PUBLICATIONS DE L'INSTITUT MATHÉMATIQUE
Nouvelle série, tome 108(122) (2020), 79–89 DOI: https://doi.org/10.2298/PIM2022079G

FURTHER RESULTS ON THE OUTER CONNECTED GEODETIC NUMBER OF A GRAPH

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Abstract. For a connected graph G of order at least two, an outer connected geodetic set S in a connected graph G is called a *minimal outer connected geodetic set* if no proper subset of S is an outer connected geodetic set of G. The *upper outer connected geodetic number* $g_{\mathrm{oc}}^{+}(G)$ of G is the maximum cardinality of a minimal outer connected geodetic set of G. We determine bounds for it and find the upper outer connected geodetic number of some standard graphs. Some realization results on the upper outer connected geodetic number of a graph are studied. The proposed method can be extended to the identification of beacon vertices towards the network fault-tolerant in wireless local access network communication. Also, another parameter *forcing outer connected geodetic number* $f_{\mathrm{og}}(G)$ of a graph G is introduced and several interesting results and realization theorem are proved.

1. Introduction

By a graph $G = (V, E)$ we mean a finite simple undirected connected graph. The order and size of G are denoted by p and q, respectively. For basic graph theoretic terminology we refer to Harary [**1**, **8**]. The *distance* $d(x, y)$ is the length of a shortest $x - y$ path in G. A $x - y$ path of length $d(x, y)$ is called $x - y$ *geodesic*. A vertex v of G is said to lie on a $x - y$ geodesic P if v is a vertex of P including the vertices x and y. For any vertex u of G, the *eccentricity* of u is $e(u) = \max\{d(u, v) : v \in V(G)\}$. The *radius* $\mathrm{rad}(G)$ and *diameter* $\mathrm{diam}(G)$ of G are defined as $\mathrm{rad}(G) = \min\{e(v) : v \in V(G)\}$ and $\mathrm{diam}(G) = \max\{e(v) : v \in V(G)\}$, respectively. The *neighborhood* of a vertex v is the set $N(v)$ consisting of all vertices u which are adjacent with v. A vertex v is called an *extreme vertex* of G if the subgraph induced by its neighbors is complete.

The *closed interval* $I[x, y]$ consists of all vertices lying on some $x - y$ geodesic of G, while for $S \subseteq V$, $I[S] = \bigcup_{x,y \in S} I[x, y]$. A set S of vertices of G is a *geodetic*

2010 *Mathematics Subject Classification:* Primary 05C12.

Key words and phrases: geodetic set, outer connected geodetic set, outer connected geodetic number, upper outer connected geodetic set, forcing outer connected geodetic number.

The first author research work was supported by National Board for Higher Mathematics (NBHM), Department of Atomic Energy (DAE), Government of India. Project No.NBHM/R.P.29/2015/Fresh/157.

set if $I[S] = V$ and the minimum cardinality of a geodetic set of G is the *geodetic number* $g(G)$ *of* G. The geodetic number of a graph and its variants have been studied by several authors in [**2**–**6**, **9**, **10**]. A set S of vertices in a graph G is said to be an *outer connected geodetic set* if S is a geodetic set of G and either $S = V$ or the subgraph induced by $V - S$ is connected. The minimum cardinality of an outer connected geodetic set of G is the *outer connected geodetic number* of G and is denoted by $g_{\mathrm{oc}}(G)$. The outer connected geodetic number of a graph was introduced and studied in [**7**]. The following theorems will be used in the sequel.

Theorem 1.1. *[**7**] Each extreme vertex of a connected graph G belongs to every outer connected geodetic set of G.*

Theorem 1.2. *[**7**] For the complete graph $K_p (p \geqslant 2)$, $g_{\mathrm{oc}}(K_p) = p$.*

Theorem 1.3. *[**7**] If T is a tree with k endvertices, then $g_{\mathrm{oc}}(T) = k$.*

Throughout this paper G denotes a connected graph with at least two vertices.

2. Upper outer connected geodetic number

Definition 2.1. An outer connected geodetic set S in a connected graph G is called a *minimal outer connected geodetic set* if no proper subset of S is an outer connected geodetic set of G. The *upper outer connected geodetic number* $g^{+}_{\mathrm{oc}}(G)$ of G is the maximum cardinality of a minimal outer connected geodetic set of G.

Example 2.1. For the graph G given in Figure 2.1, the minimal outer connected geodetic sets of G are $S = \{v_1, v_2, v_6\}$ and $S' = \{v_1, v_2, v_5, v_8\}$. It follows from the definitions of the outer connected geodetic number and the upper outer connected geodetic number, we have $g_{\mathrm{oc}}(G) = 3$ and $g^{+}_{\mathrm{oc}}(G) = 4$. Thus the outer connected geodetic number and the upper outer connected geodetic number of a graph G are different.

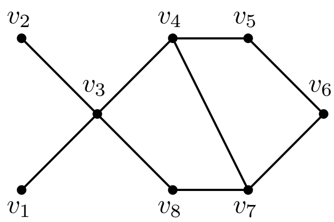

Figure 2.1: G

In an application point of view, for a Wireless Local Access Network (WLAN) modeled as a graph where vertices represent the access points and two vertices are adjacent if there are locations in the designated coverage area where the signals would interfere both access points. Beacon vertices supervise a group of vertices which lie on a geodesic joining them. The entire network can be covered with a set of Beacon vertices(SBV) with the minimum cardinality. If SBV fails, in order to make the network fault-tolerance, the rest of the vertices are able to communicate with each other. A SBV of a graph is called minimal if no proper subset S of SBV cover the network through the geodesic joining any pair of vertices in S. A fault-tolerant network can be designed with the minimal SBV to improve the Wireless Local Access Networks(WLAN) communication.

Remark 2.1. Every minimum outer connected geodetic set of G is a minimal outer connected geodetic set of G. The converse need not be true. For the graph G given in Figure 2.1, $S' = \{v_1, v_2, v_5, v_8\}$ is a minimal outer connected geodetic set and it is not a minimum outer connected geodetic set of G.

Theorem 2.1. *Every extreme vertex of a connected graph G belongs to every minimal outer connected geodetic set of G.*

Proof. Since every minimal outer connected geodetic set is an outer connected geodetic set of G, the result follows from Theorem 1.1. □

Result 2.1. *For the complete graph K_p, $g^+_{oc}(K_p) = p$.*

Result 2.2. *If T is a tree with k endvertices, then $g^+_{oc}(T) = k$.*

Result 2.3. *For the star $K_{1,p-1}(p \geqslant 1)$, $g^+_{oc}(K_{1,p-1}) = p - 1$.*

Theorem 2.2. *For any connected graph G of order $p \geqslant 2$, $2 \leqslant g_{oc}(G) \leqslant g^+_{oc}(G) \leqslant p$.*

Proof. Any outer connected geodetic set of G needs at least two vertices and so $g_{oc}(G) \geqslant 2$. Since every minimal outer connected geodetic set of G is also an outer connected geodetic set of G, it follows that $g_{oc}(G) \leqslant g^+_{oc}(G)$. Also, it is clear that $V(G)$ induces a minimal outer connected geodetic set of G and so $g^+_{oc}(G) \leqslant p$. Thus $2 \leqslant g_{oc}(G) \leqslant g^+_{oc}(G) \leqslant p$. □

Remark 2.2. The bounds in Theorem 2.2 are sharp. For any non-trivial path $P_n(n \geqslant 3)$, $g_{oc}(P_n) = 2$ and for the complete graph $K_p(p \geqslant 2)$, $g^+_{oc}(K_p) = p$. Also, all the inequalities in Theorem 2.2 can be strict. For the graph G given in Figure 2.1, $g_{oc}(G) = 3$, $g^+_{oc}(G) = 4$ and $p = 8$ so that $2 < g_{oc}(G) < g^+_{oc}(G) < p$.

The next result follows from Theorems 2.1 and 2.2.

Result 2.4. *For any connected graph G of order $p \geqslant 2$ with k extreme vertices, max $\{2, k\} \leqslant g^+_{oc}(G) \leqslant p$.*

Theorem 2.3. *For any connected graph G of order p, $g_{oc}(G) = p$ if and only if $g^+_{oc}(G) = p$.*

Proof. If $g^+_{oc}(G) = p$, then $S = V(G)$ is the unique minimal outer connected geodetic set of G. Since no proper subset of S is an outer connected geodetic set, S is the unique minimum outer connected geodetic set of G so that $g_{oc}(G) = p$. The converse follows from Theorem 2.2. □

Theorem 2.4. *If any connected graph G has a unique minimal outer connected geodetic set S then $g^+_{oc}(G) = g_{oc}(G) = |S|$.*

Proof. Since S is the unique minimal outer connected geodetic set of G and no proper subset of S is an outer connected geodetic set of G, $g^+_{oc}(G) = g_{oc}(G) = |S|$. □

The converse of Theorem 2.4 need not be true. For the cycle C_4, any set of three vertices of $V(C_4)$ is minimal, as well as a minimum outer connected geodetic set of C_4. Note that C_4 does not have a unique minimal outer connected geodetic set.

In view of Theorem 2.2, we have the following realization theorem.

Theorem 2.5. *For any two integers a, b such that* $4 \leqslant a \leqslant b \leqslant p-3$*, there is a connected graph G of order p with* $g_{oc}(G) = a$ *and* $g^+_{oc}(G) = b$*.*

Proof. We prove this theorem by considering four cases.
Case 1. $4 \leqslant a = b = p-3$. By Theorem 1.3 and Result 2.2, any tree of order p with three internal vertices has the desired properties.

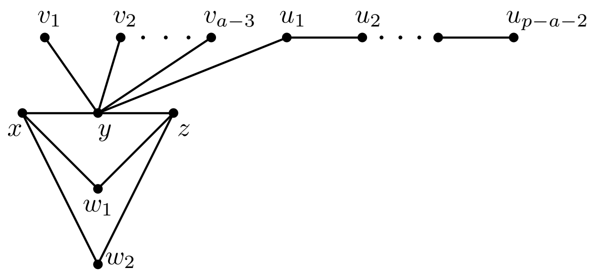

Figure 2.2: G

Case 2. $4 \leqslant a = b < p-3$. Let $P_{p-a-2} : u_1, u_2, \ldots, u_{p-a-2}$ be a path of order $p-a-2$ and let $P_3 : x, y, z$ be a path of order 3. Let G be the graph obtained from P_3 and P_{p-a-2} by adding $a-1$ new vertices v_1, v_2,. . . ,v_{a-3}, w_1, w_2 and joining each $v_i (1 \leqslant i \leqslant a-3)$ to the vertex y of P_3; and joining the vertex u_1 of P_{p-a-2} to the vertex y of P_3; and also joining the vertices w_1, w_2 to the vertices x, z of P_3. The graph G of order p is shown in Figure 2.2. Let $S = \{v_1, v_2, \ldots, v_{a-3}, u_{p-a-2}\}$ be the set of all extreme vertices of G. By Theorems 1.1 and 2.1, every outer connected geodetic set and every minimal outer connected geodetic set of G contain S. It is clear that S is not an outer connected geodetic set of G. Also for any vertex $u \in V - S$, $S \cup \{u\}$ is neither an outer connected geodetic set nor a minimal outer connected geodetic set of G. It is clear that S_1=$S \cup \{w_1, w_2\}$ is the unique minimum outer connected geodetic set and so $g_{oc}(G)$= a. It is easy to verify that, S_1 is the unique minimal outer connected geodetic set of G and so $g^+_{oc}(G) = a$.
Case 3. $4 \leqslant a < b = p-3$. Let $P_3 : x, y, z$ be a path of order 3. Let G be the graph obtained from P_3 by adding $p-3$ new vertices $v_1, v_2, \ldots, v_{a-3}, w_1, w_2, \ldots, w_{p-a}$ and joining each $v_i (1 \leqslant i \leqslant a-3)$ to the vertex y of P_3; and joining each $w_i (1 \leqslant i \leqslant p-a)$ to the vertices x, z of P_3; and also joining each w_i $(2 \leqslant i \leqslant p-a)$ with the vertex w_1. The graph G of order p is shown in Figure 2.3. Let $S = \{v_1, v_2, \ldots, v_{a-3}\}$ be the set of all extreme vertices of G. By Theorem 1.1, every outer connected geodetic set of G contains S.

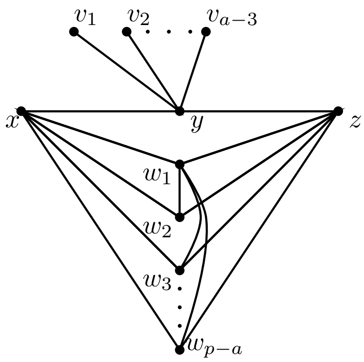

Figure 2.3: G

It is clear that S is not an outer connected geodetic set of G. Also, for any two vertices $u, v \in V - S$, $S \cup \{u, v\}$ is not an outer connected geodetic set of G. It is easy to verify that $S \cup \{x, y, z\}$ is a minimum outer connected geodetic set of G and so $g_{\mathrm{oc}}(G) = a$. The minimal outer connected geodetic sets of G are $S_1 = S \cup \{x, y, z\}$ and $S_2 = S \cup \{w_1, w_2, \ldots, w_{p-a}\}$. By the definition of the upper outer connected geodetic number of a graph G, we have $g^{+}_{\mathrm{oc}}(G) = p - 3$.

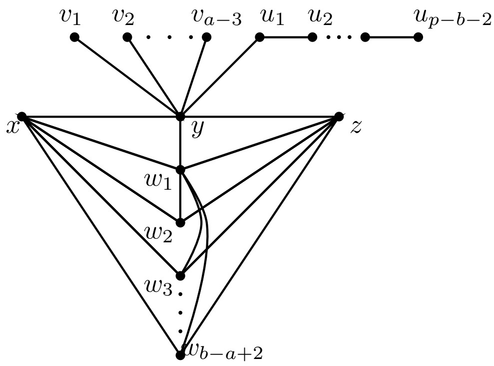

Figure 2.4: G

Case 4. $4 \leqslant a < b < p - 3$. Let $P_{p-b-2} : u_1, u_2, \ldots, u_{p-b-2}$ be a path of order $p - b - 2$ and let $P_3 : x, y, z$ be a path of order 3. Let G be the graph obtained from P_3 and P_{p-b-2} by adding $b-1$ new vertices $v_1, v_2, \ldots, v_{a-3}, w_1, w_2, \ldots, w_{b-a+2}$ and joining each $v_i (1 \leqslant i \leqslant a - 3)$ to the vertex y of P_3; and joining the vertex u_1 of P_{p-b-2} to the vertex y of P_3; and joining each $w_i (1 \leqslant i \leqslant b - a + 2)$ to the vertices x, z of P_3; and also joining each $w_i (2 \leqslant i \leqslant b - a + 2)$ to the vertex w_1; and joining the vertex y of P_3 to the vertex w_1, thereby producing the graph G of order p shown in Figure 2.4. Let $S = \{v_1, v_2, \ldots, v_{a-3}, u_{p-b-2}\}$ be the set of all extreme vertices of G. By Theorem 1.1, every outer connected geodetic set of G contains S. It is clear that S is not an outer connected geodetic set of G. Also, for any vertex $u \in V - S$, $S \cup \{u\}$ is not an outer connected geodetic set of G. It is easy to verify that $S \cup \{x, z\}$ is a minimum outer connected geodetic set of G and so $g_{\mathrm{oc}}(G) = a$. The minimal outer connected geodetic sets of G are $S' = S \cup \{x, z\}$ and

$S'' = S \cup \{w_1, w_2, \ldots, w_{b-a+2}\}$. By the definition of the upper outer connected geodetic number of a graph G, we have $g^+_{oc}(G) = b$. □

For every connected graph G, $\text{rad}(G) \leqslant \text{diam}(G) \leqslant 2\,\text{rad}(G)$. Ostrand[11] showed that every two positive integers a and b with $a \leqslant b \leqslant 2a$ are realizable as the radius and diameter respectively, of some connected graph. Now, Ostrands theorem can be extended so that the upper outer connected geodetic number can also be prescribed.

Theorem 2.6. *For any three positive integers r, d and $k \geqslant 2$ such that $r < d \leqslant 2r$, there is a connected graph G with $\text{rad}(G) = r$, $\text{diam}(d) = d$ and $g^+_{oc}(G) = k$.*

Proof. If $r = 1$, then $d = 2$. Then by Result 2.3, the star $K_{1,k}$ has the desired property.

Now, let $r \geqslant 2$ and $r < d \leqslant 2r$. Let $C_{2r} : u_1, u_2, \ldots, u_{2r}, u_1$ be a cycle of order $2r$ and let P_{d-r+1}: $v_0, v_1, \ldots, v_{d-r}$ be a path of length $d - r$. Let H be the graph obtained from C_{2r} and P_{d-r+1} by identifying the vertex v_0 of P_{d-r+1} and the vertex u_1 of C_{2r}; and joining the vertex u_{r+2} to the vertex u_r. Add $k-2$ new vertices $w_1, w_2, \ldots, w_{k-2}$ to the graph H and join each vertex $w_i (1 \leqslant i \leqslant k-2)$ to the vertex v_{d-r-1}, thereby producing the graph G shown in Figure 2.5. It is easy to verify that $r \leqslant e(x) \leqslant d$ for any vertex x in G, $e(u_1) = r$, $e(v_{d-r}) = d = e(u_{r+1})$ and $e(w_i) = d(1 \leqslant i \leqslant k-2)$. Thus $\text{rad}(G) = r$ and $\text{diam}(G) = d$. Let $S = \{w_1, w_2, \ldots, w_{k-2}, v_{d-r}, u_{r+1}\}$ be the set of all extreme vertices of G. By Theorem 2.1, every minimal outer connected geodetic set of G contains S. It is clear that S is the unique minimal outer connected geodetic set of G and so $g^+_{oc}(G) = k$. □

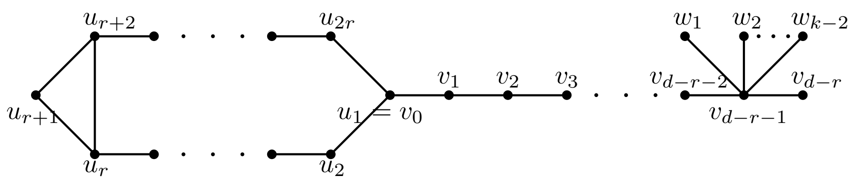

Figure 2.5: G

Problem 2.1. *For any three positive integers r, d and $k \geqslant 2$ such that $r = d \leqslant 2r$, does there exist a connected graph G with $\text{rad}(G) = r$, $\text{diam}(G) = d$ and $g^+_{oc}(G) = k$?*

Theorem 2.7. *If p, d and k are integers such that $2 \leqslant d < p$, $2 \leqslant k < p$ and $p - d - k + 1 \geqslant 0$, then there exists a connected graph G of order p, $\text{diam}(G) = d$ and $g^+_{oc}(G) = k$.*

Proof. Let $P_d : u_0, u_1, u_2, \ldots, u_d$ be a path of length d. Let $K_{p-d-k+1}$ be the complete graph of order $p - d - k + 1$. Let H be the graph obtained from the path P_d and the complete graph $K_{p-d-k+1}$ by joining each vertex of $K_{p-d-k+1}$ to the vertices u_0, u_1 and u_2 of P_d. Add $k-2$ new vertices $v_1, v_2, \ldots, v_{k-2}$ to the graph H and join each vertex $v_i (1 \leqslant i \leqslant k-2)$ to the vertex u_1 of P_d. The graph G of order p is shown in Figure 2.6. It is easy to verify that $1 \leqslant e(x) \leqslant d$ for any vertex x

in G, $e(u_0) = e(u_d) = d$. Thus $\text{diam}(G) = d$. Let $S = \{u_0, u_d, v_1, v_2, \ldots, v_{k-2}\}$ be the set of all extreme vertices of G. It is clear that S is the unique minimal outer connected geodetic set of G and it follows from Theorem 2.4 that $g^+_{oc}(G) = k$. □

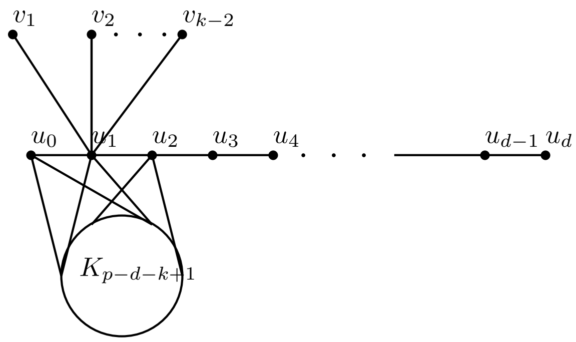

Figure 2.6: G

3. Forcing outer connected geodetic number

Definition 3.1. Let S be a minimum outer connected geodetic set of G. A subset T of a minimum outer connected geodetic set S of G is a *forcing outer connected geodetic subset* for S if S is the unique minimum outer connected geodetic set containing T. A forcing outer connected geodetic subset for S of minimum cardinality is a *minimum forcing outer connected geodetic subset* of S. The forcing outer connected geodetic number $f_{og}(S)$ in G is the cardinality of a minimum forcing outer connected geodetic subset of S. The *forcing outer connected geodetic number* of G is $f_{og}(G) = \min\{f_{og}(S)\}$, where the minimum is taken over all minimum outer connected geodetic sets S in G.

Example 3.1. For the cycle C_4, it is clear that no 2-element subset of $V(C_4)$ is an outer connected geodetic set of C_4. It is easy to verify that any 3-element subset of $V(C_4)$ is a minimum outer connected geodetic set of C_4 and so $g_{oc}(C_4) = 3$. Since no minimum outer connected geodetic set of C_4 is the unique minimum outer connected geodetic set containing any of its proper subsets, $f_{og}(C_4) = 3$.

Theorem 3.1. *For a connected graph G of order p, $0 \leqslant f_{og}(G) \leqslant g_{oc}(G) \leqslant p$.*

Proof. By the definition of the forcing outer connected geodetic number of a graph, it is clear that $f_{og}(G) \geqslant 0$. Let S be a minimum outer connected geodetic set of G. Clearly, $f_{og}(S) \leqslant |S|$ and $f_{og}(G) = min\{f_{og}(S)\}$, where the minimum is taken over all minimum outer connected geodetic sets S in G. Hence $0 \leqslant f_{og}(G) \leqslant g_{oc}(G) \leqslant p$. □

Remark 3.1. The bounds in Theorem 3.1 are sharp. By Theorem 1.3, for any path $P_n (n \geqslant 2)$, the set of endvertices of P_n is the unique minimum outer connected geodetic set of P_n and so $f_{og}(P_n) = 0$. By Theorem 1.2, for the complete graph $K_p (p \geqslant 2)$, $g_{oc}(K_p) = p$. Also, all the inequalities in Theorem 3.1 can be strict. For the graph G given in Figure 3.1 of order 7, it is clear that no 2-element subset of $V(G)$ is an outer connected geodetic set of G. The minimum outer connected geodetic sets of G are $S_1 = \{v_1, v_5, v_6\}$ and $S_2 = \{v_1, v_5, v_7\}$ so that $g_{oc}(G) = 3$. It is clear that $f_{og}(S_i) = 1 (i = 1, 2)$ and so $f_{og}(G) = 1$. Thus $0 < f_{og}(G) < g_{oc}(G) < p$.

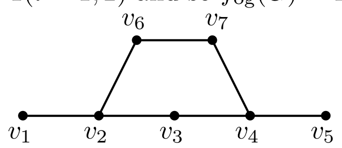

Figure 3.1: G

The following theorem is an easy consequence of the definitions of the outer connected geodetic number and the forcing outer connected geodetic number. In fact, the theorem characterizes graphs G for which the lower bound in Theorem 3.1 is attained and also graphs G for which $f_{og}(G) = 1$ and $f_{og}(G) = g_{oc}(G)$.

THEOREM 3.2. *Let G be a connected graph. Then*
(i) $f_{og}(G) = 0$ *if and only if G has a unique minimum outer connected geodetic set.*
(ii) $f_{og}(G) = 1$ *if and only if G has at least two minimum outer connected geodetic sets, one of which is a unique minimum outer connected geodetic set containing one of its elements*
(iii) $f_{og}(G) = g_{oc}(G)$ *if and only if no minimum outer connected geodetic set of G is the unique minimum outer connected geodetic set containing any of its proper subsets.*

A vertex v of a connected graph G is said to be an *outer connected geodetic vertex* of G if v belongs to every minimum outer connected geodetic set of G.

REMARK 3.2. If G has the unique minimum outer connected geodetic set S, then every vertex in S is an outer connected geodetic vertex of G. Also, if x is an extreme vertex of G, then x is an outer connected geodetic vertex of G. For the graph G given in Figure 3.1, v_1 and v_5 are the outer connected geodetic vertices of G.

The next theorem and corollary are immediate consequence of the definitions of an outer connected geodetic vertex and the forcing outer connected geodetic subset.

THEOREM 3.3. *Let Ψ_{oc} be the set of relative complements of the minimum forcing outer connected geodetic subsets in their respective minimum outer connected geodetic sets in a connected graph G. Then $\bigcap_{F \in \Psi_{oc}} F$ is the set of all outer connected geodetic vertices of G.*

COROLLARY 3.1. *Let S be a minimum outer connected geodetic set of a connected graph G. Then no outer connected geodetic vertex of G belongs to any minimum forcing outer connected geodetic subset of S.*

THEOREM 3.4. *Let M be the set of all outer connected geodetic vertices of a connected graph G. Then $f_{og}(G) \leqslant g_{oc}(G) - |M|$.*

PROOF. Let S be any minimum outer connected geodetic set of G. Then $g_{oc}(G) = |S|$, $M \subseteq S$ and S is the unique minimum outer connected geodetic set containing $S - M$. Thus $f_{og}(G) \leqslant |S - M| = |S| - |M| = g_{oc}(G) - |M|$. □

COROLLARY 3.2. *If G is a connected graph with m extreme vertices, then $f_{\rm og}(G) \leqslant g_{\rm oc}(G) - m$.*

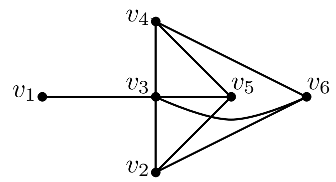

Figure 3.2: G

REMARK 3.3. The bound in Theorem 3.4 is sharp. For the graph G given in Figure 3.1, $g_{\rm oc}(G) = 3$ and $f_{\rm og}(G) = 1$. Also, $M = \{v_1, v_5\}$ is the set of all outer connected geodetic vertices of G and so $f_{\rm og}(G) = g_{\rm oc}(G) - |M|$. Also, all the inequalities in Theorem 3.4 can be strict. For the graph G given in Figure 3.2, the minimum outer connected geodetic sets of G are $S_1 = \{v_1, v_2, v_4\}$ and $S_2 = \{v_1, v_5, v_6\}$ and so $g_{\rm oc}(G) = 3$. It is clear that $f_{\rm og}(S_i) = 1 (i = 1, 2)$ and so $f_{\rm og}(G) = 1$. Also, the vertex v_1 is the only outer connected geodetic vertex of G, we have $f_{\rm og}(G) < g_{\rm oc}(G) - |M|$.

Next, we proceed to determine the forcing outer connected geodetic number of certain classes of graphs.

THEOREM 3.5. *For any complete graph $G = K_p (p \geqslant 2)$ or any non-trivial tree $G = T$, $f_{\rm og}(G) = 0$.*

PROOF. Let $G = K_p$. By Theorem 1.2, the set of all vertices of G is the unique minimum outer connected geodetic set of G and so by Theorem 3.2 (i), $f_{\rm og}(G) = 0$. If G is a non-trivial tree, then by Theorem 1.3, the set of all endvertices of G is the unique minimum outer connected geodetic set of G and so by Theorem 3.2 (i), $f_{\rm og}(G) = 0$. □

THEOREM 3.6. *For the complete bipartite graph $G = K_{m,n} (2 \leqslant m \leqslant n)$,*

$$f_{\rm og}(G) = \begin{cases} n+1 & \text{if } 2 = m \leqslant n \\ 4 & \text{if } 3 \leqslant m \leqslant n \end{cases}.$$

PROOF. Let $U = \{u_1, u_2, \dots, u_m\}$ and $W = \{w_1, w_2, \dots, w_n\}$ be the partite sets of G, where $m \leqslant n$. We prove this theorem by considering two cases.
Case 1. If $m = 2$, then it is clear that any minimum outer connected geodetic sets of G is of the form $V(G) - \{w_i\} (1 \leqslant i \leqslant n)$ or $V(G) - \{u_j\} (1 \leqslant j \leqslant m)$. It is easy to verify that, no minimum outer connected geodetic set of G is the unique minimum outer connected geodetic set containing any of its proper subsets. Then by Theorem 3.2(iii), we have $f_{\rm og}(G) = n + 1$.
Case 2. If $3 \leqslant m \leqslant n$, then any minimum outer connected geodetic set of G is obtained by choosing any two elements from U as well as W, and G has at least two minimum outer connected geodetic sets. Hence $g_{\rm oc}(G) = 4$. Clearly, no minimum outer connected geodetic set of G is the unique minimum outer connected geodetic set containing any of its proper subsets. Then by Theorem 3.2(iii), we have $f_{\rm og}(G) = g_{\rm oc}(G) = 4$. □

THEOREM 3.7. *For every pair* a, b *of integers such that* $0 \leqslant a < b$ *and* $b \geqslant a+2$, *there is a connected graph* G *with* $f_{\mathrm{og}}(G) = a$ *and* $g_{\mathrm{oc}}(G) = b$.

PROOF. If $a = 0$, let $G = K_b$. Then by Theorem 3.5, $f_{\mathrm{og}}(G) = 0$ and by Theorem 1.2, $g_{\mathrm{oc}}(G) = b$. Now, assume that $0 < a < b$. Let H be the graph obtained from the path $P_3 : u, v, w$ of order 3 and the star $K_{1,b-a-1}$ with the vertex set $V(K_{1,b-a-1}) = \{x, z_1, z_2, \dots, z_{b-a-1}\}$ by identifying the vertex w of P_3 and the central vertex x of $K_{1,b-a-1}$. Let $P_i : x_i, y_i (1 \leqslant i \leqslant a)$ be 'a' copies of the path of order 2. The graph G is obtained from H and $P_i(1 \leqslant i \leqslant a)$ by joining each x_i of P_i to the vertex of v of H and joining each y_i of P_i to the vertex of w of H. The graph G is shown in Figure 3.3. Let $S = \{z_1, z_2, \dots, z_{b-a-1}, u\}$ be the set of all endvertices of G. By Theorem 1.1, every outer connected geodetic set of G contains S. It is clear that S is not an outer connected geodetic set of G. We observe that every minimum outer connected geodetic set of G contains exactly one vertex from the set $\{x_i, y_i\}$ for every $i(1 \leqslant i \leqslant a)$. Thus $g_{\mathrm{oc}}(G) \geqslant b$. Since $S_1 = S \cup \{x_1, x_2, \dots, x_a\}$ is an outer connected geodetic set of G, it follows that $g_{\mathrm{oc}}(G) = b$.

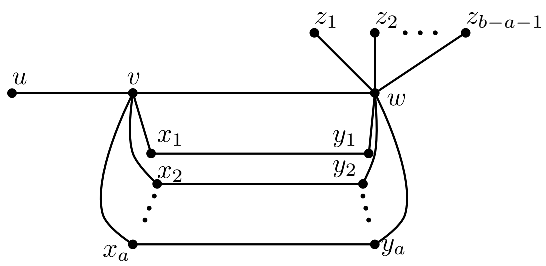

Figure 3.3: G

Next, we show that $f_{\mathrm{og}}(G) = a$. Since every minimum outer connected geodetic set of G contains S, it follows from Theorem 3.4 that $f_{\mathrm{og}}(G) \leqslant g_{\mathrm{oc}}(G) - |S| = b - (b - a) = a$. It is clear that every minimum outer connected geodetic set S' of G is of the form $S \cup \{u_1, u_2, \dots, u_a\}$, where $u_i \in \{x_i, y_i\}$ for every $i(1 \leqslant i \leqslant a)$. Let T be any proper subset of S' with $|T| < a$. Then by Corollary 3.1, there is a vertex $x \in S' - S$ such that $x \notin T$. If $x = x_i(1 \leqslant i \leqslant a)$, then $S'' = (S' - \{x_i\}) \cup \{y_i\}$ is a minimum outer connected geodetic set of G containing T. Similarly, if $x = y_j(1 \leqslant j \leqslant a)$, then $S''' = (S' - \{y_j\}) \cup \{x_j\}$ is a minimum outer connected geodetic set containing T. Thus S' is not the unique minimum outer connected geodetic set containing T and so T is not a forcing outer connected geodetic subset of S'. This is true for all minimum outer connected geodetic sets of G and so $f_{\mathrm{og}}(G) = a$. □

References

1. F. Buckley, F. Harary, *Distance in Graphs*, Addison-Wesley, Redwood City, CA, 1990.
2. F. Buckley, F. Harary, L. V. Quintas, *Extremal results on the geodetic number of a graph*, Scientia. **A2** (1998) 17–26.
3. G. Chartrand, F. Harary, P. Zhang, *On the geodetic number of a graph*, Networks. **39**(1) (2002), 1–6.
4. G. Chartrand, F. Harary, H. C. Swart, P. Zhang, *Geodomination in graphs*, Bull. Inst. Comb. Appl. **31** (2001), 51–59.

5. G. Chartrand, G. L. Johns, P. Zhang, *On the Detour Number and Geodetic Number of a Graph*, Ars Comb. **72** (2004), 3–15.
6. G. Chartrand, E. M. Palmer, P. Zhang, *The geodetic number of a graph*, A survey, Congr. Numer. **156** (2002), 37–58.
7. K. Ganesamoorthy, D. Jayanthi, *The Outer Connected Geodetic Number of a Graph*, Proc. Natl. Acad. Sci. India, Sect. A Phys. Sci. (2020). https://doi.org/10.1007/s40010-020-00661-5.
8. F. Harary, *Graph Theory*, Addision-Wesely (1969).
9. F. Harary, E. Loukakis, C. Tsouros, *The geodetic number of a graph*, Math. Comput. Modelling. **17**(11) (1993), 89–95.
10. R. Muntean, P. Zhang, *On geodomination in graphs*, Congr. Numer. **143** (2000), 161–174.
11. P. A. Ostrand, *Graphs with specified radius and diameter*, Discrete Math. **4** (1973), 71–75.

(Received 30 06 2020)

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