

FURTHER RESULTS ON MULTIPLE q -EULERIAN INTEGRALS FOR VARIOUS q -HYPERGEOMETRIC FUNCTIONS

Thomas Ernst

ABSTRACT. We continue the study of single and multiple q -Eulerian integrals in the spirit of Exton, Driver, Johnston, Pandey, Saran and Erdélyi. The method of proof is often the q -beta integral method with the correct q -power together with the q -binomial theorem. By the Totov method we can prove summation theorems as special cases of multiple q -Eulerian integrals. The Srivastava Δ notation for q -hypergeometric functions is used to enable the shortest possible form of the long formulas. The various q -Eulerian integrals are in fact meromorphic continuations of the various multiple q -functions, suitable for numerical computations. In the end of the paper a generalization of the q -binomial theorem is used to find q -analogues of a multiple integral formulas for q -Kampé de Fériet functions.

1. Introduction

First, in [6], we found several q -Eulerian integral representations and summation formulas for multiple q -hypergeometric functions in the footsteps of Koschmieder and Srivastava-Singhal. Then, in [8], we proved q -integral transformations for q -Lauricella functions in the spirit of Koschmieder and Exton. Finally, in [9], we found q -integral formulas in the spirit of Exton, Choi and Rathie. In this paper we find some more formulas of the same type. The common theme in all these papers is proof by the q -beta integral method. The parameters have restrictions on their real parts for q real, which can be removed by analytic continuation. For notation, see our book [7]. We also use the JHC q -addition, the formula [7, 6.192] with $x = 1$ will be used to switch to q -shifted factorials in the proofs.

This paper is organized as follows: In section 1 we give a general introduction and illustrate the q -beta method with some simple examples. In section 2 we extend the proofs to triple Eulerian integrals. In section 3 we find q -analogues of some of Exton's integral formulas and extend the two q -additions. In section 4 we find q -analogues of some more complicated integral formulas by Totov. This includes a generalization of a summation formula for the fourth q -Lauricella function. Then, in section 5 we prove q -Eulerian integrals with general q -real numbers, which are

often used in the the last step of the proofs. In section 6 we find a q -analogue of the Burchnell-Chaundy integral for the second q -Appell function by using their inverse operators from [7]. In section 7 we find a transformation formula between two intermediate q -Lauricella functions. A general formula in section 8 generalizes a formula by Koschmieder [14] for Lauricella functions. In section 9 we find a q -Euler integral for a q -hypergeometric functions of four variables by using general q -real numbers.

THEOREM 1.1. (*A q -analogue of [4, Theorem 2.5].*) *If $\operatorname{Re} c > \operatorname{Re} b > 0$, we have the following q -Eulerian integral for ${}_{2p+1}\phi_{2p}$:*

$${}_{2p+1}\phi_{2p} \left[\begin{matrix} a, \Delta(q; p; b) \\ \Delta(q; p; c) \end{matrix} \middle| q; x \right] = \Gamma_q \left[\begin{matrix} c \\ b, c - b \end{matrix} \right] \int_0^1 t^{b-1} \frac{(tq; q)_{c-b-1}}{(xt^p; q)_a} d_q(t).$$

PROOF.

$$\begin{aligned} \text{LHS} &= \sum_{m=0}^{\infty} \frac{\langle a; q \rangle_m \langle b; q \rangle_{pm}}{\langle 1; q \rangle_m \langle c; q \rangle_{pm}} x^m \\ &\stackrel{\text{by [7, 1.46]}}{=} \Gamma_q \left[\begin{matrix} c \\ b, c - b \end{matrix} \right] \sum_{m=0}^{\infty} \frac{\langle a; q \rangle_m}{\langle 1; q \rangle_m} x^m B_q(b + pm, c - b) \\ &\stackrel{\text{by [7, 7.55]}}{=} \Gamma_q \left[\begin{matrix} c \\ b, c - b \end{matrix} \right] \sum_{m=0}^{\infty} \frac{\langle a; q \rangle_m}{\langle 1; q \rangle_m} x^m \int_0^1 t^{b+pk-1} (tq; q)_{c-b-1} d_q(t) \\ &= \Gamma_q \left[\begin{matrix} c \\ b, c - b \end{matrix} \right] \int_0^1 t^{b-1} (tq; q)_{c-b-1} \sum_{m=0}^{\infty} \frac{\langle a; q \rangle_m}{\langle 1; q \rangle_m} (xt^p)^m d_q(t) \stackrel{\text{by [7, 7.27]}}{=} \text{RHS}. \square \end{aligned}$$

THEOREM 1.2. (*A q -analogue of [15, p.4].*) *For $|z| < 1$, $\operatorname{Re} e > \operatorname{Re} c > 0$, $\operatorname{Re} d > \operatorname{Re} b > 0$, we have the following double q -Eulerian integral for ${}_3\phi_2$:*

$$(1.1) \quad {}_3\phi_2(a, b, c; d, e | q; z) \cong \Gamma_q \left[\begin{matrix} d, e \\ d - b, e - c, b, c \end{matrix} \right] \int_0^1 \int_0^1 t^{b-1} s^{c-1} \frac{(qt; q)_{d-b-1} (qs; q)_{e-c-1}}{(zst; q)_a} d_q(t) d_q(s).$$

PROOF. Assume that $|z| < 1$, $\operatorname{Re} \delta > -1$, $\operatorname{Re} \beta > -1$, $\operatorname{Re} \gamma > -1$, $\operatorname{Re} \alpha > -1$. By the q -binomial theorem:

$$\begin{aligned} &\int_0^1 \int_0^1 t^\alpha s^\beta \frac{(qt; q)_\gamma (qs; q)_\delta}{(zst; q)_\epsilon} d_q(t) d_q(s) \\ &\stackrel{\text{by [7, 7.27]}}{=} \int_0^1 \int_0^1 t^\alpha s^\beta (qt; q)_\gamma (qs; q)_\delta \sum_{k=0}^{\infty} \frac{\langle \epsilon; q \rangle_k}{\langle 1; q \rangle_k} (zts)^k d_q(t) d_q(s) \\ &= \int_0^1 t^\alpha (qt; q)_\gamma d_q(t) \sum_{k=0}^{\infty} \frac{\langle \epsilon; q \rangle_k}{\langle 1; q \rangle_k} (zt)^k \int_0^1 s^{\beta+k} (qs; q)_\delta d_q(s) \\ &= \sum_{k=0}^{\infty} \frac{\langle \epsilon; q \rangle_k}{\langle 1; q \rangle_k} z^k \int_0^1 t^{\alpha+k} (qt; q)_\gamma d_q(t) \int_0^1 s^{\beta+k} (qs; q)_\delta d_q(s) \end{aligned}$$

$$\begin{aligned} & \text{by [7, 7.55]} \sum_{k=0}^{\infty} \frac{\langle \epsilon; q \rangle_k}{\langle 1; q \rangle_k} z^k \Gamma_q \left[\begin{matrix} \alpha + k + 1, \gamma + 1, \beta + k + 1, \delta + 1 \\ \alpha + \gamma + k + 2, \beta + \delta + k + 2 \end{matrix} \right] \\ &= \Gamma_q \left[\begin{matrix} \alpha + 1, \gamma + 1, \beta + 1, \delta + 1 \\ \alpha + \gamma + 2, \beta + \delta + 2 \end{matrix} \right] {}_3\phi_2 \left(\begin{matrix} \epsilon, \alpha + 1, \beta + 1 \\ \alpha + \gamma + 2, \beta + \delta + 2 \end{matrix} \middle| q; z \right). \end{aligned}$$

Finally, introduce new parameters a, b, c, d, e subject to

$$\begin{aligned} \epsilon &= a, \quad \alpha + \gamma + 2 = d, \quad \gamma = d - b + 1, \quad \alpha + 1 = b, \\ \beta + \delta + 2 &= e, \quad \delta = e - c + 1, \quad \beta + 1 = c. \end{aligned} \quad \square$$

COROLLARY 1.1. A q -analogue of [5, 2.4.1]:

$$\begin{aligned} & {}_2\phi_1(b, c; d|q; z) \\ & \cong \Gamma_q \left[\begin{matrix} d, d \\ d - b, d - c, b, c \end{matrix} \right] \int_0^1 \int_0^1 t^{b-1} s^{c-1} \frac{(qt; q)_{d-b-1} (qs; q)_{d-c-1}}{(zst; q)_d} d_q(t) d_q(s). \end{aligned}$$

PROOF. Put $a = d$ in (1.1). □

DEFINITION 1.1. Put

$$\rho_{n+1}(a, b; c|q; z) \equiv \sum_{k=n+1}^{\infty} \frac{\langle a, b; q \rangle_k}{\langle 1, c; q \rangle_k} z^k.$$

COROLLARY 1.2. A q -analogue of [5, 2.3.11]:

$$\begin{aligned} & \rho_{n+1}(a, b; c|q; z) \\ & \cong \frac{z^{n+1}}{\{n\}_q!} \Gamma_q \left[\begin{matrix} a + n + 1, c \\ a, b, c - b \end{matrix} \right] \int_0^1 \int_0^1 t^{b+n} \frac{(qt; q)_{c-b-1} (qs; q)_n}{(zst; q)_{a+n+1}} d_q(t) d_q(s). \end{aligned}$$

PROOF.

$$\begin{aligned} \text{LHS} &= z^{n+1} \frac{\langle a, b; q \rangle_{n+1}}{\langle 1, c; q \rangle_{n+1}} {}_3\phi_2(a + n + 1, b + n + 1, 1; c + n + 1, n + 2|q; z) \\ & \text{by (1.1)} \cong z^{n+1} \frac{\langle a, b; q \rangle_{n+1} \{n+1\}_q!}{\langle 1, c; q \rangle_{n+1} \{n\}_q!} \Gamma_q \left[\begin{matrix} c + n + 1 \\ c - b, b + n + 1, 1 \end{matrix} \right] \\ & \times \int_0^1 \int_0^1 t^{b+n} \frac{(qt; q)_{c-b-1} (qs; q)_n}{(zst; q)_{a+n+1}} d_q(t) d_q(s) = \text{RHS}. \end{aligned} \quad \square$$

COROLLARY 1.3. A q -analogue of [5, 2.4.2], for $\text{Re } c > \text{Re } s > 0$:

$${}_2\phi_1(a, b; c|q; z) \cong \Gamma_q \left[\begin{matrix} c \\ c - s, s \end{matrix} \right] \int_0^1 t^{s-1} (qt; q)_{c-s-1} {}_2\phi_1(a, b; s|q; zt) d_q(t).$$

PROOF. Put $b = e$ in (1.1) and use [7, 7.50]. □

2. Two triple Eulerian integrals for ${}_4\phi_3$

Triple Eulerian integrals for hypergeometric functions are seldom seen in the literature. The following two integral formulas, one general, and one special, seem to be new even for $q = 1$.

THEOREM 2.1. *For $|z| < 1$, $d \in \mathbb{N}$, $d \geq 2$, $\operatorname{Re} e > \operatorname{Re} b > 0$, $\operatorname{Re} f > \operatorname{Re} c > 0$, we have the following triple Eulerian integral for a general ${}_4\phi_3$:*

$$(2.1) \quad {}_4\phi_3(a, b, c, \infty; d, e, f|q; z(1-q)) \cong \Gamma_q \left[\begin{matrix} e, f \\ e-b, f-c, b, c \end{matrix} \right] \frac{\langle 1; q \rangle_{d-1}}{(1-q)^{d-1}} \\ \times \int_0^1 \int_0^1 \int_0^1 \frac{t^{b-1} s^{c-1} u^{d-2} (qt; q)_{e-b-1} (qs; q)_{f-c-1}}{(zstu; q)_a} d_q(t) d_q(s) d_q(u).$$

PROOF. By the q -binomial theorem:

$$\int_0^1 \int_0^1 \int_0^1 \frac{t^{b-1} s^{c-1} u^{d-2} (qt; q)_{e-b-1} (qs; q)_{f-c-1}}{(zstu; q)_a} d_q(t) d_q(s) d_q(u) \\ \stackrel{\text{by [7, 7.27]}}{=} \int_0^1 \int_0^1 \int_0^1 t^{b-1} s^{c-1} (qt; q)_{e-b-1} (qs; q)_{f-c-1} \\ \sum_{k=0}^{\infty} \frac{\langle a; q \rangle_k}{\langle 1; q \rangle_k} (zts)^k u^{k-2+d} d_q(t) d_q(s) d_q(u) \\ = \int_0^1 \int_0^1 t^{b+k-1} s^{c+k-1} (qt; q)_{e-b-1} (qs; q)_{f-c-1} \\ \sum_{k=0}^{\infty} \frac{\langle a; q \rangle_k}{\langle 1; q \rangle_k \{k+d-1\}_q} z^k d_q(t) d_q(s) \\ \stackrel{\text{by [7, 7.55]}}{=} \sum_{k=0}^{\infty} \frac{\langle a; q \rangle_k (1-q)^{d+k-1}}{\langle 1, d; q \rangle_k \langle 1; q \rangle_{d-1}} z^k \Gamma_q \left[\begin{matrix} b+k, e-b, c+k, f-c \\ e+k, f+k \end{matrix} \right] \\ = \frac{(1-q)^{d-1}}{\langle 1; q \rangle_{d-1}} \Gamma_q \left[\begin{matrix} b, e-b, c, f-c \\ e, f \end{matrix} \right] \sum_{k=0}^{\infty} \frac{\langle a, b, c; q \rangle_k (1-q)^k}{\langle 1, e, f, d; q \rangle_k} z^k \\ = \frac{(1-q)^{d-1}}{\langle 1; q \rangle_{d-1}} \Gamma_q \left[\begin{matrix} b, e-b, c, f-c \\ e, f \end{matrix} \right] {}_4\phi_3(a, b, c, \infty; d, e, f|q; z(1-q)). \quad \square$$

COROLLARY 2.1. *For $|z| < 1$, $d \in \mathbb{N}$, $\operatorname{Re} b > 0$, $\operatorname{Re} c > 0$, we have the following triple Eulerian integral for a special ${}_4\phi_3$:*

$${}_4\phi_3(a, b, c, \infty; d, b+1, c+1|q; z(1-q)) \\ \cong \{b\}_q \{c\}_q \{d-1\}_q \int_0^1 \int_0^1 \int_0^1 \frac{t^{b-1} s^{c-1} u^{d-2}}{(zstu; q)_a} d_q(t) d_q(s) d_q(u).$$

PROOF. Put $e = b+1$ and $f = c+1$ in (2.1). □

3. q -analogues of Exton's integral formulas

We first prove a general q -integral formula.

THEOREM 3.1. (A q -analogue of [11, p.49 2.3.1.5])

$$\begin{aligned} & \int_0^1 x^{a-1} (qx; q)_{b-1} {}_2\phi_1(c, d; f|q; yx^k) {}_{G+1}\phi_G((g); (h)|q; zx^k) d_q(x) \\ &= B_q(a, b) \Phi_{2k:2;G+1}^{2k:1;G} \left[\begin{array}{c} \Delta(q; k; a) : c, d; (g) \\ \Delta(q; k; a + b) : f; (h) \end{array} \middle| q; x, y \right]. \end{aligned}$$

PROOF. We use the formula [9, (26)]:

$$(3.1) \quad \begin{aligned} & \int_0^1 x^{\alpha-1} (qx; q)_{\beta-1} {}_2\phi_1(\gamma, \delta; \lambda|q; yx^k) d_q(x) \\ &= B_q(\alpha, \beta) {}_{2+2k}\phi_{1+2k} \left[\begin{array}{c} \gamma, \delta, \Delta(q; k; \alpha) \\ \lambda, \Delta(q; k; \alpha + \beta) \end{array} \middle| q; y \right]. \end{aligned}$$

$$\begin{aligned} \text{LHS} &= \sum_{n=0}^{\infty} \frac{\langle (g); q \rangle_n z^n}{\langle (h), 1; q \rangle_n} \int_0^1 x^{a+kn-1} (qx; q)_{b-1} {}_2\phi_1(c, d; f|q; yx^k) d_q(x) \\ &\stackrel{\text{by (3.1)}}{=} \sum_{n=0}^{\infty} \frac{\langle (g); q \rangle_n z^n}{\langle (h), 1; q \rangle_n} B_q(a + kn, b) \\ & {}_{2+2k}\phi_{1+2k} \left[\begin{array}{c} c, d, \Delta(q; k; a + kn) \\ f, \Delta(q; k; a + b + kn) \end{array} \middle| q; y \right] = \text{RHS}. \quad \square \end{aligned}$$

DEFINITION 3.1. [9] We extend the two q -additions as follows: If we write a letter γ in the form $\gamma \equiv (\alpha \oplus_q \beta)^k \vee \gamma \equiv (\alpha \boxplus_q \beta)^k$, this means the two linear functionals $\gamma^n \equiv (\alpha \oplus_q \beta)^{nk} \vee \gamma^n \equiv (\alpha \boxplus_q \beta)^{nk}$.

THEOREM 3.2. (A q -analogue of [11, p.50 2.3.1.6]). Assume that the vectors (g) and (h) have dimensions G and $G - 1$. Then we have the following q -Eulerian integral:

$$\begin{aligned} & \int_0^1 x^{a-1} (qx; q)_{b-1} {}_2\phi_1(c, d; f|q; yx^k) {}_G\phi_{G-1}((g); (h)|q; z(1 \boxplus_q xq^b)^k) d_q(x) \\ &= B_q(a, b) \Phi_{2k:2k+2;2k+G}^{2k:2k+1;2k+G-1} \left[\begin{array}{c} 2k\infty : c, d, \Delta(q; k; a); (g), \Delta(q; k; b) \\ \Delta(q; k; a + b) : f, 2k\infty; (h), 2k\infty \end{array} \middle| q; y, z \right]. \end{aligned}$$

PROOF.

$$\begin{aligned} \text{LHS} &= \sum_{m=0}^{\infty} \frac{\langle (g); q \rangle_m z^m}{\langle (h), 1; q \rangle_m} \int_0^1 x^{a-1} (qx; q)_{b+km-1} {}_2\phi_1(c, d; f|q; yx^k) d_q(x) \\ &\stackrel{\text{by (3.1)}}{=} \sum_{m=0}^{\infty} \frac{\langle (g); q \rangle_m z^m}{\langle (h), 1; q \rangle_m} B_q(a, b + km) \sum_{n=0}^{\infty} \frac{\langle c, d; q \rangle_n \langle a; q \rangle_{kn} y^n}{\langle f, 1; q \rangle_n \langle a + b + km; q \rangle_{kn}} = \text{RHS}. \quad \square \end{aligned}$$

The following table is a q -analogue of a slightly improved version of Eulerian integrals for single hypergeometric functions from [11, p.152]. The proofs are similar to above.

| $f(u)$ | $\int_0^1 u^{a-1} f(u) d_q(u)$ |
|--|---|
| $(qu; q)_{b-1} {}_0\phi_1(\cdot; a q; ux)$ | $B_q(a, b) {}_0\phi_1(\cdot; a+b q; x)$ |
| $(qu; q)_{b-1} {}_0\phi_1(\cdot; c q; ux)$ | $B_q(a, b) {}_1\phi_2(a; a+b, c q; x)$ |
| ${}_0\phi_1(\cdot; \frac{a}{2} q; u^2x)$ | $\frac{1}{\{a\}_q} {}_0\phi_1(\cdot; 1 + \frac{a}{2} q; x)$ |
| $(qu; q)_{b-1} {}_0\phi_1(\cdot; \frac{a}{2} q; u^2x)$ | $B_q(a, b) {}_3\phi_4(\frac{a}{2}, \frac{a+1}{2}, \frac{a+1}{2}; \Delta(q; 2; a+b) q; x)$ |
| $(qu; q)_{b-1} {}_0\phi_1(\cdot; c q; u^2x)$ | $B_q(a, b) {}_4\phi_5(\Delta(q; 2; a); \Delta(q; 2; a+b), c q; x)$ |
| $(qu; q)_{a-1} {}_0\phi_1(\cdot; a q; u(1 \ominus_q uq^a)x)$ | $B_q(a, a) {}_2\phi_3(2\infty; \tilde{a}, a + \frac{1}{2}, a + \frac{1}{2} q; x)$ |
| $(qu; q)_{b-1} {}_0\phi_1(\cdot; b q; u(1 \ominus_q uq^b)x)$ | $B_q(a, b) {}_0{}_3\phi_4(a, 2\infty; \Delta(q; 2; a+b) q; x)$ |
| $(qu; q)_{b-1} {}_0\phi_1(\cdot; c q; u(1 \ominus_q uq^b)x)$ | $B_q(a, b) {}_4\phi_5(a, b, 2\infty; \Delta(q; 2; a+b), c q; x)$ |
| $(qu; q)_{b-1} {}_1\phi_1(c; a q; ux)$ | $B_q(a, b) {}_1\phi_1(c; a+b q; x)$ |
| $(qu; q)_{b-1} {}_1\phi_1(c; d q; ux)$ | $B_q(a, b) {}_2\phi_2(a, c; a+b, d q; x)$ |

Finally,

$$\int_0^1 u^{a-1} (qu; q)_{b-1} {}_2\phi_1(a+b, \infty; a|q; ux(1-q)) = B_q(a, b) E_q(x).$$

$$\int_0^1 u^{a-1} {}_5\phi_4(\Delta(q; 2; a+1), \infty; \Delta(q; 2; a)|q; u^2x(1-q)) d_q(u) = \frac{1}{\{a\}_q} E_q(x).$$

4. q -analogues of Totov's integral formulas

Totov [17] has published an interesting article with many new Eulerian integrals. First we repeat an important definition.

DEFINITION 4.1. [7, p.368] The vectors $(a), (b), (g_i), (h_i), (a'), (b'), (g'_i), (h'_i)$ have dimensions $A, B, G, H, A', B', G', H'$. Let

$$1 + B + B' + H + H' - A - A' - G - G' \geq 0.$$

Then the generalized q -Kampé de Fériet function is defined by

$$\Phi_{B+B':H+H'}^{A+A':G+G'} \left[\begin{array}{c} (\hat{a}) : (\hat{g}_1); \dots; (\hat{g}_n) \\ (\hat{b}) : (\hat{h}_1); \dots; (\hat{h}_n) \end{array} \middle| \vec{q}; \vec{x} \middle| \begin{array}{c} (a') : (g'_1); \dots; (g'_n) \\ (b') : (h'_1); \dots; (h'_n) \end{array} \right] \equiv$$

$$\sum_{\vec{m}} \frac{\langle \hat{a}; q_0 \rangle_m (a')(q_0, m) \prod_{j=1}^n \langle \hat{g}_j; q_j \rangle_{m_j} \langle (g'_j)(q_j, m_j) x_j^{m_j} \rangle}{\langle \hat{b}; q_0 \rangle_m (b')(q_0, m) \prod_{j=1}^n \langle \hat{h}_j; q_j \rangle_{m_j} \langle (h'_j)(q_j, m_j) \langle 1; q_j \rangle_{m_j} \rangle} \times$$

$$(-1)^{\sum_{j=1}^n m_j (1+H+H'-G-G'+B+B'-A-A')} \times$$

$$\text{QE} \left((B+B'-A-A') \binom{m}{2}, q_0 \right) \prod_{j=1}^n \text{QE} \left((1+H+H'-G-G') \binom{m_j}{2}, q_j \right),$$

where

$$\hat{a} \equiv a \vee \tilde{a} \vee \frac{\tilde{m}}{n} a \vee_k \tilde{a} \vee \Delta(q; l; \lambda).$$

It is assumed that there are no zero factors in the denominator. We assume that $(a')(q_0, m), (g'_j)(q_j, m_j), (b')(q_0, m), (h'_j)(q_j, m_j)$ contain factors of the form $\langle a(\hat{k}); q \rangle_k, \langle s; q \rangle_k, \langle s(k); q \rangle_k$ or $\text{QE}(f(\vec{m}))$.

Our next aim is to find Eulerian q -integrals for certain classes of the above multiple series. We start with the article by Totov.

THEOREM 4.1. *A q -analogue of [17, (12), p.47].*

$$(4.1) \quad \Phi_{2;0}^{2;1} \left[\begin{matrix} \alpha, \beta : \gamma_1; \dots; \gamma_n \\ \lambda, \mu : \cdot \end{matrix} \middle| q; \vec{x} \right] = \Gamma_q \left[\begin{matrix} \lambda, \mu \\ \alpha, \beta, \lambda - \alpha, \mu - \beta \end{matrix} \right] \\ \int_0^1 \int_0^1 u^{\alpha-1} v^{\beta-1} (qu; q)_{\lambda-\alpha-1} (qv; q)_{\mu-\beta-1} \prod_{k=1}^n \frac{1}{(uvx_k; q)_{\gamma_k}} d_q(u) d_q(v).$$

PROOF. Put

$$C \equiv \Gamma_q \left[\begin{matrix} \lambda, \mu \\ \alpha, \beta, \lambda - \alpha, \mu - \beta \end{matrix} \right].$$

Then

$$\begin{aligned} \text{LHS} &= \sum_{\vec{m}=\vec{0}}^{\infty} \frac{\langle \alpha, \beta; q \rangle_{\vec{m}} \langle \gamma_1; q \rangle_{m_1} \dots \langle \gamma_n; q \rangle_{m_n}}{\langle \lambda, \mu; q \rangle_{\vec{m}} \langle 1; q \rangle_{m_1} \dots \langle 1; q \rangle_{m_n}} x_1^{m_1} \dots x_n^{m_n} \text{ by [7, 1.46], } 2 \times \text{ [7, 7.55]} \\ &= C \sum_{\vec{m}=\vec{0}}^{\infty} \frac{\langle \vec{\gamma}; q \rangle_{\vec{m}}}{\langle \vec{1}; q \rangle_{\vec{m}}} \vec{x}^{\vec{m}} \int_{\vec{0}}^{\vec{1}} u^{\alpha+m-1} v^{\beta+m-1} (qu; q)_{\lambda-\alpha-1} (qv; q)_{\mu-\beta-1} d_q(\vec{u}) \\ &= C \int_{\vec{0}}^{\vec{1}} u^{\alpha-1} v^{\beta-1} (qu; q)_{\lambda-\alpha-1} (qv; q)_{\mu-\beta-1} \sum_{\vec{m}=\vec{0}}^{\infty} \frac{\langle \vec{\gamma}; q \rangle_{\vec{m}}}{\langle \vec{1}; q \rangle_{\vec{m}}} \prod_{i=1}^n (uvx_i)^{m_i} d_q(\vec{u}) \end{aligned}$$

by [7, 7.27] RHS. □

We have the following special case:

LEMMA 4.1. *A q -analogue of [17, p.48]. Put $\gamma = \sum_{i=1}^n \gamma_i$. Then*

$$(4.2) \quad \Phi_{2;0}^{2;1} \left[\begin{matrix} \alpha, \beta : \gamma_1; \dots; \gamma_n \\ \lambda, \mu : \cdot \end{matrix} \middle| q; x, xq^{-\gamma_2}, xq^{-\gamma_2-\gamma_3}, \dots, xq^{-\gamma_2-\dots-\gamma_n} \right] \\ = \Gamma_q \left[\begin{matrix} \lambda, \mu \\ \alpha, \beta, \lambda - \alpha, \mu - \beta \end{matrix} \right] \\ \int_0^1 \int_0^1 u^{\alpha-1} v^{\beta-1} (qu; q)_{\lambda-\alpha-1} (qv; q)_{\mu-\beta-1} \frac{1}{(uvxq^{-\gamma_2-\dots-\gamma_n}; q)_{\gamma}} d_q(u) d_q(v).$$

COROLLARY 4.1. *A q -analogue of [17, p.48]. If $\gamma = 0$, then*

$$\Phi_{2;0}^{2;1} \left[\begin{matrix} \alpha, \beta : \gamma_1; \dots; \gamma_n \\ \lambda, \mu : \cdot \end{matrix} \middle| q; x, xq^{-\gamma_2}, xq^{-\gamma_2-\gamma_3}, \dots, xq^{-\gamma_2-\dots-\gamma_n} \right] = 1.$$

PROOF. Use lemma (4.2) and the q -beta integral. □

COROLLARY 4.2. *A q -analogue of [17, p.49]. If $\gamma = -1$, then*

$$\Phi_{2;0}^{2;1} \left[\begin{matrix} \alpha, \beta : \gamma_1; \dots; \gamma_n \\ \lambda, \mu : \cdot \end{matrix} \middle| q; x, xq^{-\gamma_2}, xq^{-\gamma_2-\gamma_3}, \dots, xq^{-\gamma_2-\dots-\gamma_n} \right] \\ = 1 - xq^{-\gamma_2-\dots-\gamma_n-1} \frac{\{\alpha\}_q \{\beta\}_q}{\{\lambda\}_q \{\mu\}_q}.$$

COROLLARY 4.3. A q -analogue of [17, (13), p.49].

$$(4.3) \quad \Phi_{2:0}^{2:1} \left[\begin{array}{c} \alpha, \beta : \gamma_1; \dots; \gamma_n \\ \lambda, \mu : \cdot \end{array} \middle| q; \vec{x} \right] = \Gamma_q \left[\begin{array}{c} \lambda \\ \alpha, \lambda - \alpha \end{array} \right] \int_0^1 u^{\alpha-1} (qu; q)_{\lambda-\alpha-1} \Phi_D^{(n)}(\beta, \gamma_1, \dots, \gamma_n; \mu | q; ux_1, \dots, ux_n) d_q(u).$$

PROOF. Use formula (4.1) together with [8, (38)]. \square

THEOREM 4.2. A q -analogue of [17, (17), p.52] and a generalization of [7, 10.128].

$$\Phi_{2:0}^{2:1} \left[\begin{array}{c} \alpha, \beta : \gamma_1; \dots; \gamma_n \\ \lambda, \mu : \cdot \end{array} \middle| q; x, xq^{-\gamma_2}, xq^{-\gamma_2-\gamma_3}, \dots, xq^{-\gamma_2-\dots-\gamma_n} \right] = {}_3\phi_2 \left[\begin{array}{c} \alpha, \beta, \gamma \\ \lambda, \mu \end{array} \middle| q; xq^{-\gamma_2-\dots-\gamma_n} \right].$$

PROOF.

$$\begin{aligned} \text{LHS} &\stackrel{\text{by (4.3)}}{=} \Gamma_q \left[\begin{array}{c} \lambda \\ \alpha, \lambda - \alpha \end{array} \right] \int_0^1 u^{\alpha-1} (qu; q)_{\lambda-\alpha-1} \\ &\Phi_D^{(n)}(\beta, \gamma_1, \dots, \gamma_n; \mu | q; ux, uxq^{-\gamma_2}, uxq^{-\gamma_2-\gamma_3}, \dots, uxq^{-\gamma_2-\dots-\gamma_n}) d_q(u) \\ &\stackrel{\text{by [7, 10.128]}}{=} \Gamma_q \left[\begin{array}{c} \lambda \\ \alpha, \lambda - \alpha \end{array} \right] \int_0^1 u^{\alpha-1} (qu; q)_{\lambda-\alpha-1} \\ &\sum_{m=0}^{\infty} \frac{\langle \gamma, \beta; q \rangle_m}{\langle \mu, 1; q \rangle_m} (ux)^m q^{m(-\gamma_2-\dots-\gamma_n)} d_q(u) \stackrel{\text{by [7, 7.55]}}{=} \text{RHS}. \end{aligned} \quad \square$$

After these obvious proofs we continue with the introduction of some new q -numbers.

5. Formulas with general q -real numbers

The following two definitions prepare for the next formulas.

DEFINITION 5.1. A generalization of the q -binomial theorem [9, 10]. The following formula applies to a q -deformed hyper-rhombus of length 1 in \mathbb{R}^n .

$$(5.1) \quad (1 \boxplus_q q^a x_1 \boxplus_q \dots \boxplus_q q^a x_n)^{-a} = \sum_{\vec{m}=\vec{0}}^{\vec{\infty}} \frac{\langle a; q \rangle_m \vec{x}^{\vec{m}}}{\langle \vec{1}; q \rangle_{\vec{m}}}, \quad a \in \mathbb{R}^*.$$

DEFINITION 5.2. Let $f(x) \in \mathbb{R}[[x]]$. The multiple q -integral I_n of order n is defined by

$$\begin{aligned} I_n &\equiv \int_0^{a_1} \dots (n) \dots \int_0^{a_n} f(a_1, \dots, a_n, q) d_q(x_1) \dots d_q(x_n) \\ &\equiv \prod_{i=1}^n (a_i(1-q)) \sum_{\vec{m}=\vec{0}}^{\vec{\infty}} f(x_1 q^{m_1}, \dots, x_n q^{m_n}, q) q^m, \quad 0 < |q| < 1, \{a_i\}_{i=1}^n \in \mathbb{R}. \end{aligned}$$

THEOREM 5.1. A q -integral representation of the first q -Lauricella function [9, (22), p.183], [10].

(5.2)

$$\mathbb{B}_q(\vec{b}, \overrightarrow{c-\vec{b}}) \Phi_A^{(n)}(a, \vec{b}; \vec{c}|q; \vec{x}) = \int_0^1 \dots (n) \dots \int_0^1 u_1^{b_1-1} \dots u_n^{b_n-1} (qu_1; q)_{c_1-b_1-1} \dots (qu_n; q)_{c_n-b_n-1} (1 \boxplus_q q^a u_1 x_1 \boxplus_q \dots \boxplus_q q^a u_n x_n)^{-a} d_q(u_1) \dots d_q(u_n).$$

THEOREM 5.2. A q -analogue of [15, p.7 (2.10)]. Put

$$C \equiv \Gamma_q \left[\begin{matrix} \gamma, \delta, \delta' \\ \alpha, \beta_1, \beta'_1, \gamma - \alpha, \delta - \beta_1, \delta' - \beta'_1 \end{matrix} \right].$$

Then

$$\Phi_{1:1}^{1:2} \left[\begin{matrix} \alpha : \beta_1, \beta_2; \beta'_1, \beta'_2 \\ \gamma : \delta; \delta' \end{matrix} \middle| q; x, y \right] = \int_0^1 \dots (3) \dots \int_0^1 u^{\alpha-1} (qu; q)_{\gamma-\alpha-1} v^{\beta_1-1} (qv; q)_{\delta-\beta_1-1} \omega^{\beta'_1-1} (q\omega; q)_{\delta'-\beta'_1-1} \frac{1}{(uvx; q)_{\beta_2} (u\omega y; q)_{\beta'_2}} d_q(u) d_q(v) d_q(\omega).$$

PROOF.

$$\begin{aligned} \text{LHS} &= \sum_{\vec{m}=\vec{0}}^{\infty} \frac{\langle \alpha; q \rangle_{m_1+m_2} \langle \beta_1, \beta_2; q \rangle_{m_1} \langle \beta'_1, \beta'_2; q \rangle_{m_2}}{\langle \gamma; q \rangle_{m_1+m_2} \langle \delta, 1; q \rangle_{m_1} \langle \delta', 1; q \rangle_{m_2}} x^{m_1} y^{m_2} \\ &\stackrel{\text{by [7, 1.46]}}{=} \sum_{\vec{m}=\vec{0}}^{\infty} \frac{\langle \beta_2; q \rangle_{m_1} \langle \beta'_2; q \rangle_{m_2}}{\langle 1; q \rangle_{m_1} \langle 1; q \rangle_{m_2}} x^{m_1} y^{m_2} \Gamma_q(\gamma) \Gamma_q(\delta) \Gamma_q(\delta') \\ &\quad \times \Gamma_q \left[\begin{matrix} \alpha + m_1 + m_2, \gamma - \alpha, \beta_1 + m_1, \beta'_1 + m_2 \\ \alpha, \gamma + m_1 + m_2, \delta + m_1, \delta' + m_2 \end{matrix} \right] \\ &\stackrel{\text{by [7, 7.55]}}{=} C \int_{\vec{0}}^{\vec{1}} \sum_{\vec{m}=\vec{0}}^{\infty} \frac{\langle \beta_2; q \rangle_{m_1} \langle \beta'_2; q \rangle_{m_2}}{\langle 1; q \rangle_{m_1} \langle 1; q \rangle_{m_2}} (uvx)^{m_1} (u\omega y)^{m_2} u^{\alpha-1} (qu; q)_{\gamma-\alpha-1} \\ &\quad v^{\beta_1-1} (qv; q)_{\delta-\beta_1-1} \omega^{\beta'_1-1} (q\omega; q)_{\delta'-\beta'_1-1} d_q(u) d_q(v) d_q(\omega) \stackrel{\text{by [7, 7.27]}}{=} \text{RHS}. \quad \square \end{aligned}$$

THEOREM 5.3. A q -analogue of [1, (32) p.154]. Put

$$C \equiv \Gamma_q \left[\begin{matrix} \gamma_1, \dots, \gamma_n \\ \alpha_1, \gamma_1 - \alpha_1, \dots, \alpha_n, \gamma_n - \alpha_n \end{matrix} \right].$$

Then

$$\Phi_{n:0}^{n:1} \left[\begin{matrix} \alpha_1, \dots, \alpha_n : \beta; \beta' \\ \gamma_1, \dots, \gamma_n : \cdot \end{matrix} \middle| q; x, y \right] = C \int_0^1 \dots (n) \dots \int_0^1 \prod_{i=1}^n [u_i^{\alpha_i-1} (qu_i; q)_{\gamma_i-\alpha_i-1} d_q(u_i)] \frac{1}{(u_1 \dots u_n x; q)_{\beta} (u_1 \dots u_n y; q)_{\beta'}}.$$

PROOF.

$$\begin{aligned}
\text{LHS} &= \sum_{\vec{m}=\vec{0}}^{\infty} \frac{\langle \vec{\alpha}; q \rangle_{m_1+m_2} \langle \beta; q \rangle_{m_1} \langle \beta'; q \rangle_{m_2}}{\langle \vec{\gamma}; q \rangle_{m_1+m_2} \langle 1; q \rangle_{m_1} \langle 1; q \rangle_{m_2}} x^{m_1} y^{m_2} \\
&\stackrel{\text{by [7, 1.46]}}{=} C \sum_{\vec{m}=\vec{0}}^{\infty} \frac{\langle \beta; q \rangle_{m_1} \langle \beta'; q \rangle_{m_2}}{\langle 1; q \rangle_{m_1} \langle 1; q \rangle_{m_2}} x^{m_1} y^{m_2} \\
&\quad \times \Gamma_q \left[\begin{matrix} \alpha_1 + m_1 + m_2, \gamma_1 - \alpha_1, \dots, \alpha_n + m_1 + m_2, \gamma_n - \alpha_n \\ \gamma_1 + m_1 + m_2, \dots, \gamma_n + m_1 + m_2 \end{matrix} \right] \\
&\stackrel{\text{by } n \times [7, 7.55]}{=} C \int_0^1 \sum_{\vec{m}=\vec{0}}^{\infty} \frac{\langle \beta; q \rangle_{m_1} \langle \beta'; q \rangle_{m_2}}{\langle 1; q \rangle_{m_1} \langle 1; q \rangle_{m_2}} (u_1 \dots u_n x)^{m_1} (u_1 \dots u_n y)^{m_2} \\
&\quad \prod_{i=1}^n \left[u_i^{\alpha_i-1} (qu_i; q)_{\gamma_i-\alpha_i-1} d_q(u_i) \right] \stackrel{\text{by [7, 7.27]}}{=} \text{RHS}. \quad \square
\end{aligned}$$

THEOREM 5.4. A q -analogue of [1, p. 154]. Put

$$C \equiv \Gamma_q \left[\begin{matrix} \delta_1, \dots, \delta_n, \delta'_1, \dots, \delta'_n \\ \beta_1, \dots, \beta_n, \beta'_1, \dots, \beta'_n, \delta_1 - \beta_1, \dots, \delta_n - \beta_n, \delta'_1 - \beta'_1, \dots, \delta'_n - \beta'_n \end{matrix} \right].$$

Then

$$\begin{aligned}
\Phi_{1:n}^{1:n+1} \left[\begin{matrix} \alpha : \beta_1, \dots, \beta_n, \infty; \beta'_1, \dots, \beta'_n, \infty \\ \infty : \delta_1, \dots, \delta_n; \delta'_1, \dots, \delta'_n \end{matrix} \middle| q; x, y \right] &= C \int_0^1 \dots (2n) \dots \int_0^1 \\
(1 \boxplus_q q^\alpha u_1 \dots u_n x \boxminus_q q^\alpha v_1 \dots v_n y)^{-\alpha} & \\
\prod_{i=1}^n \left[u_i^{\beta_i-1} v_i^{\beta'_i-1} (qu_i; q)_{\delta_i-\beta_i-1} (qv_i; q)_{\delta'_i-\beta'_i-1} d_q(u_i) d_q(v_i) \right] &.
\end{aligned}$$

PROOF.

$$\begin{aligned}
\text{LHS} &= \sum_{m_1, m_2=0}^{\infty} \frac{\langle \alpha; q \rangle_{m_1+m_2} \langle \vec{\beta}; q \rangle_{m_1} \langle \vec{\beta}'; q \rangle_{m_2}}{\langle \vec{\delta}, 1; q \rangle_{m_1} \langle \vec{\delta}', 1; q \rangle_{m_2}} x^{m_1} y^{m_2} \\
&\stackrel{\text{by [7, 1.46]}}{=} C \sum_{m_1, m_2=0}^{\infty} \frac{\langle \alpha; q \rangle_{m_1+m_2}}{\langle 1; q \rangle_{m_1} \langle 1; q \rangle_{m_2}} x^{m_1} y^{m_2} \Gamma_q \left[\begin{matrix} \beta_1 + m_1, \dots, \beta_n + m_1 \\ \delta_1 + m_1, \dots, \delta_n + m_1 \end{matrix} \right] \\
&\quad \Gamma_q \left[\begin{matrix} \beta'_1 + m_2, \dots, \beta'_n + m_2, \delta_1 - \beta_1, \dots, \delta_n - \beta_n, \delta'_1 - \beta'_1, \dots, \delta'_n - \beta'_n \\ \delta'_1 + m_2, \dots, \delta'_n + m_2 \end{matrix} \right] \\
&\stackrel{\text{by } 2n \times [7, 7.55]}{=} C \int_0^1 \dots (2n) \dots \int_0^1 \\
&\quad \sum_{m_1, m_2=0}^{\infty} \frac{\langle \alpha; q \rangle_{m_1+m_2}}{\langle 1; q \rangle_{m_1} \langle 1; q \rangle_{m_2}} (u_1 \dots u_n x)^{m_1} (v_1 \dots v_n y)^{m_2} \\
&\quad \prod_{i=1}^n \left[u_i^{\beta_i-1} v_i^{\beta'_i-1} (qu_i; q)_{\delta_i-\beta_i-1} (qv_i; q)_{\delta'_i-\beta'_i-1} d_q(u_i) d_q(v_i) \right] \stackrel{\text{by (5.1)}}{=} \text{RHS}. \quad \square
\end{aligned}$$

THEOREM 5.5. A q -analogue of [15, p.8]. Put

$$C \equiv \Gamma_q \left[\begin{matrix} \delta, \delta', \gamma_1, \dots, \gamma_n \\ \beta, \beta', \delta - \beta, \delta' - \beta', \alpha_1, \gamma_1 - \alpha_1, \dots, \alpha_n, \gamma_n - \alpha_n \end{matrix} \right].$$

Then

$$\begin{aligned} \Phi_{n+1:n+1}^{n+1:n} \left[\begin{matrix} \alpha, \alpha_1, \dots, \alpha_n, \beta_n : \beta, \infty; \beta', \infty \\ \gamma_1, \dots, \gamma_n, \infty : \delta; \delta' \end{matrix} \middle| q; x, y \right] &= C \int_0^1 \dots (n) \dots \int_0^1 \\ &\int_0^1 \int_0^1 d_q(u) d_q(v) (1 \boxplus_q q^\alpha u u_1 \dots u_n x \boxminus_q q^\alpha v u_1 \dots u_n y)^{-\alpha} u^{\beta-1} v^{\beta'-1} \\ &(qu; q)_{\delta-\beta-1} (qv; q)_{\delta'-\beta'-1} \prod_{i=1}^n [u_i^{\alpha_i-1} (qu_i; q)_{\gamma_i-\alpha_i-1} d_q(u_i)]. \end{aligned}$$

PROOF.

$$\begin{aligned} \text{LHS} &= \sum_{m_1, m_2=0}^{\infty} \frac{\langle \alpha, \vec{\alpha}; q \rangle_{m_1+m_2} \langle \beta; q \rangle_{m_1} \langle \beta'; q \rangle_{m_2}}{\langle \vec{\gamma}; q \rangle_{m_1+m_2} \langle \delta, 1; q \rangle_{m_1} \langle \delta', 1; q \rangle_{m_2}} x^{m_1} y^{m_2} \stackrel{\text{by [7, 1.46]}}{=} \sum_{m_1, m_2=0}^{\infty} \\ &\frac{\langle \alpha; q \rangle_{m_1+m_2}}{\langle 1; q \rangle_{m_1} \langle 1; q \rangle_{m_2}} x^{m_1} y^{m_2} \Gamma_q \left[\begin{matrix} \beta + m_1, \alpha_1 + m_1 + m_2, \dots, \alpha_n + m_1 + m_2 \\ \delta + m_1, \gamma_1 + m_1 + m_2, \dots, \gamma_n + m_1 + m_2 \end{matrix} \right] \\ &\Gamma_q \left[\begin{matrix} \beta' + m_2, \delta, \delta', \gamma_1 - \alpha_1, \dots, \gamma_n - \alpha_n \\ \beta, \beta', \delta' + m_2, \alpha_1, \dots, \alpha_n \end{matrix} \right] \stackrel{\text{by [7, 7.55]}}{=} C \\ &\int_0^1 \dots (n+2) \dots \int_0^1 \sum_{m_1, m_2=0}^{\infty} \frac{\langle \alpha; q \rangle_{m_1+m_2}}{\langle 1; q \rangle_{m_1} \langle 1; q \rangle_{m_2}} (u u_1 \dots u_n x)^{m_1} (v u_1 \dots u_n y)^{m_2} \\ &\prod_{i=1}^n [u_i^{\beta_i-1} v_i^{\beta'_i-1} (qu_i; q)_{\delta_i-\beta_i-1} (qv_i; q)_{\delta'_i-\beta'_i-1} d_q(u_i)] d_q(u) d_q(v) \stackrel{\text{by (5.1)}}{=} \text{RHS}. \square \end{aligned}$$

6. A q -Burchall-Chaundy integral for the second q -Appell function

We next find an expression for the second q -Appell function by using formulas from our first book [7].

THEOREM 6.1. A q -analogue of [2, (57)].

$$\begin{aligned} \Phi_2(a; b, b'; c, c' | q; x, y) &= \sum_{r=0}^{\infty} \frac{\langle b'; q \rangle_r}{\langle 1; q \rangle_r} (ux)^r (vy)^r q^{r(a+r-1)} \\ &\Gamma_q \left[\begin{matrix} c, c' \\ b, c-b, a, c'-a \end{matrix} \right] \int_0^1 \int_0^1 u^{b-1} v^{a-1} (qu; q)_{c-b-1} (qv; q)_{c'-a-1} \\ &\sum_{m=0}^{\infty} \frac{\langle a+r; q \rangle_m}{\langle 1; q \rangle_m} (ux)^m \sum_{n=0}^{\infty} \frac{\langle b'+r; q \rangle_n}{\langle 1; q \rangle_n} (vy)^n d_q(u) d_q(v). \end{aligned}$$

PROOF. We use the formula

$$(6.1) \quad [7, 10.212] \Phi_2(a; b, b'; c, c' | q; x_1, x_2) = \sum_{r=0}^{\infty} \frac{\langle a, b, b'; q \rangle_r}{\langle 1, c, c'; q \rangle_r} x_1^r x_2^r q^{r(a+r-1)} \times \\ 2\phi_1(a+r, b+r; c+r | q; x_1) 2\phi_1(a+r, b'+r; c'+r | q; x_2). \\ \text{LHS} \stackrel{\text{by [7, 7.50], (6.1)}}{=} \sum_{r=0}^{\infty} \frac{\langle a, b, b'; q \rangle_r}{\langle 1, c, c'; q \rangle_r} x_1^r x_2^r q^{r(a+r-1)} \Gamma_q \left[\begin{matrix} c+r, c'+r \\ b+r, c-b, a+r, c'-a \end{matrix} \right] \\ \int_0^1 u^{b+r-1} \frac{(qu; q)_{c-b-1}}{(ux; q)_{a+r}} d_q(u) \int_0^1 v^{a+r-1} \frac{(qv; q)_{c'-a-1}}{(vy; q)_{b'+r}} d_q(v) \stackrel{\text{by [7, 1.46, 7.27]}}{=} \text{RHS}. \quad \square$$

7. Karlsson's intermediate Lauricella functions

Per Karlsson [12] has introduced four intermediate Lauricella functions with corresponding convergence regions. Later, Chandel Singh and Vishwakarma [3] discovered a transformation of one of these functions by Euler integrals and a general Beta function operator. The purpose of this section is to q -deform this formula. First we define our intermediate Lauricella function.

DEFINITION 7.1. A q -analogue of [12, p.212].

$$\begin{aligned} {}^{(k)}\Phi_{\text{CD}}^{(n)}(a, b, b_1, \dots, b_k; c, c_{k+1}, \dots, c_n | q; x_1, \dots, x_n) &\equiv \\ &\equiv \sum_{\vec{m}} \frac{\langle a; q \rangle_m \langle b; q \rangle_{m_{k+1} + \dots + m_n} \prod_{j=1}^k \langle b_j; q \rangle_{m_j} \vec{x}^{\vec{m}}}{\langle c; q \rangle_{m_1 + \dots + m_k} \langle \vec{1}; q \rangle_{\vec{m}} \prod_{j=k+1}^n \langle c_j; q \rangle_{m_j}}. \end{aligned}$$

The convergence region for $q = 1$ is

$$\max(|x_1| \dots |x_k|) + (\sqrt{|x_{k+1}|} + \dots + \sqrt{|x_n|})^2 < 1.$$

This function contains $\Phi_{\text{C}}^{(n)}(a, b; c_1, \dots, c_n | q; x_1, \dots, x_n)$ in the special case $k = 0$, and $\Phi_{\text{D}}^{(n)}(a, b_1, \dots, b_n; c | q; x_1, \dots, x_n)$ in the special case $k = n$.

THEOREM 7.1. A q -analogue of [3, 2.1]. A transformation formula between two intermediate q -Lauricella functions.

$$\begin{aligned} {}^{(k)}\Phi_{\text{CD}}^{(n)}(\alpha, \nu, \beta_1, \dots, \beta_k; \gamma, \gamma_{k+1}, \dots, \gamma_n | q; x_1, \dots, x_n) &= \Gamma_q \left[\begin{matrix} \vec{\nu} \\ \vec{\beta}, \mu, \vec{\beta} \end{matrix} \right] \\ &\int_0^1 \dots (k) \dots \int_0^1 \prod_{i=1}^k \left[t_i^{\beta_i-1} (qt_i; q)_{\nu_i-\beta_i-1} \right] d_q(t_1) \dots d_q(t_k) \times \\ &{}^{(k)}\Phi_{\text{CD}}^{(n)}(\alpha, \nu, \nu_1, \dots, \nu_k; \gamma, \gamma_{k+1}, \dots, \gamma_n | q; x_1 t_1, \dots, x_k t_k, x_{k+1}, \dots, x_n). \end{aligned}$$

8. A general q -integral formula

DEFINITION 8.1. Assume that \vec{x} and \vec{m} have dimension n . Let

$$F(\vec{x}, \vec{\beta}_1, \vec{\beta}_2) \equiv \sum_{\vec{m}} \frac{g_1(\vec{m}) \langle \vec{\beta}_1; q \rangle_{\vec{m}}}{\langle \vec{1}, \vec{\beta}_2; q \rangle_{\vec{m}}} \vec{x}^{\vec{m}}$$

where $g_1(\vec{m})$ contains arbitrary functions independent of \vec{x} .

We assume that $\vec{\beta}_1, \vec{\mu}_1$ and \vec{s} have dimension s and $\vec{\beta}_2, \vec{\mu}_2$ and \vec{t} have dimension t , where $s, t \leq n$.

THEOREM 8.1. *Put*

$$C \equiv \Gamma_q \left[\begin{matrix} \vec{\mu}_1, \vec{\beta}_2 \\ \vec{\beta}_1, \vec{\mu}_1 - \vec{\beta}_1, \vec{\mu}_2, \vec{\beta}_2 - \vec{\mu}_2 \end{matrix} \right].$$

Then we have

$$F(\vec{x}, \vec{\beta}_1, \vec{\beta}_2) \cong C \int_{\vec{s}=0}^{\vec{1}} \int_{\vec{t}=0}^{\vec{1}} \vec{s}^{\vec{\beta}_1 - \vec{1}} (q\vec{s}; q)_{\vec{\mu}_1 - \vec{\beta}_1 - \vec{1}} \vec{t}^{\vec{\mu}_2 - \vec{1}} (q\vec{t}; q)_{\vec{\beta}_2 - \vec{\mu}_2 - \vec{1}} F(\vec{x}\vec{s}\vec{t}, \vec{\mu}_1, \vec{\mu}_2) d_q(\vec{s}) d_q(\vec{t}).$$

PROOF. We compute the right-hand side. The infinite sums for the two q -integrals are denoted by \vec{k}_1 and \vec{k}_2 . For simplicity, we sometimes write $\vec{k} \equiv \vec{k}_1 + \vec{k}_2$.

$$\begin{aligned} \text{RHS} &\stackrel{\text{by [7, 6.54]}}{=} C \sum_{\vec{m}, \vec{k}=0}^{\infty} \frac{\langle \vec{\mu}_1; q \rangle_{\vec{m}} \vec{x}^{\vec{m}}}{\langle \vec{\mu}_2, \vec{1}; q \rangle_{\vec{m}}} (1-q)^{s+t} \frac{q^{k((\vec{\beta}_1, \vec{\mu}_2) + \vec{m})}}{\langle \vec{1} + \vec{k}_1; q \rangle_{\vec{\mu}_1 - \vec{\beta}_1 - \vec{1}} \langle \vec{1} + \vec{k}_2; q \rangle_{\vec{\beta}_2 - \vec{\mu}_2 - \vec{1}}} \\ &\stackrel{\text{by [7, 6.10]}}{=} C \sum_{\vec{m}, \vec{k}=0}^{\infty} \frac{\langle \vec{\mu}_1; q \rangle_{\vec{m}} \vec{x}^{\vec{m}}}{\langle \vec{\mu}_2, \vec{1}; q \rangle_{\vec{m}}} (1-q)^{s+t} \frac{q^{k((\vec{\beta}_1, \vec{\mu}_2) + \vec{m})}}{\langle \vec{1}; q \rangle_{\vec{k}} \langle \vec{\mu}_1 - \vec{\beta}_1; q \rangle_{\vec{k}_1} \langle \vec{\beta}_2 - \vec{\mu}_2; q \rangle_{\vec{k}_2} \langle \vec{1}, \vec{1}; q \rangle_{\infty}} \\ &\stackrel{\text{by [7, 2} \times \text{7.27]}}{=} C \sum_{\vec{m}=0}^{\infty} \frac{\langle \vec{\mu}_1; q \rangle_{\vec{m}} \vec{x}^{\vec{m}}}{\langle \vec{\mu}_2, \vec{1}; q \rangle_{\vec{m}}} (1-q)^{s+t} \frac{\langle \vec{\mu}_1 + \vec{m}, \vec{\beta}_2 + \vec{m}, \vec{1}, \vec{1}; q \rangle_{\infty}}{\langle \vec{\mu}_1 - \vec{\beta}_1, \vec{\beta}_2 - \vec{\mu}_2, \vec{\beta}_1 + \vec{m}, \vec{\mu}_2 + \vec{m}; q \rangle_{\infty}} \\ &\stackrel{\text{by [7, 1.46]}}{=} \text{LHS}. \quad \square \end{aligned}$$

9. A q -analogue of Sharma and Parihar

Our next aim is to find a q -analogue of an Euler integral for a hypergeometric function of four variables.

THEOREM 9.1. *A q -analogue of Sharma and Parihar [16, p.62]. Put*

$$\Phi_6^{(4)} \equiv \sum_{m,n,p,r=0}^{\infty} \frac{\langle \alpha_1; q \rangle_{m+p+r} \langle \alpha_2; q \rangle_n \langle \beta_1; q \rangle_{m+n} \langle \beta_2; q \rangle_p \langle \beta_3; q \rangle_r}{\langle 1, \gamma_1; q \rangle_m \langle 1, \gamma_4; q \rangle_n \langle 1, \gamma_2; q \rangle_p \langle 1, \gamma_3; q \rangle_r} x^m t^n y^p z^r.$$

Then we have

$$\begin{aligned} \Phi_6^{(4)} &= \Gamma_q \left[\begin{matrix} \gamma_2, \gamma_3, \gamma_4 \\ \beta_2, \beta_3, \gamma_2 - \beta_2, \gamma_3 - \beta_3, \alpha_2, \gamma_4 - \alpha_2 \end{matrix} \right] \sum_{m=0}^{\infty} \frac{\langle \alpha_1, \beta_1; q \rangle_m}{\langle 1, \gamma_1; q \rangle_m} x^m \\ &\int_0^{\vec{1}} u^{\beta_2-1} v^{\beta_3-1} \omega^{\alpha_2-1} (qu; q)_{\gamma_2-\beta_2-1} (qv; q)_{\gamma_3-\beta_3-1} \\ &\frac{(q\omega; q)_{\gamma_4-\alpha_2-1}}{(t\omega; q)_{\beta_1+m}} (1 \boxminus_q q^{\alpha_1+m} u y \boxminus_q q^{\alpha_1+m} v z)^{-\alpha_1-m} d_q(u) d_q(v) d_q(\omega). \end{aligned}$$

PROOF.

$$\begin{aligned}
\text{LHS} &= \sum_{m,n=0}^{\infty} \frac{\langle \alpha_1; q \rangle_m \langle \alpha_2; q \rangle_n \langle \beta_1; q \rangle_{m+n}}{\langle 1, \gamma_1; q \rangle_m \langle 1, \gamma_4; q \rangle_n} x^m t^n \\
&\Phi_2^{(n)}(\alpha_1 + m, \beta_2, \beta_3; \gamma_2, \gamma_3 | q; y, z) \stackrel{\text{by (5.2)}}{=} \Gamma_q \left[\begin{matrix} \gamma_2, \gamma_3 \\ \beta_2, \beta_3, \gamma_2 - \beta_2, \gamma_3 - \beta_3 \end{matrix} \right] \\
&\sum_{m,n=0}^{\infty} \frac{\langle \alpha_1; q \rangle_m \langle \alpha_2; q \rangle_n \langle \beta_1; q \rangle_{m+n}}{\langle 1, \gamma_1; q \rangle_m \langle 1, \gamma_4; q \rangle_n} x^m t^n \int_0^1 \int_0^1 u^{\beta_2-1} v^{\beta_3-1} (qu; q)_{\gamma_2-\beta_2-1} \\
&(qv; q)_{\gamma_3-\beta_3-1} (1 \ominus_q q^{\alpha_1+m} uy \ominus_q q^{\alpha_1+m} vz)^{-\alpha_1-m} d_q(u) d_q(v) \\
&\stackrel{\text{by [7, 7.50]}}{=} \text{RHS}. \quad \square
\end{aligned}$$

10. Discussion

In this paper we have proved q -analogues of some Eulerian integrals in Appell and Kampé de Fériet [1] and Erdélyi et al. [5]. We have found q -analogues and improved some of Exton's integral representations. We have generalized the reduction theorem for a q -Lauricella function by Totovs method. We have q -deformed some multiple integrals by Karlsson and by Sharma and Parihar. And last but not least, we have proved a quite general transformation formula. Although the steps in all proofs are identical, the formulas are disparate, which shows the richness of the subject. A general algorithm to produce such proofs by the beta integral method for hypergeometric identities, written in *Mathematica*, is presented in [13].

All these (multiple) q -Euler integrals are meromorphic continuations of the corresponding multiple q -functions. Equivalent meromorphic continuations with the same poles and zeros can sometimes be obtained by using the Jackson q -analogue of the Euler–Pfaff–Kummer transformation [7, 7.62]. This will be made in another paper.

References

1. P. Appell, J. Kampé de Fériet, *Fonctions hypergéométriques et hypersphériques*, Paris 1926.
2. J. L. Burchall, T. W. Chaundy *Expansions of Appell's double hypergeometric functions II*, Quart. J. Math. (Oxford) **12** (1941), 112–128
3. R. C. Singh Chandel, P. K. Vishwakarma, *Fractional integration and integral representations of Karlsson's multiple hypergeometric function and its confluent forms*, Jnanbbha **20** (1990), 101–110
4. K. A. Driver, S. J. Johnston, *An integral representation of some hypergeometric functions*, ETNA, Electron. Trans. Numer. Anal. **25** (2006), 115–120.
5. A. Erdélyi, W. Magnus, F. Oberhettinger, F. G. Tricomi, *Higher transcendental functions. Vols. I*; Based, in part, on notes left by Harry Bateman. McGraw-Hill, New York–Toronto–London, 1953
6. T. Ernst, *Multiple q -hypergeometric transformations involving q -integrals*, Proc. 9th Annual Conf. Society for Special Functions and their Applications (SSFA) **9** (2011), 91–99.
7. ———, *A comprehensive treatment of q -calculus*, Birkhäuser (2012)
8. ———, *On the symmetric q -Lauricella functions*, Proc. Jangjeon Math. Soc. **19**(2) (2016), 319–344

9. ———, *On Eulerian q -integrals for single and multiple q -hypergeometric series*, Commun. Korean Math. Soc. **33**(1) (2018), 179–196
10. ———, *A new semantics for special functions*, submitted to De Gruyter
11. H. Exton, *Handbook of hypergeometric integrals. Theory, applications, tables, computer programs*, Chichester; Halsted Press [Wiley], New York–London–Sydney, 1978.
12. P. Karlsson, *On intermediate Lauricella functions*, Jnanbbha **16** (1986), 211–222
13. C. Krattenthaler, K. Srinivasa Rao, *Automatic generation of hypergeometric identities by the beta integral method*, J. Comput. Appl. Math. **160**(1–2) (2003), 159–173
14. L. Koschmieder, *Integrals with hypergeometric integrands II*, Univ. Nac. Tucuman. Revista A. **9** (1952), 63–78 (in Spanish)
15. R. C. Pandey, S. Saran, *On the hypergeometric functions of higher order in two variables*, Proc. Rajasthan Acad. Sci. **10**(1) (1963), 3–13
16. C. Sharma, C. L. Parihar, *Integral representations of Euler's type for hypergeometric functions of four variables*, J. Indian Acad. Math. **14**(1) (1992), 60–69
17. G. Totov *A hypergeometric function of n variables*, God. Vissh. Tekh. Uchebn. Zaved., Mat. **1**(2) (1965), 37–77; in Bulgarian, Russian summary.

Department of Mathematics
Uppsala University
Uppsala, Sweden
thomas@math.uu.se

(Received 24 05 2020)