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ON DISCONTINUITY AT FIXED POINT VIA POWER QUASI CONTRACTION

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Dedicated to Professor Hari Mohan Srivastava on the occasion of his 80th birthday

Abstract. We extend the scope of the study of fixed point theorems of power quasi contractions from the class of continuous mappings to a wider class of mappings which also include discontinuous mappings. As a by-product, we provide a new answer to an open problem posed by Rhoades.

1. Introduction

Ćirić [**6**] generalized the Banach contraction principle by taking a convex combination of distances between the four points x, y, Tx and Ty and named it as quasi-contraction.

Definition 1.1. A self-mapping T of a metric space (X, d) is said to be quasi-contraction iff

$$d(Tx,Ty) \leqslant c\max\{d(x,y), d(x,Tx), d(y,Ty), d(x,Ty), d(y,Tx)\},$$

for all x, y in X, where $0 \leqslant c < 1$.

Theorem 1.1. [**6**] *Let T be a quasi-contraction on the metric space (X, d) into itself and let X be T-orbitally complete. Then T has a unique fixed point in X.*

Remark 1.1. It may be observed that quasi-contraction does not require the continuity of the mapping T for the existence of the fixed point. However, a mapping T satisfying quasi-contraction turns out to be continuous at the fixed point. To see this, suppose that $z = Tz$ is a fixed point of T and $x_n \to z$. Then

$$\begin{aligned} d(Tx_n, z) &= d(Tx_n, Tz) \\ &\leqslant c\max\Big\{d(x_n,z), d(x_n,Tx_n), d(z,Tz), d(x_n,Tz), d(z,Tx_n)\Big\}. \end{aligned}$$

Now we consider the following two non-trivial cases.

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Case 1. If $d(Tx_n, z) \leqslant cd(x_n, Tx_n) \leqslant c[d(x_n, z) + d(z, Tx_n)]$,
that is, $(1 - c)d(Tx_n, z) \leqslant cd(x_n, z)$, then $Tx_n \to z = Tz$.

Case 2. If $d(Tx_n, z) \leqslant cd(z, Tx_n)$, that is, $(1 - c)d(Tx_n, z) \leqslant 0$,
then $Tx_n \to z = Tz$.

Thus in both cases we get $Tx_n \to z = Tz$ and T is continuous at the fixed point z.

In 1979, Fisher [**7**] generalized the notion of quasi-contraction and proved a power contraction version of Ćirić's fixed point theorem. In this paper we will call quasi-contraction (in the sense of Fisher) as power quasi-contraction. The quasi-contraction (in the sense of Ćirić) may be considered as power quasi-contraction of order 1 (for $p = q = 1$).

Definition 1.2. A self-mapping T of a metric space (X, d) is said to be power quasi-contraction or quasi-contraction (in the sense of Fisher) if and only if

$$d(T^p x, T^q y) \leqslant c \max\big\{d(T^r x, T^s y), d(T^r x, T^{r'} x), d(T^s y, T^{s'} y) : \\ 0 \leqslant r,\ r' \leqslant p \text{ and } 0 \leqslant s,\ s' \leqslant q\big\},$$

for all x, y in X, where $0 \leqslant c < 1$, for some fixed positive integers p and q and by T^p we mean the composition of T by itself p times.

Theorem 1.2. [**7**] *Let T be a power quasi-contraction on the complete metric space (X, d) into itself and T be continuous. Then T has a unique fixed point in X.*

Remark 1.2. If we consider q (or p) $= 1$ in Theorem 1.2 (above), then the condition of continuity on T is not necessary to ensure the existence of a fixed point [**7**].

Recall that the set $O(x; T) = \{T^n x : n = 0, 1, 2, \dots\}$ is called the orbit of the self-mapping T at the point $x \in X$.

Definition 1.3. A self-mapping T of a metric space (X, d) is called orbitally continuous at a point $z \in X$ if for any sequence $\{x_n\} \subset O(x; T)$ (for some $x \in X$) $x_n \to z$ implies $Tx_n \to Tz$ as $n \to \infty$.

Remark 1.3. Continuous self-mapping of a metric space are orbitally continuous, however, the converse need not be true (see Example 1.5 below).

In a recent work, Pant and Pant [**21**] gave a weaker version of continuity notion and called it as k-continuity.

Definition 1.4. A self-mapping T of a metric space (X, d) is called k-continuous [**21**], $k = 1, 2, 3, \dots$, if $T^k x_n \to Tz$, whenever $\{x_n\}$ is a sequence in X such that $T^{k-1} x_n \to z$.

Remark 1.4. It is important to note that for a self-mapping T of a metric space (X, d), the notion of 1-continuity coincides with continuity. However,

$$\text{1-continuity} \Rightarrow \text{2-continuity} \Rightarrow \text{3-continuity} \Rightarrow \ \cdots,$$

but not conversely [**21**].

EXAMPLE 1.1. Let $X = [0, 2]$ and d be the usual metric on X. Define $T : X \to X$ by

$$T(x) = 1 \text{ if } x \in [0, 1], \quad T(x) = 0 \text{ if } x \in (1, 2].$$

Then $Tx_n \to t \Rightarrow T^2x_n \to t$ since $Tx_n \to t$ implies $t = 0$ or $t = 1$ and $T^2x_n \to 1 = T1$ for all n. Hence T is 2-continuous. However, T is not continuous or 1-continuous at $x = 1$.

The study of contractive conditions which are discontinuous at the fixed point is presently an active area of research. The question regarding the existence of a contractive definition which is strong enough to generate a fixed point, but not so strong as to force the mapping to be continuous at the fixed point was reiterated by Rhoades in [**24**] as an existing open problem. A few answers to the open question have been given in [**2**–**5**, **8**, **17**–**22**]. In this paper, we provide one more affirmative answer (independent to the previous answers) to the open question posed by Rhoades. Our results improve and generalize many results existing in the literature [**5**–**7**, **12**, **13**, **15**, **16**, **26**].

2. Main results

Our first main result is the following.

THEOREM 2.1. *Let T be a self-mapping of a complete metric space (X, d) such that for each $x, y \in X$;*

$$d(T^p x, T^q y) \leqslant c \max\big\{d(T^r x, T^s y), d(T^r x, T^{r'} x), d(T^s y, T^{s'} y) : 0 \leqslant r,\ r' \leqslant p,\ 0 \leqslant s,\ s' \leqslant q\big\}, \tag{2.1}$$

for all x, y in X, where $0 \leqslant c < 1$, for some fixed positive integers p and q. If T is orbitally continuous or k-continuous for $k \geqslant 1$, then T possesses a unique fixed point in X.

PROOF. The case $k = 1$, i.e., T being continuous is analogous to the Fisher fixed point theorem. We, therefore, take up the case when $k \geqslant 2$. Choose c such that $\delta = (c/(1 - c)) \geqslant 1$. Also, assume that $p \geqslant q$. Let x be an arbitrary point and define the points inductively by $x_0 = x, x_{n+1} = T^n x_0 = Tx_n$ for all $n \in \mathbb{N} \bigcup \{0\}$. The sequence of points $\{T^n x : n \in \mathbb{N}\}$ is bounded. If not, then the set of real numbers $\{d(T^n x, T^q x) : n \in \mathbb{N}\}$ is unbounded and so there exists an integer n such that

$$d(T^n x, T^q x) > \delta \max\{d(T^i x, T^q x) : 0 \leqslant i \leqslant p\}. \tag{2.2}$$

Let n be the smallest such n so that

$$d(T^n x, T^q x) \geqslant \max\{d(T^r x, T^q x) : 0 \leqslant r \leqslant n\}. \tag{2.3}$$

Since $\delta \geqslant 1$ from (2.2), it is clear that $n > p$. From (2.2) and (2.3) we get

$$\begin{aligned}
(1 - c)\, d(T^n x, T^q x) &> c \max\{d(T^i x, T^q x) : 0 \leqslant i \leqslant p\} \\
&\geqslant c \max\{d(T^i x, T^r y) - d(T^r x, T^q x) : 0 \leqslant i \leqslant p, 0 \leqslant r \leqslant n\} \\
&\geqslant c \max\{d(T^i x, T^r y) - d(T^n x, T^q x) : 0 \leqslant i \leqslant p, 0 \leqslant r \leqslant n\},
\end{aligned}$$

i.e.,

$$d(T^n x, T^q x) > c \max\{d(T^i x, T^r y) : 0 \leqslant i \leqslant p, 0 \leqslant r \leqslant n\}. \tag{2.4}$$

We will show that

$$d(T^n x, T^q x) > c \max\{d(T^i x, T^r y) : 0 \leqslant i, r \leqslant n\}. \tag{2.5}$$

For if not $d(T^n x, T^q x) \leqslant c \max\{d(T^i x, T^r y) : 0 \leqslant i, r \leqslant n\}$ and so

$$d(T^n x, T^q x) \leqslant c \max\{d(T^i x, T^r y) : p < i, r < n\}, \tag{2.6}$$

on using (2.4). In view of (2.1) and (2.6) we get

$$d(T^n x, T^q x) \leqslant c^h \max\{d(T^i x, T^r y) : p < i, r < n\} \tag{2.7}$$

for $h = 1, 2, \ldots$. On letting $h \to \infty$ in (2.7), we have $d(T^n x, T^q x) = 0$, contradicting the definition of n. Hence inequality (2.5) holds.

However, on using (2.1), we now have

$$\begin{aligned} d(T^n x, T^q y) \leqslant c \max\{&d(T^r x, T^s y), d(T^r x, T^{r'} x), d(T^s y, T^{s'} y) : \\ &n - p \leqslant r,\ r' \leqslant n \text{ and } 0 \leqslant s,\ s' \leqslant q\}, \end{aligned}$$

which implies

$$d(T^n x, T^q y) \leqslant c \max\{d(T^r x, T^s y) : 0 \leqslant r, s \leqslant n\},$$

which is not possible because of (2.5). Hence $\{T^n x : n = 1, 2, \ldots\}$ is bounded.

We now put

$$M = \sup\{d(T^r x, T^s x) : r, s = 0, 1, 2, \ldots\} < \infty.$$

Then, for arbitrary $\epsilon > 0$, choose N so that $c^N M < \epsilon$. It follows that for $m, n > N, \max\{p, q\}$ and on using inequality (2.1) N times we get

$$d(T^m x, T^n x) < c^N M < \epsilon.$$

Thus the sequence $\{T^n x : n \in \mathbb{N}\}$ is a Cauchy sequence in X. Since X is complete, there exists a point $z \in X$ such that $x_{n+1} = T^n x \to z$ as $n \to \infty$. Also, for each $k \geqslant 1$ we have $T^k x_n \to z$.

Suppose that T is orbitally continuous. Since $T^n x \to z$, orbital continuity implies that $TT^n x \to Tz$. This gives $z = Tz$ and z is a fixed point of T. Uniqueness of the fixed point follows from (i).

Finally, suppose that T is $k-$continuous. Since $T^{k-1} x_n \to z$, k-continuity of T implies that $lim_{n\to\infty} T^k x_n = Tz$. This yields $z = Tz$, that is, z is a fixed point of T. Uniqueness of the fixed point follows easily. □

We now give an example to show that power quasi-contraction is strong enough to generate a fixed point but the fixed point may be discontinuous at the fixed point.

EXAMPLE 2.1. Let $X = [-2, 4]$ and d be the usual metric on X. Define $T : X \to X$ by

$$T(x) = \begin{cases} 1/2, & \text{if } -2 \leqslant x < 0; \\ 1, & \text{if } 0 \leqslant x \leqslant 1; \\ 0, & \text{if } 1 < x \leqslant 2; \\ -3/2, & \text{if } 2 < x \leqslant 4. \end{cases}$$

Then T satisfies all the conditions of Theorem 2.1 and has a unique fixed point $x = 1$ at which T is discontinuous. It may be noted that T is 3-continuous, since $T^3x = 1$ for all $x \in X$.

In 1981, Istrătescu [**12**] extended the well-known Banach contraction principle by introducing a convexity condition, namely, convex contraction mapping of order m where m is a positive integer $\geqslant 2$. Meanwhile, a more complete study which consists data dependency, well-posedness, Ulam-Hyers stability, limit shadowing property and Ostrowski property for convex contraction mapping of order 2 was recently proposed in [**16**](see also [**25**]). In [**9**] Georgescu initiated study of iterated function system consisting of generalized convex contractions.

DEFINITION 2.1. A continuous function $T : X \to X$, where (X, d) is a metric space is called convex contraction of order m (also called generalized convex contraction [**9**]) if there exist $m \in \mathbb{N}$ and $a_0, a_1, \ldots, a_{m-1} \geqslant 0$ such that $\sum_{i=0}^{m-1} a_j < 1$ and

$$d(T^m x, T^m y) \leqslant \sum_{j=0}^{m-1} a_j d(T^j x, T^j y),$$

for each $x, y \in X$, where by T^j we mean the composition of T by itself j times.

THEOREM 2.2. [**12**] *Let T be a convex contraction of order m self-mapping of a complete metric space (X, d). Then T has a unique fixed point in X.*

It is easy to observe that convex contraction of order $m-1$ is convex contraction of order m, but converse need not be true [**26**].

We now prove a fixed point theorem for convex contraction of order m without assuming continuity requirement on T.

THEOREM 2.3. *Let T be a self-mapping of a complete metric space (X, d) such that for each $x, y \in X$;*

$$d(T^m x, T^m y) \leqslant \sum_{j=0}^{m-1} a_j d(T^j x, T^j y), \tag{2.8}$$

where $a_0, a_1, \ldots, a_{m-1} \geqslant 0$ such that $\sum_{i=0}^{m-1} a_j < 1$. If T is orbitally continuous or k-continuous for $k \geqslant 1$ then T possesses a unique fixed point in X.

PROOF. The proof is straightforward since (2.8) is a particular case of (2.1). □

Putting $m = 2$ in Theorem 2.3, we get the following result which is an extended version of Istrătescu's result for convex contraction mapping of order 2.

COROLLARY 2.1. *Let T be a self-mapping of a complete metric space (X, d) such that for each $x, y \in X$;*

$$d(T^2x, T^2y) \leqslant a_0 d(x, y) + a_1 d(Tx, Ty),$$

where $a_0, a_1 \geqslant 0$ such that $a_0 + a_1 < 1$. If T is orbitally continuous or k-continuous for $k \geqslant 1$ then T possesses a unique fixed point in X.

3. Mappings for which $F(T) = F(T^n)$

DEFINITION 3.1. [**14**] Let T be a self-mapping of metric space (X, d) with a nonempty fixed point set $F(T)$. Then T is said to have property P if $F(T^n) = F(T)$, for each $n \in \mathbb{N}$. Equivalently, it has no nontrivial periodic points.

THEOREM 3.1. *Let T be a convex contraction mapping of order $m(m \geqslant 2)$ on a complete metric space (X, d). Then T and T^m have a unique fixed point, i.e., $F(T) = F(T^m)$.*

PROOF. Since T is a convex contraction mapping of order m on a complete metric space (X, d), then $F(T) = \{z\}$, i.e., z is a unique fixed point for T. It is clear that $F(T) \subset F(T^2) \subset \cdots \subset F(T^m)$. Let $v \in F(T^m)$ and $v \neq z$. We show that $T(v) = v$. If $Tv \neq v$, then (since $T^m v = v(m \geqslant 2)$)

$$\begin{aligned} d(Tv, v) &= d(T^m Tv, T^m v) \\ &\leqslant a_0 d(Tv, v) + a_1 d(T^2 v, Tv) + \cdots + a_{m-1} d(T^m v, T^{m-1} v), \\ &= a_0 d(Tv, v) + a_1 d(v, Tv) + \cdots + a_{m-1} d(Tv, v), \\ &= (a_0 + a_1 + \cdots + a_{m-1}) d(Tv, v) < d(Tv, v), \end{aligned}$$

a contradiction. Hence $F(T) = F(T^m)$. □

REMARK 3.1. Using the notion of k-continuity, a host of theorems proved in [**1**, **10**, **11**, **13**, **15**, **16**, **26**] can also be extended to a wider class of mappings which are discontinuous at the fixed point.

REMARK 3.2. We show that power quasi-contraction and convex contraction mapping of higher orders are sufficient to guarantee the existence of a fixed point, but not so strong as to force the mapping to be continuous at the fixed point. Thus provide one more answer to the open problem of Rhoades [**24**].

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