\phi X, \phi Y) = 0$$

from which it follows that $\mu = 0$ and $g(\phi X, \phi Y) \neq 0$. Thus we have the following:

Theorem 3.4. *A $*$-conformal η-Ricci soliton in an ϵ-Kenmotsu manifold whose Ricci tensor is cyclic parallel becomes a $*$-conformal Ricci soliton.*

Now by considering (3.6) along with $\mu = 0$, we get from (3.4) that

$$S(Y, Z) = -\epsilon(n - 1)g(Y, Z). \tag{3.12}$$

Thus we have the following:

Corollary 3.2. *If an n-dimensional ϵ-Kenmotsu manifold admits $*$-conformal η-Ricci soliton and the manifold has a cyclic parallel Ricci tensor, then the manifold is an Einstein manifold of the form* (3.12).

4. $*$-conformal η-Ricci solitons in ϵ-Kenmotsu manifolds satisfying $R(\xi, X) \cdot S = 0$

Let an n-dimensional ϵ-Kenmotsu manifold admitting $*$-conformal η-Ricci soliton satisfies $R(\xi, X) \cdot S = 0$. Then we have

$$S(R(\xi, X)Y, Z) + S(Y, R(\xi, X)Z) = 0 \tag{4.1}$$

for all $X, Y, Z \in \chi(M)$. By using (2.7) in (4.1), we have

$$S(\eta(Y)X - \epsilon g(X, Y)\xi, Z) + S(Y, \eta(Z)X - \epsilon g(X, Z)\xi) = 0$$

which by taking $Z = \xi$ and using (3.5) takes the form

$$S(X, Y) = -\epsilon\Big(n - 1 + \epsilon\lambda + \mu - \frac{\epsilon}{2}\Big(p + \frac{2}{n}\Big)\Big)g(X, Y). \tag{4.2}$$

Now from (3.4) and (4.2), we obtain

$$\mu(g(X,Y) - \epsilon\eta(X)\eta(Y)) = 0 \implies \mu g(\phi X, \phi Y) = 0$$

from which it follows that $\mu = 0$ and $g(\phi X, \phi Y) \neq 0$. Thus we have the following:

THEOREM 4.1. *A $*$-conformal η-Ricci soliton in an ϵ-Kenmotsu manifold satisfying $R(\xi, X) \cdot S = 0$ becomes a $*$-conformal Ricci soliton.*

By virtue of (3.6), (4.2) becomes

$$S(X,Y) = -\epsilon(n-1)g(X,Y). \tag{4.3}$$

Thus we have the following:

COROLLARY 4.1. *If an n-dimensional ϵ-Kenmotsu manifold admitting $*$-conformal η-Ricci soliton satisfies $R(\xi, X) \cdot S = 0$, then the manifold is an Einstein manifold of the form* (4.3)*.*

5. $*$-conformal η-Ricci solitons in ϵ-Kenmotsu manifolds satisfying $S(\xi, X) \cdot R = 0$

Let an n-dimensional ϵ-Kenmotsu manifold admitting $*$-conformal η-Ricci soliton satisfies $S(\xi, X) \cdot R = 0$. Then we have

$$(X \wedge_S \xi)R(U,V)Z + R((X \wedge_S \xi)U, V)Z + R(U, (X \wedge_S \xi)V)Z \tag{5.1}$$

$$+ R(U,V)(X \wedge_S \xi)Z = 0 \tag{5.2}$$

for any $X, U, V, Z \in \chi(M)$, where the endomorphism $X \wedge_S \xi$ is defined by

$$(X \wedge_S \xi)Z = S(\xi, Z)X - S(X,Z)\xi. \tag{5.3}$$

Using definition (5.3), (5.1) becomes

$$S(\xi, R(U,V)Z)X - S(X, R(U,V)Z)\xi + S(\xi,U)R(X,V)Z - S(X,U)R(\xi,V)Z$$
$$+ S(\xi,V)R(U,X)Z - S(X,V)R(U,\xi)Z + S(\xi,Z)R(U,V)X - S(X,Z)R(U,V)\xi = 0.$$

Taking the inner product of above equation with ξ, we have

$$\begin{aligned} &S(\xi, R(U,V)Z)\eta(X) - S(X, R(U,V)Z) + S(\xi,U)\eta(R(X,V)Z) \\ &\quad - S(X,U)\eta(R(\xi,V)Z) + S(\xi,V)\eta(R(U,X)Z) - S(X,V)\eta(R(U,\xi)Z) \\ &\qquad + S(\xi,Z)\eta(R(U,V)X) - S(X,Z)\eta(R(U,V)\xi) = 0, \end{aligned}$$

$\epsilon \neq 0$, which by putting $V = Z = \xi$ and using (2.6)-(2.8) reduces to

$$\begin{aligned} &S(X,U) - \eta(U)S(X,\xi) - \eta(X)S(\xi,U) + S(\xi,\xi)\eta(X)\eta(U) \\ &\qquad + \epsilon S(\xi,\xi)g(X,U) - S(\xi,\xi)\eta(X)\eta(U) = 0. \end{aligned} \tag{5.4}$$

In view of (3.5), (5.4) takes the form

$$\begin{aligned} S(X,U) = \epsilon&\Big(n - 1 + \epsilon\lambda + \mu - \frac{\epsilon}{2}\Big(p + \frac{2}{n}\Big)\Big)g(X,U) \\ &- 2\Big(n - 1 + \epsilon\lambda + \mu - \frac{\epsilon}{2}\Big(p + \frac{2}{n}\Big)\Big)\eta(X)\eta(U). \end{aligned} \tag{5.5}$$

Now taking $X = U = \xi$ in (5.5) and using (2.10), we find

$$\lambda + \epsilon\mu = \frac{1}{2}\Big(p + \frac{2}{n}\Big). \tag{5.6}$$

Thus we can state the following:

THEOREM 5.1. *If an n-dimensional ϵ-Kenmotsu manifold admitting $*$-conformal η-Ricci soliton satisfies $S(\xi, X) \cdot R = 0$, then the scalars λ and μ are related by $\lambda + \epsilon\mu = \frac{1}{2}(p + \frac{2}{n})$.*

Now from equations (5.5) and (5.6), we get

$$S(X, U) = \epsilon(n-1)g(X, U) - 2(n-1)\eta(X)\eta(U). \tag{5.7}$$

Thus we have the following:

COROLLARY 5.1. *If an n-dimensional ϵ-Kenmotsu manifold admitting $*$-conformal η-Ricci soliton satisfies $S(\xi, X) \cdot R = 0$, then the manifold is an η-Einstein manifold of the form* (5.7).

6. $*$-conformal η-Ricci solitons in ϵ-Kenmotsu manifolds satisfying $C(\xi, X) \cdot S = 0$

Let an n-dimensional ϵ-Kenmotsu manifold admitting $*$-conformal η-Ricci soliton satisfies $C(\xi, X) \cdot S = 0$. Then we have

$$S(C(\xi, X)Y, Z) + S(Y, C(\xi, X)Z) = 0. \tag{6.1}$$

From (2.12), we find

$$C(\xi, X)Y = \Big(1 + \frac{\epsilon r}{n(n-1)}\Big)(\eta(Y)X - \epsilon g(X, Y)\xi). \tag{6.2}$$

By making use of (6.2) in (6.1), we have

$$\Big(1+\frac{\epsilon r}{n(n-1)}\Big)[\eta(Y)S(X,Z)-\epsilon g(X,Y)S(\xi,Z)+\eta(Z)S(X,Y)-\epsilon g(X,Z)S(Y,\xi)] = 0$$

which by putting $Z = \xi$ and using (2.1), (2.2) and (3.5) reduces to

$$\Big(1 + \frac{\epsilon r}{n(n-1)}\Big)\Big[S(X, Y) + \epsilon\Big(n - 1 + \epsilon\lambda + \mu - \frac{\epsilon}{2}\Big(p + \frac{2}{n}\Big)\Big)g(X, Y)\Big] = 0.$$

Therefore we have either $r = -\epsilon n(n-1)$, or

$$S(X, Y) = -\epsilon\Big(n - 1 + \epsilon\lambda + \mu - \frac{\epsilon}{2}\Big(p + \frac{2}{n}\Big)\Big)g(X, Y). \tag{6.3}$$

From the equations (3.4) and (6.3), we obtain

$$\mu(g(X, Y) - \epsilon\eta(X)\eta(Y)) = 0 \implies \mu g(\phi X, \phi Y) = 0$$

from which it follows that $\mu = 0$ and $g(\phi X, \phi Y) \neq 0$. Thus we have the following:

THEOREM 6.1. *If an n-dimensional ϵ-Kenmotsu manifold admitting $*$-conformal η-Ricci soliton satisfying $C(\xi, X) \cdot S = 0$, then either the scalar curvature is constant or $*$-conformal η-Ricci soliton on the manifold becomes a $*$-conformal Ricci soliton.*

By virtue of (3.6), (6.3) turns to

$$S(X,Y) = -\epsilon(n-1)g(X,Y). \tag{6.4}$$

Thus we have the following:

Corollary 6.1. *If an n-dimensional ϵ-Kenmotsu manifold admitting ∗-conformal η-Ricci soliton satisfies $C(\xi,X)\cdot S = 0$, then the manifold is an Einstein manifold of the form* (6.4).

7. ϕ-concircularly flat ϵ-Kenmotsu manifolds admitting ∗-conformal η-Ricci solitons

Definition 7.1. An ϵ-Kenmotsu manifold is said to be ϕ-concircularly flat if

$$\phi^2 C(\phi X,\phi Y)\phi Z = 0 \tag{7.1}$$

for all X, Y, Z on M.

Let M be an n-dimensional ϕ-concircularly flat ϵ-Kenmotsu manifold admitting ∗-conformal η-Ricci soliton. Therefore from (7.1), it follows that

$$g(C(\phi X,\phi Y)\phi Z,\phi W) = 0. \tag{7.2}$$

In view of (2.12), (7.2) turns to

$$\begin{aligned} g[R(\phi X,\phi Y)\phi Z,\phi W] = \frac{r}{n(n-1)}&[g(\phi Y,\phi Z)g(\phi X,\phi W)\\ &- g(\phi X,\phi Z)g(\phi Y,\phi W)]. \end{aligned} \tag{7.3}$$

Let $\{e_1, e_2, ..., e_{n-1}, \xi\}$ be a local orthonormal basis of the vector fields on M. Using that $\{\phi e_1, \phi e_2, ..., \phi e_{n-1}, \xi\}$ is also a local orthonormal basis. If we put $X = W = e_i$ in (7.3) and sum up with respect to i $(1 \leqslant i \leqslant n-1)$, then we have

$$\begin{aligned} \sum_{i=1}^{n-1} g[R(\phi e_i,\phi Y)\phi Z,\phi e_i] = \frac{r}{n(n-1)}\sum_{i=1}^{n-1}&[g(\phi Y,\phi Z)g(\phi e_i,\phi e_i)\\ &- g(\phi e_i,\phi Z)g(\phi Y,\phi e_i)]. \end{aligned} \tag{7.4}$$

It can be easily verified that

$$\sum_{i=1}^{n-1} g[R(\phi e_i,\phi Y)\phi Z,\phi e_i] = S(\phi Y,\phi Z) + \epsilon g(\phi Y,\phi Z), \tag{7.5}$$

$$\sum_{i=1}^{n-1} g(\phi e_i,\phi Z)g(\phi Y,\phi e_i) = g(\phi Y,\phi Z), \tag{7.6}$$

$$\sum_{i=1}^{n-1} g(\phi e_i,\phi e_i) = (n-1). \tag{7.7}$$

By using (7.5)–(7.7) in (7.4), we obtain

$$S(\phi Y,\phi Z) = \Big[\frac{r(n-2)}{n(n-1)} - \epsilon\Big]g(\phi Y,\phi Z). \tag{7.8}$$

By virtue of (3.4) and (3.6), we find

$$S(\phi Y,\phi Z)=\epsilon(\mu-n+1)g(\phi Y,\phi Z), \tag{7.9}$$

$$r=\sum_{i=1}^{n}S(e_i,e_i)=\epsilon(n\mu-\mu-n^2+n). \tag{7.10}$$

By using (7.9) and (7.10), (7.8) gives $\epsilon(n-1)\mu g(\phi Y,\phi Z)=0$ from which it follows that $\mu=0$ and $\epsilon(n-1)g(\phi Y,\phi Z)\neq 0$. Thus we have the following:

Theorem 7.1. *A $*$-conformal η-Ricci soliton in ϕ-concircularly flat ϵ-Kenmotsu manifolds becomes a $*$-conformal Ricci soliton.*

Now by using (3.6) along with $\mu=0$ in (3.4), we obtain

$$S(X,Y)=-\epsilon(n-1)g(X,Y). \tag{7.11}$$

Thus we have the following:

Corollary 7.1. *If an n-dimensional ϕ-concircularly flat ϵ-Kenmotsu manifold admits $*$-conformal η-Ricci soliton, then the manifold is an Einstein manifold of the form* (7.11).

Example 7.1. We consider the 7-dimensional manifold

$$M=\{(x_1,x_2,x_3,x_4,x_5,x_6,z\neq 0)\in\mathbb{R}^7\},$$

where $(x_1,x_2,x_3,x_4,x_5,x_6,z)$ are the standard coordinates in $\mathbb{R}^7$. Let e_1, e_2, e_3, e_4, e_5, e_6 and e_7 be the vector fields on M given by

$$e_1=z\frac{\partial}{\partial x_1},\ e_2=z\frac{\partial}{\partial x_2},\ e_3=z\frac{\partial}{\partial x_3},\ e_4=z\frac{\partial}{\partial x_4},$$
$$e_5=z\frac{\partial}{\partial x_5},\ e_6=z\frac{\partial}{\partial x_6},\ e_7=-\epsilon z\frac{\partial}{\partial z}=\xi.$$

Let g be the indefinite Riemannian metric defined by $g(e_i,e_j)=0$, $i\neq j$, $i,j=1,2,3,4,5,6,7$ and

$$g(e_1,e_1)=g(e_2,e_2)=g(e_3,e_3)=g(e_4,e_4)=g(e_5,e_5)=g(e_6,e_6)=1, g(e_7,e_7)=\epsilon.$$

Let η be the 1-form on M defined by $\eta(X)=\epsilon g(X,e_7)=\epsilon g(X,\xi)$ for all $X\in\chi(M)$. Let ϕ be the $(1,1)$-tensor field on M defined by

$$\phi e_1=e_2,\ \phi e_2=-e_1,\ \phi e_3=e_4,\ \phi e_4=-e_3,\ \phi e_5=e_6,\ \phi e_6=-e_5,\ \phi e_7=0.$$

The linearity property of ϕ and g yields

$$\eta(e_7)=1,\ \ \phi^2X=-X+\eta(X)\xi,\ \ g(\phi X,\phi Y)=g(X,Y)-\epsilon\eta(X)\eta(Y)$$

for all $X,Y\in\chi(M)$. Thus for $e_7=\xi$, the structure $(\phi,\xi,\eta,g,\epsilon)$ defines an indefinite almost contact metric structure on M. Now, by direct computations, we obtain

$$[e_1,e_2]=[e_1,e_3]=[e_1,e_4]=[e_1,e_5]=[e_1,e_6]=[e_2,e_3]=[e_2,e_4]=[e_2,e_5]=0,$$
$$[e_2,e_6]=[e_3,e_4]=[e_3,e_5]=[e_3,e_6]=[e_4,e_5]=[e_4,e_6]=[e_5,e_6]=0,$$
$$[e_1,e_7]=\epsilon e_1,[e_2,e_7]=\epsilon e_2,[e_3,e_7]=\epsilon e_3,[e_4,e_7]=\epsilon e_4,[e_5,e_7]=\epsilon e_5,[e_6,e_7]=\epsilon e_6.$$

The Riemannian connection ∇ of the metric g is given by

$$2g(\nabla_X Y, Z) = Xg(Y, Z) + Yg(Z, X) - Zg(X, Y) \\ - g(X, [Y, Z]) + g(Y, [Z, X]) + g(Z, [X, Y]),$$

which is known as Koszul's formula. Using Koszul's formula, we can easily calculate

$$\nabla_{e_1} e_1 = -e_7,\ \nabla_{e_1} e_2 = 0, \nabla_{e_1} e_3 = 0, \nabla_{e_1} e_4 = 0, \nabla_{e_1} e_5 = 0, \nabla_{e_1} e_6 = 0, \nabla_{e_1} e_7 = \epsilon e_1,$$
$$\nabla_{e_2} e_1 = 0,\ \nabla_{e_2} e_2 = -e_7, \nabla_{e_2} e_3 = 0, \nabla_{e_2} e_4 = 0, \nabla_{e_2} e_5 = 0, \nabla_{e_2} e_6 = 0, \nabla_{e_2} e_7 = \epsilon e_2,$$
$$\nabla_{e_3} e_1 = 0, \nabla_{e_3} e_2 = 0, \nabla_{e_3} e_3 = -e_7, \nabla_{e_3} e_4 = 0, \nabla_{e_3} e_5 = 0, \nabla_{e_3} e_6 = 0, \nabla_{e_3} e_7 = \epsilon e_3,$$
$$\nabla_{e_4} e_1 = 0,\ \nabla_{e_4} e_2 = 0, \nabla_{e_4} e_3 = 0, \nabla_{e_4} e_4 = -e_7, \nabla_{e_4} e_5 = 0, \nabla_{e_4} e_6 = 0, \nabla_{e_4} e_7 = \epsilon e_4,$$
$$\nabla_{e_5} e_1 = 0,\ \nabla_{e_5} e_2 = 0, \nabla_{e_5} e_3 = 0, \nabla_{e_5} e_4 = 0, \nabla_{e_5} e_5 = -e_7, \nabla_{e_5} e_6 = 0, \nabla_{e_5} e_7 = \epsilon e_5,$$
$$\nabla_{e_6} e_1 = 0,\ \nabla_{e_6} e_2 = 0, \nabla_{e_6} e_3 = 0, \nabla_{e_6} e_4 = 0, \nabla_{e_6} e_5 = 0, \nabla_{e_6} e_6 = -e_7, \nabla_{e_6} e_7 = \epsilon e_6,$$
$$\nabla_{e_7} e_1 = 0,\ \nabla_{e_7} e_2 = 0,\ \nabla_{e_7} e_3 = 0,\ \nabla_{e_7} e_4 = 0,\ \nabla_{e_7} e_5 = 0,\ \nabla_{e_7} e_6 = 0,\ \nabla_{e_7} e_7 = 0.$$

Using the above relations, for any vector field X on M, it follows that

$$\nabla_X \xi = \epsilon(X - \eta(X)\xi)$$

for any $\xi \in \chi(M)$. Hence the manifold M under the consideration is an ϵ-Kenmotsu manifold of dimension seven. From the above results, it is not difficult to find

$$R(X, Y)Z = -\epsilon(g(Y, Z)X - g(X, Z)Y)$$

from which it follows that $S(Y, Z) = -6\epsilon g(Y, Z)$ and hence $r = -42\epsilon$. From the equation (3.4), we have

$$\sum_{i=1}^{7} \epsilon_i S(e_i, e_i) = -\Big[6\epsilon + \lambda - \frac{1}{2}\Big(p + \frac{2}{n}\Big)\Big] \sum_{i=1}^{7} \epsilon_i g(Y, Z) - \sum_{i=1}^{7} \epsilon_i \mu \eta(Y)\eta(Z)$$

where $\epsilon_i = g(e_i, e_i)$. This implies

$$\lambda + \frac{1}{7}\epsilon\mu = \frac{1}{2}\Big(p + \frac{2}{7}\Big). \tag{7.12}$$

From equations (3.6) and (7.12), we obtain $\mu = 0$. Therefore, the data (g, ξ, λ, μ) for $\lambda = \frac{1}{2}(p + \frac{2}{7})$ and $\mu = 0$ defines a $*$-conformal Ricci soliton on the manifold $(M, \phi, \xi, \eta, g, \epsilon)$.

Acknowledgement. The authors are thankful to the editor and anonymous referees for their valuable suggestions in the improvement of the paper".

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(Received 16 05 2020)

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