PUBLICATIONS DE L'INSTITUT MATHÉMATIQUE
Nouvelle série, tome 108(122) (2020), 47–61 DOI: https://doi.org/10.2298/PIM2022047D

SUBCLASSES OF ANALYTIC FUNCTIONS WITH RESPECT TO SYMMETRIC AND CONJUGATE POINTS CONNECTED WITH *Q*-ANALOGUE OF THE BESSEL FUNCTION

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Abstract. By using q-analogue of the Bessel function, we introduce new subclasses of starlike functions with respect to symmetric and conjugate points and obtain some a useful properties of these subclasses.

1. Introduction

Srivastava [**23**] presented and motivated about brief expository overview of the classical $q-$analysis versus the so-called (p, q)-analysis with an obviously redundant additional parameter p. We also briefly consider several other families of such extensively and widely-investigated linear convolution operators as (for example) the Dziok–Srivastava, Srivastava–Wright and Srivastava–Attiya linear convolution operators (see also [**21, 22**]), together with their extended and generalized versions. The theory of (p, q)-analysis has important role in many areas of mathematics and physics. Our usages here of the q-calculus and the fractional q-calculus in geometric function theory of complex analysis are believed to encourage and motivate significant further developments on these and other related topics (see also Srivastava and Karlsson [**24**, pp. 350–351]). Our main objective in this survey-cum-expository article is based chiefly upon the fact that the recent and future usages of the classical q-calculus and the fractional q-calculus in geometric function theory of complex analysis have the potential to encourage and motivate significant further researches on many of these and other related subjects. Jackson [**13, 14**] was the first that gave some application of q-calculus and introduced the q-analogue of derivative and integral operator (see also [**1**]).

Let $\mathcal{A}$ denote the class of all analytic functions of the form

$$f(z) = z + \sum_{k=2}^{\infty} a_k z^k,\ z \in \mathbb{U} := \{z \in \mathbb{C} : |z| < 1\},$$

2010 *Mathematics Subject Classification:* Primary 30C50, 30C45; Secondary 11B65, 47B38.
Key words and phrases: analytic function, Bessel function of first kind, q-derivative, symmetric points, conjugate points.

If $k \in \mathcal{A}$ is given by

$$k(z) = z + \sum_{k=2}^{\infty} b_k z^k, \ z \in \mathbb{U},$$

then, the *Hadamard (or convolution) product* of f and k is defined by

$$(f * k)(z) := z + \sum_{k=2}^{\infty} a_k b_k z^k, \ z \in \mathbb{U}.$$

If f and F are analytic functions in $\mathbb{U}$, we say that f *is subordinate to* F, written as $f \prec F$ or $f(z) \prec F(z)$ if there exists a *Schwarz function* w, which is analytic in $\mathbb{U}$, with $w(0) = 0$, and $|w(z)| < 1$ for all $z \in \mathbb{U}$, such that $f(z) = F(w(z))$, $z \in \mathbb{U}$. Furthermore, if the function F is univalent in $\mathbb{U}$, then we have the following equivalence (see [**5**] and [**16**]):

$$f(z) \prec F(z) \Leftrightarrow f(0) = F(0) \text{ and } f(\mathbb{U}) \subset F(\mathbb{U}). \tag{1.1}$$

Sakaguchi [**18**] introduced a class S_s^* of *functions starlike with respect to symmetric points*, which consists of functions $f \in \mathcal{A}$ satisfying the inequality

$$\operatorname{Re}\left(\frac{zf'(z)}{f(z) - f(-z)}\right) > 0, \quad z \in \mathbb{U},$$

Obviousely the class of univalent functions and starlike with respect to symmetric points include the classes of convex functions and odd functions starlike with respect to the origin (see [**18**]).

Also, Aouf et al. [**2**] introduced and studied the class $S_{s,n}^* T(1,1)$ of *functions n-starlike with respect to symmetric points*, which consists of functions $f \in \mathcal{A}$ with $a_k \leqslant 0$ for $k \geqslant 2$, and satisfying the inequality

$$\operatorname{Re}\left(\frac{D^{n+1}f(z)}{D^n f(z) - D^n f(-z)}\right) > 0, \quad z \in \mathbb{U},$$

where D^n is the *Sălăgean operator* [**19**].

El-Ashwah and Thomas [**6**] introduced and studied the class namely S_c^* consisting of functions starlike with respect to conjugate points if it satisfies the following condition:

$$\operatorname{Re}\left(\frac{zf'(z)}{f(z) + \overline{f(\overline{z})}}\right) > 0, \quad z \in \mathbb{U},$$

and, Aouf et al. [**2**] introduced and studied the class $S_{c,n}^* T(1,1)$ of *functions n-starlike with respect to conjugate points*, which consists of functions $f \in \mathcal{A}$ with $a_k \leqslant 0$ for $k \geqslant 2$, and satisfying the inequality

$$\operatorname{Re}\left(\frac{D^{n+1}f(z)}{D^n f(z) + D^n \overline{f(\overline{z})}}\right) > 0, \ z \in \mathbb{U}.$$

Definition 1.1. [**15**] Let Ω be the family of functions $w(z)$ which are analytic in $\mathbb{U}$ and satisfy the conditions $w(0) = 0$ and $|w(z)| < 1$ for $z \in \mathbb{U}$. Next, for arbitrary fixed numbers A and B, such that $-1 \leqslant B < A \leqslant 1$, denote by $\mathcal{P}[A, B]$ the family of functions $p(z) = 1 + b_1 z + b_2 z^2 + \cdots$, is analytic in $\mathbb{U}$ and such that

$p(z) \in \mathcal{P}[A, B]$, if and only if $p(z) \prec \frac{1+Az}{1+Bz}$ or $p(z) = \frac{1+Aw(z)}{1+Bw(z)}$, for some function $w(z) \in \Omega$ and for every $z \in \mathbb{U}$.

Recalling that the function of the form $\frac{1+Az}{1+Bz}$ maps conformally $\mathbb{U}$ onto a disc symmetrical with respect to the real axis, which is centered at the point $\frac{1-AB}{1-B^2}$ $(B \neq \pm 1)$, and with radius equal to $\frac{A-B}{1-B^2}$ $(B \neq \pm 1)$.

Janowski [**15**] introduced the subclass of starlike functions as follows:

$$\mathcal{S}^*(A, B) = \Big\{ f(z) \in \mathcal{A} : \frac{zf'(z)}{f(z)} \prec \frac{1+Az}{1+Bz} (-1 \leqslant B < A \leqslant 1; z \in \mathbb{U}) \Big\},$$

and, Nasr and Aouf [**17**] introduced the subclass of $\mathcal{S}$ denoted by $\mathcal{S}(b)$ is said to be in the class of starlike functions of complex order b, if

$$\Big\{ f(z) \in \mathcal{A} : \mathrm{Re} \Big\{ 1 + \frac{1}{b} \Big(\frac{zf'(z)}{f(z)} - 1 \Big) \Big\} > 0 \ \ (b \in \mathbb{C} \setminus \{0\}, \ \ z \in \mathbb{U}) \Big\}.$$

Goel and Mehrok [**11**] introduced a subclass of $\mathcal{S}_s^\star$ denoted by $\mathcal{S}_s^\star(A, B)$, defined as follows

$$\Big\{ f(z) \in \mathcal{A} : \frac{2zf'(z)}{f(z) - f(-z)} \prec \frac{1+Az}{1+Bz} \ \ (-1 \leqslant B < A \leqslant 1, \ z \in \mathbb{U}) \Big\}.$$

For $-1 \leqslant B < A \leqslant 1$, $b \in \mathbb{C}^*$, $z \in \mathbb{U}$. Aouf et al. [**3**] with $\Phi(z) = \frac{1+Az}{1+Bz}$, $n = 0$ and Arif et al. [**4**] introduced a subclass of $\mathcal{S}_s^\star$ denoted by $\mathcal{S}_s^*(b, A, B)$ defined as follows

$$\Big\{ f(z) \in \mathcal{A} : 1 + \frac{1}{b} \Big(\frac{2zf'(z)}{f(z) - f(-z)} - 1 \Big) \prec \frac{1+Az}{1+Bz} \ \Big\}, \tag{1.2}$$

and, Arif et al. [**4**] introduced another subclass of $\mathcal{S}_s^\star$ denoted by $\mathcal{C}_s^*(b, A, B)$ defined as follows

$$\Big\{ f(z) \in \mathcal{A} : 1 + \frac{1}{b} \Big(\frac{2(zf'(z))'}{(f(z) - f(-z))'} - 1 \Big) \prec \frac{1+Az}{1+Bz} \Big\}.$$

Srivastava [**23, 25**] made use of various operators of q-calculus and fractional q-calculus and recalling the definition and notations. The q-shifted factorial is defined for $\lambda, q \in \mathbb{C}$ and $n \in \mathbb{N}_0 = \mathbb{N} \cup \{0\}$ as follows

$$(\lambda; q)_k = \begin{cases} 1 & k = 0, \\ (1-\lambda)(1-\lambda q) \ldots (1-\lambda q^{k-1}) & k \in \mathbb{N}. \end{cases}.$$

By using the q-gamma function $\Gamma_q(z)$, we get

$$(q^\lambda; q)_k = \frac{(1-q)^k \, \Gamma_q(\lambda + k)}{\Gamma_q(\lambda)}, \ \ (k \in \mathbb{N}_0),$$

where (see [**10**])

$$\Gamma_q(z) = (1-q)^{1-z} \frac{(q;q)_\infty}{(q^z;q)_\infty}, \ \ (|q| < 1).$$

Also, we note that $(\lambda; q)_\infty = \prod_{k=0}^\infty (1 - \lambda q^k)$, $(|q| < 1)$, and, the q-gamma function $\Gamma_q(z)$ is known $\Gamma_q(z+1) = [z]_q \Gamma_q(z)$, where $[k]_q$ denotes the basic q-number defined by

$$[k]_q := \begin{cases} \frac{1-q^k}{1-q}, & k \in \mathbb{C}, \\ 1 + \sum_{j=1}^{k-1} q^j, & k \in \mathbb{N}. \end{cases} \tag{1.3}$$

Using definition formula (1.3) we have the next two products:

(i) For any non negative integer k, the *q-shifted factorial* is given by

$$[k]_q! := \begin{cases} 1, & \text{if } k = 0, \\ \prod_{n=1}^{k} [n]_q, & \text{if } k \in \mathbb{N}. \end{cases}$$

(ii) For any positive number r, the *q-generalized Pochhammer symbol* is defined by

$$[r]_{q,k} := \begin{cases} 1, & \text{if } k = 0, \\ \prod_{n=r}^{r+k-1} [n]_q, & \text{if } k \in \mathbb{N}. \end{cases}$$

It is known in terms of the classical (Euler's) gamma function $\Gamma(z)$, that

$$\Gamma_q(z) \to \Gamma(z) \text{ as } q \to 1^-.$$

Also, we observe that

$$\lim_{q\to 1^-} \left\{ \frac{(q^\lambda; q)_k}{(1-q)^k} \right\} = (\lambda)_k,$$

where $(\lambda)_k$ is the familiar Pochhammer symbol defined by

$$(\lambda)_k = \begin{cases} 1, & \text{if } k = 0, \\ \lambda(\lambda+1)\ldots(\lambda+k-1), & \text{if } k \in \mathbb{N}. \end{cases}$$

The *Bessel function of the first kind of order ν* is defined by the infinite series

$$J_\nu(z) := \sum_{k\geqslant 0} \frac{(-1)^k (\frac{z}{2})^{2k+\nu}}{k!\Gamma(k+\nu+1)}, \quad (z \in \mathbb{C},\ \nu \in \mathbb{R}),$$

where Γ stands for the *Gamma function.* Recently, Szász and Kupán [**27**] investigated the univalence of the normalized Bessel function of the first kind $g_\nu : \mathbb{U} \to \mathbb{C}$ defined by (see also [**12, 20**])

$$g_\nu(z) := 2^\nu \Gamma(\nu+1) z^{1-\frac{\nu}{2}} J_\nu(z^{\frac{1}{2}}) = z + \sum_{k=2}^{\infty} \frac{(-1)^{k-1}\Gamma(\nu+1)}{4^{k-1}(k-1)!\Gamma(k+\nu)} z^k, \quad (z \in \mathbb{U}, \nu \in \mathbb{R}).$$

For $0 < q < 1$, the *q-derivative operator* for g_ν is defined by

$$\begin{aligned} D_q g_\nu(z) = D_q\left[z + \sum_{k=2}^{\infty} \frac{(-1)^{k-1}\Gamma(\nu+1)}{4^{k-1}(k-1)!\Gamma(k+\nu)} z^k\right] &:= \frac{g_\nu(qz) - g_\nu(z)}{z(q-1)} \\ = 1 + \sum_{k=2}^{\infty} \frac{(-1)^{k-1}\Gamma(\nu+1)}{4^{k-1}(k-1)!\Gamma(k+\nu)} [k,q] z^{k-1}, \quad z \in \mathbb{U}, \end{aligned}$$

For $\nu > 0$, $\lambda > -1$, and $0 < q < 1$, El-Deeb and Bulboaca [**8**] (see also [**7, 9, 26**]) define the function $\mathcal{I}_{\nu,q}^{\lambda} : \mathbb{U} \to \mathbb{C}$ by $\mathcal{I}_{\nu,q}^{\lambda}(z) * \mathcal{M}_{q,\lambda+1}(z) = z\, D_q g_\nu(z)$, $z \in \mathbb{U}$, where the function $\mathcal{M}_{q,\lambda+1}$ is given by

$$\mathcal{M}_{q,\lambda+1}(z) := z + \sum_{k=2}^{\infty} \frac{[\lambda+1]_{q,k-1}}{[k-1]_q!} z^k,\ z \in \mathbb{U}$$

Then

$$\mathcal{I}_{\nu,q}^{\lambda}(z) := z + \sum_{k=2}^{\infty} \frac{(-1)^{k-1}\Gamma(\nu+1)}{4^{k-1}(k-1)!\Gamma(k+\nu)} \frac{[k]_q!}{[\lambda+1]_{q,k-1}} z^k, \quad z \in \mathbb{U},$$
$$(\nu > 0,\ \lambda > -1,\ 0 < q < 1).$$

El-Deeb and Bulboaca [**8**] used the definition of q-derivative along with the idea of convolutions to introduce the linear operator $\mathcal{N}_{\nu,q}^{\lambda} : \mathcal{A} \to \mathcal{A}$ defined by

$$\mathcal{N}_{\nu,q}^{\lambda} f(z) := \mathcal{I}_{\nu,q}^{\lambda}(z) * f(z) = z + \sum_{k=2}^{\infty} \psi_k a_k z^k,\ z \in \mathbb{U},$$
$$(\nu > 0,\ \lambda > -1,\ 0 < q < 1), \tag{1.4}$$

where

$$\psi_k := \frac{(-1)^{k-1}\Gamma(\nu+1)}{4^{k-1}(k-1)!\Gamma(k+\nu)} \cdot \frac{[k]_q!}{[\lambda+1]_{q,k-1}}. \tag{1.5}$$

From definition relation (1.4), we can easily verify that the next relations hold for all $f \in \mathcal{A}$:

(i) $[\lambda+1, q]\mathcal{N}_{\nu,q}^{\lambda} f(z) = [\lambda, q]\mathcal{N}_{\nu,q}^{\lambda+1} f(z) + q^{\lambda} z \partial_q(\mathcal{N}_{\nu,q}^{\lambda+1} f(z))$, $z \in \mathbb{U}$;

(ii) $\lim_{q\to 1^-} \mathcal{N}_{\nu,q}^{\lambda} f(z) = \mathcal{I}_{\nu,1}^{\lambda} * f(z) =: \mathcal{I}_{\nu}^{\lambda} f(z) = z + \sum_{k=2}^{\infty} \phi_k\, a_k z^k$, $z \in \mathbb{U}$, where

$$\phi_k := \frac{k!}{(\lambda+1)_{k-1}} \frac{(-1)^{k-1}\Gamma(\nu+1)}{4^{k-1}(k-1)!\Gamma(k+\nu)} \tag{1.6}$$

The class defined in (1.2) could be generalized by introducing the next class of functions, defined with the aid of the $\mathcal{N}_{\nu,q}^{\lambda}$ operator.

DEFINITION 1.2. Let the function $f \in \mathcal{A}$ is said to be in the class $\mathcal{S}_s^{\lambda,\nu,q}(\gamma, A, B)$ if and only if

$$1 + \frac{1}{\gamma}\left[\frac{2z(\mathcal{N}_{\nu,q}^{\lambda} f(z))'}{\mathcal{N}_{\nu,q}^{\lambda} f(z) - \mathcal{N}_{\nu,q}^{\lambda} f(-z)} - 1\right] \prec \frac{1+Az}{1+Bz},$$
$$(-1 \leqslant A \leqslant B \leqslant 1,\ \nu > 0,\ \lambda > -1,\ 0 < q < 1,\ \gamma \in \mathbb{C}^*).$$

Putting $q \to 1^-$ in the class $\mathcal{S}_s^{\lambda,\nu,q}(\gamma, A, B)$, we obtain that

$$\lim_{q\to 1^-} \mathcal{S}_s^{\lambda,\nu,q}(\gamma, A, B) := \mathcal{G}_s^{\lambda,\nu}(\gamma, A, B),$$

where

$$\mathcal{G}_s^{\lambda,\nu}(\gamma, A, B) := \left\{1 + \frac{1}{\gamma}\left[\frac{2z(\mathcal{I}_{\nu}^{\lambda} f(z))'}{\mathcal{I}_{\nu}^{\lambda} f(z) - \mathcal{I}_{\nu}^{\lambda} f(-z)} - 1\right] \prec \frac{1+Az}{1+Bz}\right\},$$

$$(-1 \leqslant A \leqslant B \leqslant 1,\ \nu > 0,\ \lambda > -1,\ \gamma \in \mathbb{C}^*).$$

Also, by using the $\mathcal{N}_{\nu,q}^{\lambda}$ operator, we define another class as follows.

DEFINITION 1.3. Let the function $f \in \mathcal{A}$ is said to be in the class $\mathcal{S}_c^{\lambda,\nu,q}(\gamma, A, B)$ if and only if

$$1 + \frac{1}{\gamma}\left[\frac{2z(\mathcal{N}_{\nu,q}^{\lambda} f(z))'}{\mathcal{N}_{\nu,q}^{\lambda} f(z) + \overline{\mathcal{N}_{\nu,q}^{\lambda} f(\overline{z})}} - 1\right] \prec \frac{1+Az}{1+Bz},$$
$$(-1 \leqslant A \leqslant B \leqslant 1,\ \nu > 0,\ \lambda > -1,\ 0 < q < 1,\ \gamma \in \mathbb{C}^*).$$

Putting $q \to 1^-$ in the class $\mathcal{S}_c^{\lambda,\nu,q}(\gamma, A, B)$, we obtain that

$$\lim_{q \to 1^-} \mathcal{S}_c^{\lambda,\nu,q}(\gamma, A, B) := \mathcal{G}_c^{\lambda,\nu}(\gamma, A, B),$$

where

$$\mathcal{G}_c^{\lambda,\nu}(\gamma, A, B) := \left\{1 + \frac{1}{\gamma}\left[\frac{2z(\mathcal{I}_{\nu}^{\lambda} f(z))'}{\mathcal{I}_{\nu}^{\lambda} f(z) + \overline{\mathcal{I}_{\nu}^{\lambda} f(\overline{z})}} - 1\right] \prec \frac{1+Az}{1+Bz}\right\},$$
$$(-1 \leqslant A \leqslant B \leqslant 1,\ \nu > 0,\ \lambda > -1,\ \gamma \in \mathbb{C}^*).$$

The following lemmas will be needed to prove our results.

LEMMA 1.1. [**11**, Lemma 2] *If* $p(z) = 1 + p_1 z + p_2 z^2 + \cdots \in \mathcal{P}[A, B]$*, then*

$$|p_n| \leqslant A - B.$$

LEMMA 1.2. [**11**, Lemma 3] *If* N *be analytic and* M *starlike functions in* $\mathbb{U}$ *with* $N(0) = M(0) = 0$*, then*

$$\frac{|N'(z)/M'(z) - 1|}{|A - B(N'(z)/M'(z))|} < 1,\ -1 \leqslant A \leqslant B \leqslant 1$$

implies

$$\frac{|N(z)/M(z) - 1|}{|A - B(N(z)/M(z))|} < 1, \quad (z \in \mathbb{U}).$$

2. Properties of the subclass $\mathcal{S}_s^{\lambda,\nu,q}(\gamma, A, B)$

Unless otherwise mentioned, we shall assume in the reminder of this paper that $-1 \leqslant B \leqslant A \leqslant 1$, $\nu > 0$, $\lambda > -1$, $0 < q < 1$, $\gamma \in \mathbb{C}^*$, and the powers are understood as principle values. Throughout this work, we use the following notation

$$\prod_{i=k}^{k-1} A(i) = 1.$$

THEOREM 2.1. *Let* $f(z) \in \mathcal{S}_s^{\lambda,\nu,q}(\gamma, A, B)$*, then the following condition*

$$1 + \frac{1}{\gamma}\left[\frac{z(\mathcal{N}_{\nu,q}^{\lambda} \psi(z))'}{\mathcal{N}_{\nu,q}^{\lambda} \psi(z)} - 1\right] \prec \frac{1+Az}{1+Bz}, \tag{2.1}$$

is satisfied for the odd function ψ*, where*

$$\psi(z) := \frac{f(z) - f(-z)}{2}. \tag{2.2}$$

PROOF. If $f \in \mathcal{S}_s^{\lambda,\nu,q}(\gamma, A, B)$, then there exists $h \in \mathcal{P}[A, B]$, such that

$$h(z) = 1 + \frac{1}{\gamma}\left[\frac{2z(\mathcal{N}_{\nu,q}^{\lambda} f(z))'}{\mathcal{N}_{\nu,q}^{\lambda} f(z) - \mathcal{N}_{\nu,q}^{\lambda} f(-z)} - 1\right], \tag{2.3}$$

It follows that

$$\gamma(h(z) - 1) = \frac{2z(\mathcal{N}_{\nu,q}^{\lambda} f(z))'}{\mathcal{N}_{\nu,q}^{\lambda} f(z) - \mathcal{N}_{\nu,q}^{\lambda} f(-z)} - 1,$$

$$\gamma(h(-z) - 1) = \frac{-2z(\mathcal{N}_{\nu,q}^{\lambda} f(-z))'}{\mathcal{N}_{\nu,q}^{\lambda} f(z) - \mathcal{N}_{\nu,q}^{\lambda} f(-z)} - 1,$$

which implies that

$$\frac{h(z) + h(-z)}{2} = 1 + \frac{1}{\gamma}\left[\frac{z(\mathcal{N}_{\nu,q}^{\lambda} \psi(z))'}{\mathcal{N}_{\nu,q}^{\lambda} \psi(z)} - 1\right]. \tag{2.4}$$

On the other hand, $h(z) \prec \frac{1+Az}{1+Bz}$, and $\frac{1+Az}{1+Bz}$ is univalent, so by (1.1), we have

$$\frac{h(z) + h(-z)}{2} \prec \frac{1 + Az}{1 + Bz}.$$

It follows (2.1). □

Taking $q \to 1^-$in Theorem 2.1, we obtain the following corollary:

COROLLARY 2.1. *Let $f(z) \in \mathcal{G}_s^{\lambda,\nu}(\gamma, A, B)$, then the following condition*

$$1 + \frac{1}{\gamma}\left[\frac{z(\mathcal{I}_{\nu}^{\lambda} \psi(z))'}{\mathcal{I}_{\nu}^{\lambda} \psi(z)} - 1\right] \prec \frac{1 + Az}{1 + Bz}.$$

is satisfied for the odd function ψ given by (2.2).

THEOREM 2.2. *A function $f \in \mathcal{S}_s^{\lambda,\nu,q}(\gamma, A, B)$, if and only if there exists $p \in \mathcal{P}[A, B]$ such that*

$$(\mathcal{N}_{\nu,q}^{\lambda} f(z))' = (\gamma(h(z) - 1) + 1) \exp\left(\frac{\gamma}{2}\int_0^z \frac{h(t) + h(-t) - 2}{t} dt\right). \tag{2.5}$$

PROOF. From Theorem 2.1, we have (2.4), it implies

$$\frac{(\mathcal{N}_{\nu,q}^{\lambda} \psi(z))'}{\mathcal{N}_{\nu,q}^{\lambda} \psi(z)} = \frac{1}{z} + \frac{\gamma}{2}\Big(\frac{h(z) + h(-z) - 2}{z}\Big).$$

Integrating the above equation,

$$\mathcal{N}_{\nu,q}^{\lambda} \psi(z) = z \exp\left(\frac{\gamma}{2}\int_0^z \frac{h(t) + h(-t) - 2}{t} dt\right). \tag{2.6}$$

Since $f \in \mathcal{S}_s^{\lambda,\nu,q}(\gamma, A, B)$, then from (2.3), we obtain

$$z(\mathcal{N}_{\nu,q}^{\lambda} f(z))' = (\gamma(h(z) - 1) + 1)\mathcal{N}_{\nu,q}^{\lambda} \psi(z)$$

Using (2.6) and above equation, we get (2.5). □

Taking $q \to 1^-$in Theorem 2.2, we obtain the following corollary:

Corollary 2.2. *A function* $f \in \mathcal{G}_s^{\lambda,\nu}(\gamma, A, B)$*, if and only if there exists* $p \in \mathcal{P}[A, B]$ *such that*

$$(\mathcal{I}_\nu^\lambda f(z))' = (\gamma(h(z)-1)+1)\exp\left(\frac{\gamma}{2}\int_0^z \frac{h(t)+h(-t)-2}{t}dt\right).$$

Theorem 2.3. *If* $f(z) \in \mathcal{S}_s^{\lambda,\nu,q}(\gamma, A, B)$*, then for all* $n \geqslant 1$*,*

$$|a_{2n}| \leqslant \frac{|\gamma|(A-B)}{2^n n!|\psi_{2n}|}\prod_{k=1}^{n-1}(|\gamma|(A-B)+2k), \tag{2.7}$$

$$|a_{2n+1}| \leqslant \frac{|\gamma|(A-B)}{2^n n!|\psi_{2n+1}|}\prod_{k=1}^{n-1}(|\gamma|(A-B)+2k). \tag{2.8}$$

where ψ_k*, for all* $k \geqslant 2$ *are given by* (1.5).

Proof. Since $f \in \mathcal{S}_s^{\lambda,\nu,q}(\gamma, A, B)$, Definition 1.2 yields

$$1+\frac{1}{\gamma}\left[\frac{2z(\mathcal{N}_{\nu,q}^\lambda f(z))'}{\mathcal{N}_{\nu,q}^\lambda f(z)-\mathcal{N}_{\nu,q}^\lambda f(-z)}-1\right] = \frac{1+Aw(z)}{1+Bw(z)}. \tag{2.9}$$

Assuming that

$$h(z) = 1+\sum_{k=1}^{\infty} c_k z^k = \frac{1+Aw(z)}{1+Bw(z)} \tag{2.10}$$

In view of (2.9) and (2.10), we get

$$2z(\mathcal{N}_{\nu,q}^\lambda f(z))' = \left(\mathcal{N}_{\nu,q}^\lambda f(z)-\mathcal{N}_{\nu,q}^\lambda f(-z)\right)\left(1+\gamma\sum_{k=1}^{\infty} c_k z^k\right).$$

It follows from (1.4) that

$$\begin{aligned}
&z+2\psi_2 a_2 z^2+3\psi_3 a_3 z^3+4\psi_4 a_4 z^4+\cdots+2n\psi_{2n}a_{2n}z^{2n}\\
&\quad+(2n+1)\psi_{2n+1}a_{2n+1}z^{2n+1}+\cdots\\
&\quad=(z+\psi_3 a_3 z^3+\psi_5 a_5 z^5+\cdots+\psi_{2n-1}a_{2n-1}z^{2n-1}+\psi_{2n+1}a_{2n+1}z^{2n+1}+\cdots)\\
&\qquad\cdot(1+\gamma c_1 z+\gamma c_2 z^2+\cdots)
\end{aligned}$$

Equating the coefficients of the like powers of z, we obtain

$$2\psi_2 a_2 = \gamma c_1, \;\; 2\psi_3 a_3 = \gamma c_2, \tag{2.11}$$

$$4\psi_4 a_4 = \gamma c_3+\gamma c_1\psi_3 a_3, \;\; 4\psi_5 a_5 = \gamma c_4+\gamma c_2\psi_3 a_3, \tag{2.12}$$

$$2n\psi_{2n}a_{2n} = \gamma c_{2n-1}+\gamma c_{2n-3}\psi_3 a_3+\gamma c_{2n-5}\psi_5 a_5+\cdots+\gamma c_1\psi_{2n-1}a_{2n-1}, \tag{2.13}$$

$$2n\psi_{2n+1}a_{2n+1} = \gamma c_{2n}+\gamma c_{2n-2}\psi_3 a_3+\gamma c_{2n-4}\psi_5 a_5+\ldots+\gamma c_2\psi_{2n-1}a_{2n-1}. \tag{2.14}$$

We prove (2.7) and (2.8) using mathematical induction. Using Lemma 1.1, (2.11) and (2.12) respectively, we get

$$|a_2| \leqslant \frac{|\gamma|}{2|\psi_2|}(A-B)|, \;\; a_3| \leqslant \frac{|\gamma|}{2|\psi_3|}(A-B),$$

$$|a_4| \leqslant \frac{|\gamma|(A-B)}{8|\psi_4|}(2+|\gamma|(A-B)),\ \ |a_5| \leqslant \frac{|\gamma|(A-B)}{8|\psi_5|}(2+|\gamma|(A-B)).$$

It follows that (2.7) and (2.8) hold for $n = 1,\ 2$. Equation (2.13) in conjunction with Lemma 1.1 yields

$$|a_{2n}| \leqslant \frac{|\gamma|(A-B)}{2n|\psi_{2n}|}\left(1+\sum_{r=1}^{n-1}|\psi_{2r+1}||a_{2r+1}|\right)$$

Next, we assume that (2.7) and (2.8) hold for $3, 4, \ldots, n-1$. Thus the above inequality yields

$$|a_{2n}| \leqslant \frac{|\gamma|(A-B)}{2n|\psi_{2n}|}\left(1+\sum_{r=1}^{n-1}\frac{|\gamma|(A-B)}{2^r r!}\prod_{k=1}^{r-1}(|\gamma|(A-B)+2k)\right). \tag{2.15}$$

To complete the proof it is sufficient to show that

$$\begin{aligned}\frac{|\gamma|(A-B)}{2m|\psi_{2m}|}&\left(1+\sum_{r=1}^{m-1}\frac{|\gamma|(A-B)}{2^r r!}\prod_{k=1}^{r-1}(|\gamma|(A-B)+2k)\right)\\ &= \frac{|\gamma|(A-B)}{2^m m!|\psi_{2m}|}\prod_{k=1}^{m-1}(|\gamma|(A-B)+2k),\quad m=3,4,\ldots,n\end{aligned} \tag{2.16}$$

It is easy to see that (2.16) is valid for $m = 3$. Now, suppose that (2.16) is true for $4, \ldots, m-1$. Then (2.15) follows that

$$\begin{aligned}&\frac{|\gamma|(A-B)}{2m|\psi_{2m}|}\left(1+\sum_{r=1}^{m-1}\frac{|\gamma|(A-B)}{2^r r!}\prod_{k=1}^{r-1}(|\gamma|(A-B)+2k)\right)\\
&\quad= \frac{|\gamma|(A-B)}{2m|\psi_{2m}|}\left(1+\sum_{r=1}^{m-2}\frac{|\gamma|(A-B)}{2^r r!}\prod_{k=1}^{r-1}(|\gamma|(A-B)+2k)\right.\\
&\qquad\qquad\left.+\frac{|\gamma|(A-B)}{2^{m-1}(m-1)!}\prod_{k=1}^{m-2}(|\gamma|(A-B)+2k)\right)\\
&\quad= \frac{(m-1)|\psi_{2m-2}|}{m|\psi_{2m}|}\left(\frac{|\gamma|(A-B)}{2^{m-1}(m-1)!|\psi_{2m-2}|}\prod_{k=1}^{m-2}(|\gamma|(A-B)+2k)\right.\\
&\qquad\qquad\left.+\frac{|\gamma|(A-B)}{2m|\psi_{2m}|}\frac{|\gamma|(A-B)}{2^{m-1}(m-1)!}\prod_{k=1}^{m-2}(|\gamma|(A-B)+2k)\right)\\
&\quad= \frac{(m-1)|\psi_{2m-2}|}{m|\psi_{2m}|}\left(\frac{|\gamma|(A-B)}{2^{m-1}(m-1)!|\psi_{2m-2}|}\prod_{k=1}^{m-2}(|\gamma|(A-B)+2k)\right.\\
&\qquad\qquad\left.+\frac{|\gamma|(A-B)}{2m|\psi_{2m}|}\frac{|\gamma|(A-B)}{2^{m-1}(m-1)!}\prod_{k=1}^{m-2}(|\gamma|(A-B)+2k)\right)\end{aligned}$$

$$= \frac{(m-1)|\gamma|(A-B)}{2^{m-1}m!|\psi_{2m}|}\prod_{k=1}^{m-2}(|\gamma|(A-B)+2k)$$
$$+ \frac{|\gamma|(A-B)}{2|\psi_{2m}|}\frac{|\gamma|(A-B)}{2^{m-1}m!}\prod_{k=1}^{m-2}(|\gamma|(A-B)+2k)$$
$$= \frac{|\gamma|(A-B)}{2^{m-1}m!|\psi_{2m}|}\prod_{k=1}^{m-2}(|\gamma|(A-B)+2k)\Big((m-1)+\frac{|\gamma|(A-B)}{2}\Big)$$
$$= \frac{|\gamma|(A-B)}{2^{m-1}m!|\psi_{2m}|}\prod_{k=1}^{m-2}(|\gamma|(A-B)+2k)\Big(\frac{|\gamma|(A-B)+2(m-1)}{2}\Big)$$
$$= \frac{|\gamma|(A-B)}{2^{m}m!|\psi_{2m}|}\prod_{k=1}^{m-1}(|\gamma|(A-B)+2k.$$

That is, (2.16) holds for $m = n$. From (2.15) and (2.16) we obtain (2.7). Similary we can prove (2.8). This completes the proof of Theorem 2.3. □

Taking $q \to 1^-$in Theorem 2.3, we obtain the following corollary:

Corollary 2.3. *Let $f(z) \in \mathcal{G}_s^{\lambda,\nu}(\gamma, A, B)$, then for all $n \geqslant 1$,*

$$|a_{2n}| \leqslant \frac{|\gamma|(A-B)}{2^n n!|\phi_{2n}|}\prod_{k=1}^{n-1}(|\gamma|(A-B)+2k),$$
$$|a_{2n+1}| \leqslant \frac{|\gamma|(A-B)}{2^n n!|\phi_{2n+1}|}\prod_{k=1}^{n-1}(|\gamma|(A-B)+2k),$$

where ϕ_k, for all $k \geqslant 2$ are given by (1.6).

Theorem 2.4. *If the function $f \in \mathcal{S}_s^{\lambda,\nu,q}(\gamma, A, B)$, then $F \in \mathcal{S}_s^{\lambda,\nu,q}(\gamma, A, B)$, where*

$$F(z) = \frac{2}{z}\int_0^z f(t)\,dt \tag{2.17}$$

Proof. From (2.17) it is easy to see that

$$1+\frac{1}{\gamma}\left[\frac{2z(\mathcal{N}_{\nu,q}^{\lambda}F(z))'}{\mathcal{N}_{\nu,q}^{\lambda}F(z)-\mathcal{N}_{\nu,q}^{\lambda}F(-z)}-1\right]$$
$$= \frac{2z\mathcal{N}_{\nu,q}^{\lambda}f(z)+(\gamma-3)\int_0^z\mathcal{N}_{\nu,q}^{\lambda}f(t)\,dt+(\gamma-1)\int_0^z\mathcal{N}_{\nu,q}^{\lambda}f(-t)\,dt}{\gamma(\int_0^z\mathcal{N}_{\nu,q}^{\lambda}f(t)dt+\int_0^z\mathcal{N}_{\nu,q}^{\lambda}f(-t)\,dt)}.$$

Define N and M be the numerator and denominator functions respectively. Therefore,

$$\frac{zM'(z)}{M(z)} = \frac{z\mathcal{N}_{\nu,q}^{\lambda}f(z)-z\mathcal{N}_{\nu,q}^{\lambda}f(-z)}{\int_0^z\mathcal{N}_{\nu,q}^{\lambda}f(t)\,dt+\int_0^z\mathcal{N}_{\nu,q}^{\lambda}f(-t)\,dt} \tag{2.18}$$
$$= \frac{1}{2}\Big(\frac{2zG'(z)}{G(z)-G(-z)}+\frac{2(-z)G'(-z)}{G(-z)-G(z)}\Big)$$

where $G(z) = \int_0^z \mathcal{N}_{\nu,q}^{\lambda} f(t)\,dt$. Since $f \in \mathcal{S}_s^{\lambda,\nu,q}(\gamma, A, B)$, it follows that

$$1 + \frac{1}{\gamma}\Big[\frac{2zG''(z)}{G'(z) - G'(-z)} - 1\Big] \prec \frac{1 + Az}{1 + Bz},$$

and $G(z) \in \mathcal{C}_s^*(b, A, B) \subset \mathcal{S}_s^*(b, A, B) \subset \mathcal{S}_s^*$. From (2.18), it follows that $M(z)$ is starlike functions. In addition to

$$\frac{N'(z)}{M'(z)} = 1 + \frac{1}{\gamma}\Bigg[\frac{2z(\mathcal{N}_{\nu,q}^{\lambda} f(z))'}{\mathcal{N}_{\nu,q}^{\lambda} f(z) - \mathcal{N}_{\nu,q}^{\lambda} f(-z)} - 1\Bigg].$$

Thus $\frac{N'(z)}{M'(z)} = \frac{1+Aw(z)}{1+Bw(z)}$. It follows that

$$\Big|\Big(\frac{N'(z)}{M'(z)} - 1\Big)\Big| < \Big|A - B\Big(\frac{N'(z)}{M'(z)}\Big)\Big|.$$

From Lemma 1.2, we have

$$\Big|\Big(\frac{N(z)}{M(z)} - 1\Big)\Big| < \Big|A - B\Big(\frac{N(z)}{M(z)}\Big)\Big|,$$

and this implies that $F \in \mathcal{S}_s^{\lambda,\nu,q}(\gamma, A, B)$. □

Taking $q \to 1^-$ in Theorem 2.4, we obtain the following corollary:

COROLLARY 2.4. *If the function $f \in \mathcal{G}_s^{\lambda,\nu}(\gamma, A, B)$, then F given by* (2.17) *belongs to the class $\mathcal{G}_s^{\lambda,\nu}(\gamma, A, B)$.*

3. The subclass $\mathcal{S}_c^{\lambda,\nu,q}(\gamma, A, B)$

THEOREM 3.1. *Let $f \in \mathcal{S}_c^{\lambda,\nu,q}(\gamma, A, B)$, then for all $n \geqslant 1$,*

$$|a_{2n}| \leqslant \frac{|\gamma|(A - B)}{(2n-1)!|\psi_{2n}|} \prod_{k=1}^{2n-2} (|\gamma|(A - B) + k), \tag{3.1}$$

$$|a_{2n+1}| \leqslant \frac{|\gamma|(A - B)}{2n!|\psi_{2n+1}|} \prod_{k=1}^{2n-1} (|\gamma|(A - B) + k), \tag{3.2}$$

where ψ_k, for all $k \geqslant 2$ are given by (1.5).

PROOF. Since $f \in \mathcal{S}_c^{\lambda,\nu,q}(\gamma, A, B)$, Definition 1.3 yields

$$1 + \frac{1}{\gamma}\Bigg[\frac{2z(\mathcal{N}_{\nu,q}^{\lambda} f(z))'}{\mathcal{N}_{\nu,q}^{\lambda} f(z) + \mathcal{N}_{\nu,q}^{\lambda} \overline{f(\overline{z})}} - 1\Bigg] = \frac{1 + Aw(z)}{1 + Bw(z)}. \tag{3.3}$$

Assuming that

$$h(z) = 1 + \sum_{k=1}^{\infty} c_k z^k = \frac{1 + Aw(z)}{1 + Bw(z)}, \tag{3.4}$$

from (3.3) and (3.4), we obtain

$$2z(\mathcal{N}_{\nu,q}^{\lambda} f(z))' = (\mathcal{N}_{\nu,q}^{\lambda} f(z) + \mathcal{N}_{\nu,q}^{\lambda} \overline{f(\overline{z})})\Bigg(1 + \gamma\sum_{k=1}^{\infty} c_k z^k\Bigg).$$

It follows from (1.4) that

$$z+2\psi_2a_2z^2+3\psi_3a_3z^3+4\psi_4a_4z^4+\cdots+2n\psi_{2n}a_{2n}z^{2n}$$
$$+(2n+1)\psi_{2n+1}a_{2n+1}z^{2n+1}+\cdots$$
$$=(z+\psi_2a_2z^2+\psi_3a_3z^3+\psi_4a_4z^4+\cdots+\psi_{2n}a_{2n}z^{2n}+\psi_{2n+1}a_{2n+1}z^{2n+1}+\cdots)$$
$$\cdot(1+\gamma c_1z+\gamma c_2z^2+\cdots)$$

Equating the coefficients of the like powers of z, we obtain

$$\psi_2a_2=\gamma c_1,\ \ 2\psi_3a_3=\gamma c_2+\gamma c_1\psi_2a_2, \tag{3.5}$$

$$3\psi_4a_4=\gamma c_3+\gamma c_2\psi_2a_2+\gamma c_1\psi_3a_3,\ \ 4\psi_5a_5=\gamma c_4+\gamma c_3\psi_2a_2+\gamma c_2\psi_3a_3+\gamma c_1\psi_4a_4, \tag{3.6}$$

$$(2n-1)\psi_{2n}a_{2n}=\gamma c_{2n-1}+\gamma c_{2n-2}\psi_2a_2+\cdots+\gamma c_2\psi_{2n-2}a_{2n-2}+\gamma c_1\psi_{2n-1}a_{2n-1}, \tag{3.7}$$

$$2n\psi_{2n+1}a_{2n+1}=\gamma c_{2n}+\gamma c_{2n-1}\psi_2a_2+\cdots+\gamma c_2\psi_{2n-1}a_{2n-1}+\gamma c_1\psi_{2n}a_{2n}. \tag{3.8}$$

Using Lemma 1.1, (3.5) and (3.6) respectively, we get

$$|a_2|\leqslant\frac{|\gamma|}{|\psi_2|}(A-B),\ \ |a_3|\leqslant\frac{|\gamma|(A-B)}{2|\psi_3|}(1+|\gamma|(A-B)),$$
$$|a_4|\leqslant\frac{|\gamma|(A-B)}{2.3|\psi_4|}(1+|\gamma|(A-B))(2+|\gamma|(A-B)),$$
$$|a_5|\leqslant\frac{|\gamma|(A-B)}{2.3.4|\psi_5|}(1+|\gamma|(A-B))(2+|\gamma|(A-B))(3+|\gamma|(A-B)).$$

It follows that (3.1) and (3.2) hold for $n=1,2$. Equation (3.7) in conjunction with Lemma 1.1 yields

$$|a_{2n}|\leqslant\frac{|\gamma|(A-B)}{(2n-1)|\psi_{2n}|}\left(1+\sum_{r=1}^{n-1}|\psi_{2r}||a_{2r}|+\sum_{r=1}^{n-1}|\psi_{2r+1}||a_{2r+1}|\right)$$

Next, we assume that (3.1) and (3.2) hold for $3,4,\ldots,n-1$. Thus the above inequality yields

$$|a_{2n}|\leqslant\frac{|\gamma|(A-B)}{(2n-1)|\psi_{2n}|}\left(1+\sum_{r=1}^{n-1}\frac{|\gamma|(A-B)}{(2r-1)!}\prod_{i=1}^{2r-2}(i+|\gamma|(A-B))+\sum_{r=1}^{n-1}\frac{|\gamma|(A-B)}{2r!}\prod_{i=1}^{2r-1}(i+|\gamma|(A-B))\right). \tag{3.9}$$

In order to complete the proof it is sufficient to show that

$$\frac{|\gamma|(A-B)}{(2m-1)|\psi_{2m}|}\left(1+\sum_{r=1}^{m-1}\frac{|\gamma|(A-B)}{(2r-1)!}\prod_{i=1}^{2r-2}(i+|\gamma|(A-B))+\sum_{r=1}^{m-1}\frac{|\gamma|(A-B)}{2r!}\prod_{i=1}^{2r-1}(i+|\gamma|(A-B))\right) \tag{3.10}$$

$$= \frac{|\gamma|(A-B)}{(2m-1)!|\psi_{2m}|} \prod_{i=1}^{2m-2} (i+|\gamma|(A-B)).$$

It is easy to see that (3.10) is valid for $m = 3$. Now, suppose that (3.10) is true for $4, \dots, m-1$. Then (3.9) follows that

$$\begin{aligned}
&\frac{|\gamma|(A-B)}{(2m-1)|\psi_{2m}|}\Bigg(1+\sum_{r=1}^{m-1}\frac{|\gamma|(A-B)}{(2r-1)!}\prod_{i=1}^{2r-2}(|\gamma|(A-B)+i) \\
&\qquad +\sum_{r=1}^{m-1}\frac{|\gamma|(A-B)}{2r!}\prod_{i=1}^{2r-1}(|\gamma|(A-B)+i)\Bigg) \\
&= \frac{(2m-3)|\psi_{2m-2}|}{(2m-1)|\psi_{2m}|}\Bigg[\frac{|\gamma|(A-B)}{(2m-3)|\psi_{2m-2}|}\Bigg(1+\sum_{r=1}^{m-2}\frac{|\gamma|(A-B)}{(2r-1)!}\prod_{i=1}^{2r-2}(|\gamma|(A-B)+i) \\
&\qquad +\sum_{r=1}^{m-2}\frac{|\gamma|(A-B)}{2r!}\prod_{i=1}^{2r-1}(|\gamma|(A-B)+i)\Bigg)\Bigg] \\
&\quad + \frac{|\gamma|(A-B)}{(2m-1)|\psi_{2m}|}\left(\frac{|\gamma|(A-B)}{(2m-3)!}\prod_{i=1}^{2m-4}(|\gamma|(A-B)+i)\right) \\
&\quad + \frac{|\gamma|(A-B)}{(2m-1)|\psi_{2m}|}\left(\frac{|\gamma|(A-B)}{(2m-2)!}\prod_{i=1}^{2m-3}(|\gamma|(A-B)+i)\right) \\
&= \frac{(2m-3)|\psi_{2m-2}|}{(2m-1)|\psi_{2m}|}\left(\frac{|\gamma|(A-B)}{(2m-3)!|\psi_{2m-2}|}\prod_{i=1}^{2m-4}(|\gamma|(A-B)+i)\right) \\
&\quad + \frac{|\gamma|(A-B)}{(2m-1)|\psi_{2m}|}\left(\frac{|\gamma|(A-B)}{(2m-3)!}\prod_{i=1}^{2m-4}(|\gamma|(A-B)+i)\right) \\
&\quad + \frac{|\gamma|(A-B)}{(2m-1)|\psi_{2m}|}\left(\frac{|\gamma|(A-B)}{(2m-2)!}\prod_{i=1}^{2m-3}(|\gamma|(A-B)+i)\right) \\
&= \frac{1}{(2m-1)|\psi_{2m}|}\left(\frac{|\gamma|(A-B)}{(2m-3)!}\prod_{i=1}^{2m-4}(|\gamma|(A-B)+i)\right)(|\gamma|(A-B)+(2m-3) \\
&\quad + \frac{|\gamma|(A-B)}{|\psi_{2m}|}\left(\frac{|\gamma|(A-B)}{(2m-1)!}\prod_{i=1}^{2m-3}(|\gamma|(A-B)+i)\right) \\
&= \left(\frac{|\gamma|(A-B)}{(2m-1)!|\psi_{2m}|}\prod_{i=1}^{2m-3}(|\gamma|(A-B)+i)\right)(|\gamma|(A-B)+2m-2) \\
&= \frac{|\gamma|(A-B)}{(2m-1)!|\psi_{2m}|}\prod_{i=1}^{2m-2}(|\gamma|(A-B)+i).
\end{aligned}$$

That is, (3.10) is holds for $m = n$. From (3.9) and (3.10), we obtain (3.1).

Similarly we can prove (3.2). □

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(Received 24 06 2020)
(Revised 28 10 2020)