

A (p, ν) -EXTENSION OF SRIVASTAVA'S TRIPLE HYPERGEOMETRIC FUNCTION H_C

S. A. Dar and R. B. Paris

ABSTRACT. We obtain a (p, ν) -extension of Srivastava's triple hypergeometric function $H_C(\cdot)$ by employing the extended Beta function $B_{p, \nu}(x, y)$ introduced in Parmar et al. [J. Class. Anal. **11** (2017), 91–106]. We give some of the main properties of this extended function, which include several integral representations, the Mellin transform, a differential formula, recursion formulas and a bounded inequality.

1. Introduction and preliminaries

In the present paper, we employ the following notations:

$$\mathbf{N} := \{1, 2, \dots\}, \quad \mathbf{N}_0 := \mathbf{N} \cup \{0\}, \quad \mathbf{Z}_0^- := \mathbf{Z}^- \cup \{0\},$$

where the symbols \mathbf{N} and \mathbf{Z} denote the set of integer and natural numbers; as usual, the symbols \mathbf{R} and \mathbf{C} denote the set of real and complex numbers, respectively.

Hypergeometric functions of a single variable have a long history and arise in numerous branches of mathematics and physics. The Gauss hypergeometric function is defined for $b_1, b_2 \in \mathbf{C}$, $c_1 \in \mathbf{C} \setminus \mathbf{Z}_0^-$ by

$$(1.1) \quad {}_2F_1 \left(\begin{matrix} b_1, b_2 \\ c_1 \end{matrix}; z \right) = \sum_{n=0}^{\infty} \frac{(b_1)_n (b_2)_n}{(c_1)_n} \frac{z^n}{n!} \quad (|z| < 1),$$

where $(a)_n$ denotes the Pochhammer symbol (or the shifted factorial) defined by $(a)_n = \Gamma(a+n)/\Gamma(a) = a(a+1)\dots(a+n-1)$. Extensions of this function to include p numerator parameters b_j ($1 \leq j \leq p$) and q denominator parameters c_j ($1 \leq j \leq q$) also find wide application; see [17]. Triple hypergeometric functions (that is functions of three variables x , y and z) have been introduced and investigated by Srivastava and Karlsson [21, Chapter 3] who provide a table of 205 distinct such functions. In [18, 19], Srivastava introduced the triple hypergeometric functions H_A, H_B and H_C of the second order. It is known that H_C and H_B are

2010 *Mathematics Subject Classification*: 33C60; 33C65; 33C70; 33B15; 33C05; 33C45; 33C10.

Key words and phrases: Srivastava's triple hypergeometric functions, Beta and Gamma functions, modified Bessel function, bounded inequality.

generalizations of Appell's hypergeometric functions F_1 and F_2 , while H_A is the generalization of both F_1 and F_2 .

In the present study, we shall confine our attention to Srivastava's triple hypergeometric function H_C given by [21, p. 43, 1.5(11) to 1.5(13)] (see also [18] and [20, p. 68])

$$(1.2) \quad H_C(b_1, b_2, b_3; c_1; x, y, z) := \sum_{m,n,k=0}^{\infty} \frac{(b_1)_{m+k} (b_2)_{m+n} (b_3)_{n+k}}{(c_1)_{m+n+k}} \frac{x^m y^n z^k}{m! n! k!}$$

$$= \sum_{m,n,k=0}^{\infty} \frac{(b_2)_{m+n} (b_3)_{n+k}}{(c_1)_n} \frac{B(b_1+m+k, c_1+n-b_1)}{B(b_1, c_1+n-b_1)} \frac{x^m y^n z^k}{m! n! k!}.$$

Here $B(\alpha, \beta)$ denotes the classical Beta function defined by [13, (5.12.1)]

$$(1.3) \quad B(\alpha, \beta) = \begin{cases} \int_0^1 t^{\alpha-1} (1-t)^{\beta-1} dt, & (\operatorname{Re}(\alpha) > 0, \operatorname{Re}(\beta) > 0) \\ \frac{\Gamma(\alpha)\Gamma(\beta)}{\Gamma(\alpha+\beta)}, & ((\alpha, \beta) \in \mathbf{C} \setminus \mathbf{Z}_0^-). \end{cases}$$

The convergence region for the hypergeometric series $H_C(\cdot)$ is given in [11, p. 243] as $|x| < \alpha$, $|y| < \beta$, $|z| < \gamma$, where α, β, γ satisfy the relation

$$(1.4) \quad \alpha + \beta + \gamma - 2\sqrt{(1-\alpha)(1-\beta)(1-\gamma)} < 2.$$

We shall also find it convenient to introduce an additional parameter a into $H_C(\cdot)$ in the form

$$(1.5) \quad H_C^{(a)}(b_1, b_2, b_3; c_1; x, y, z) := \sum_{m,n,k=0}^{\infty} \frac{(b_2)_{m+n} (b_3)_{n+k}}{(c_1)_n} \frac{B(b_1+a+m+k, c_1+a+n-b_1)}{B(b_1, c_1+n-b_1)} \frac{x^m y^n z^k}{m! n! k!}.$$

which reduces to (1.2) when $a = 0$.

In 1997, Chaudhry et al. [1, Eq. (1.7)] introduced a p -extension of the Beta function $B(x, y)$ given by

$$B(x, y; p) = \int_0^1 t^{x-1} (1-t)^{y-1} \exp\left[\frac{-p}{t(1-t)}\right] dt, \quad (\operatorname{Re}(p) > 0).$$

Also, Chaudhry et al. [2] employed this function to extend the Gauss hypergeometric series ${}_2F_1(\cdot)$ and its integral representations. A further extension of the Beta function has been given by Choi et al. [8] in the form

$$B_{p,q}(x, y) = \int_0^1 t^{x-1} (1-t)^{y-1} \exp\left[-\frac{p}{t} - \frac{q}{1-t}\right] dt \quad (\operatorname{Re}(p) > 0, \operatorname{Re}(q) > 0),$$

which reduces to $B(x, y; p)$ when $p = q$. Recently, Parmar et al. [15] have given a different extension of the Beta function in the form

$$(1.6) \quad B_{p,\nu}(x, y) = \sqrt{\frac{2p}{\pi}} \int_0^1 t^{x-\frac{3}{2}} (1-t)^{y-\frac{3}{2}} K_{\nu+\frac{1}{2}}\left(\frac{p}{t(1-t)}\right) dt,$$

where $\operatorname{Re}(p) > 0$, $\nu \geq 0$ and $K_\nu(z)$ is the modified Bessel function (sometimes known as the Macdonald function) of order ν . When $\nu = 0$, (1.6) reduces to $B(x, y; p)$, since $K_{\frac{1}{2}}(z) = \sqrt{\pi/(2z)}e^{-z}$.

Many authors have studied integral representations of Srivastava's triple hypergeometric function $H_C(\cdot)$ defined in (1.2); see [3–7]. Our aim in this paper is to introduce a (p, ν) -extension of this function, which we denote by $H_{C,p,\nu}(\cdot)$, based on the extended Beta function in (1.6). The Appell hypergeometric function of two variables defined by

$$F_1(b_1, b_2, b_3; c_1; x, y) = \sum_{m,n=0}^{\infty} \frac{(b_2)_n (b_3)_m B(b_1 + m + n, c_1 - b_1)}{B(b_1, c_1 - b_1)} \frac{x^m y^n}{m! n!}$$

where $(|x| < 1, |y| < 1)$ has been extended by replacement of the numerator Beta function (of the same arguments) with $B(x, y; p)$ in [14] and with $B_{p,\nu}(x, y)$ in [9]. Similar extensions of H_A and H_B have been carried out in [10, 16].

The plan of this paper is as follows. The extended function $H_{C,p,\nu}(\cdot)$ is defined in Section 2 and some integral representations are presented involving the modified Bessel function and the Gauss hypergeometric function ${}_2F_1$. The main properties of $H_{C,p,\nu}(\cdot)$ namely, its Mellin transform, a differential formula, a bounded inequality and recursion formulas are established in Sections 3 to 6. Some concluding remarks are made in Section 7.

2. The (p, ν) -extended Srivastava triple hypergeometric function $H_{C,p,\nu}(\cdot)$

Srivastava introduced the triple hypergeometric function $H_C(\cdot)$, together with its integral representations, in [18] and [20]. Here we consider the following (p, ν) -extension of this function, which we denote by $H_{C,p,\nu}(\cdot)$, based on the extended Beta function $B_{p,\nu}(x, y)$ defined in (1.6). This is given by

$$(2.1) \quad H_{C,p,\nu}(b_1, b_2, b_3; c_1; x, y, z) \\ = \sum_{m,n,k=0}^{\infty} \frac{(b_2)_{m+n} (b_3)_{n+k}}{(c_1)_n} \frac{B_{p,\nu}(b_1 + m + k, c_1 + n - b_1)}{B(b_1, c_1 + n - b_1)} \frac{x^m y^n z^k}{m! n! k!},$$

where the parameters $b_1, b_2, b_3 \in \mathbf{C}$ and $c_1 \in \mathbf{C} \setminus \mathbf{Z}_0^-$. The region of convergence is $|x| < \alpha$, $|y| < \beta$, $|z| < \gamma$, where α, β, γ satisfy (1.4). This definition clearly reduces to the original function when $\nu = 0$.

An integral representation for $H_{C,p,\nu}(\cdot)$ involving the Gauss hypergeometric function ${}_2F_1$ defined in (1.1) can be given. We have

THEOREM 2.1. *The following integral representation of the function $H_{C,p,\nu}(\cdot)$ holds for $\operatorname{Re}(p) > 0$, $\operatorname{Re}(b_j) > 0$ ($j = 1, 2, 3$) and $\operatorname{Re}(c_1 - b_1) > 0$:*

$$(2.2) \quad H_{C,p,\nu}(b_1, b_2, b_3; c_1; x, y, z) \\ = \frac{\Gamma(c_1)}{\Gamma(b_1)\Gamma(c_1 - b_1)} \sqrt{\frac{2p}{\pi}} \int_0^1 t^{b_1 - \frac{3}{2}} (1-t)^{c_1 - b_1 - \frac{3}{2}} K_{\nu + \frac{1}{2}} \left(\frac{p}{t(1-t)} \right)$$

$$(2.3) \quad \times (1 - xt)^{-b_2} (1 - zt)^{-b_3} {}_2F_1\left(\begin{matrix} b_2, b_3 \\ c_1 - b_1 \end{matrix}; \frac{(1-t)y}{(1-xt)(1-zt)}\right) dt,$$

where $|x| < 1$, $|y| < 1$ and $|z| < 1$.

PROOF. The proof of integral representation (2.3) follows by use of the extended beta function (1.6) in (2.1), a change in the order of integration and summation (with uniform convergence of the integral) to find

$$\begin{aligned} H_{C,p,\nu}(b_1, b_2, b_3; c_1; x, y, z) &= \frac{\Gamma(c_1)}{\Gamma(b_1)\Gamma(c_1 - b_1)} \sqrt{\frac{2p}{\pi}} \int_0^1 t^{b_1 - \frac{3}{2}} (1-t)^{c_1 - b_1 - \frac{3}{2}} K_{\nu + \frac{1}{2}}\left(\frac{p}{t(1-t)}\right) \\ &\quad \times \sum_{m,n,k=0}^{\infty} \frac{(b_2)_{m+n} (b_3)_{n+k}}{(c_1 - b_1)_n} \frac{(xt)^m}{m!} \frac{(y(1-t))^n}{n!} \frac{(zt)^k}{k!} dt. \end{aligned}$$

Making use of the result $(a)_{m+n} = (a)_n (a+n)_m$ we can express the treble sum as

$$\begin{aligned} &\sum_{n=0}^{\infty} \frac{(b_2)_n (b_3)_n}{(c_1 - b_1)_n} \frac{(y(1-t))^n}{n!} \sum_{m=0}^{\infty} (b_2 + n)_m \frac{(xt)^m}{m!} \sum_{k=0}^{\infty} (b_3 + n)_k \frac{(zt)^k}{k!} \\ &= (1 - xt)^{-b_2} (1 - zt)^{b_3} \sum_{n=0}^{\infty} \frac{(b_2)_n (b_3)_n}{(c_1 - b_1)_n} \frac{X^n}{n!}, \quad X = \frac{y(1-t)}{(1-xt)(1-zt)}, \end{aligned}$$

where we have employed the binomial theorem

$$\sum_{n=0}^{\infty} \frac{(a)_n w^n}{n!} = (1-w)^{-a} \quad (a \in \mathbf{C}, |w| < 1)$$

to evaluate the sums over m and k . Identification of the sum over n as a Gauss hypergeometric function by (1.1), then yields (2.3). \square

The following variants of (2.3) can be obtained by making appropriate transformations of the integration variable. We have

$$(2.4) \quad \frac{\Gamma(b_1)\Gamma(c_1 - b_1)}{\Gamma(c_1)} \sqrt{\frac{\pi}{2p}} H_{C,p,\nu}(b_1, b_2, b_3; c_1; x, y, z) \\ = \int_0^{\infty} \xi^{b_1 - \frac{3}{2}} (1 + \xi)^{b_2 + b_3 - c_1 + 1} \Omega_1^{-b_2} \Omega_2^{-b_3} K_{\nu + \frac{1}{2}}\left(\frac{p}{\sigma_1 \sigma_2}\right) {}_2F_1\left(\begin{matrix} b_2, b_3 \\ c_1 - b_1 \end{matrix}; \sigma_3 y\right) d\xi,$$

where

$$\sigma_1 = \frac{\xi}{1 + \xi}, \quad \sigma_2 = \frac{1}{1 + \xi}, \quad \sigma_3 = \frac{1 + \xi}{\sigma_1 \sigma_2}, \quad \Omega_1 = 1 + (1 - x)\xi, \quad \Omega_2 = 1 + (1 - z)\xi;$$

$$(2.5) \quad \frac{\Gamma(b_1)\Gamma(c_1 - b_1)}{\Gamma(c_1)} \sqrt{\frac{\pi}{2p}} H_{C,p,\nu}(b_1, b_2, b_3; c_1; x, y, z) \\ = \frac{(\beta - \gamma)^{b_1 - \frac{1}{2}} (\alpha - \gamma)^{c_1 - b_1 - \frac{1}{2}}}{(\beta - \alpha)^{c_1 - b_2 - b_3 - 2}} \times \int_{\alpha}^{\beta} \frac{(\xi - \alpha)^{b_1 - \frac{3}{2}} (\beta - \xi)^{c_1 - b_1 - \frac{3}{2}}}{(\xi - \gamma)^{c_1 - b_2 - b_3 - 1}} \Omega_1^{-b_2} \Omega_2^{-b_3}$$

$$\times K_{\nu+\frac{1}{2}}\left(\frac{p}{\sigma_1\sigma_2}\right) {}_2F_1\left(\begin{matrix} b_2, b_3 \\ c_1 - b_1 \end{matrix}; \sigma_3 y\right) d\xi,$$

where, with $\gamma < \alpha < \beta$,

$$\sigma_1 = \frac{(\beta - \gamma)(\xi - \alpha)}{(\beta - \alpha)(\xi - \gamma)}, \quad \sigma_2 = \frac{(\alpha - \gamma)(\beta - \xi)}{(\beta - \alpha)(\xi - \gamma)}, \quad \sigma_3 = \frac{(\alpha - \gamma)(\beta - \alpha)(\beta - \xi)(\xi - \gamma)}{\Omega_1\Omega_2},$$

$$\Omega_1 = (\beta - \alpha)(\xi - \gamma) - x(\beta - \gamma)(\xi - \alpha), \quad \Omega_2 = (\beta - \alpha)(\xi - \gamma) - z(\beta - \gamma)(\xi - \alpha);$$

$$(2.6) \quad \frac{\Gamma(b_1)\Gamma(c_1 - b_1)}{\Gamma(c_1)} \sqrt{\frac{\pi}{2p}} H_{C,p,\nu}(b_1, b_2, b_3; c_1; x, y, z)$$

$$= 2 \int_0^{\pi/2} (\sin^2 \xi)^{b_1-1} (\cos^2 \xi)^{c_1-b_1-1} \Omega_1^{-b_2} \Omega_2^{-b_3}$$

$$\times K_{\nu+\frac{1}{2}}\left(\frac{p}{\sigma_1\sigma_2}\right) {}_2F_1\left(\begin{matrix} b_2, b_3 \\ c_1 - b_1 \end{matrix}; \sigma_3 y\right) d\xi,$$

where

$$\sigma_1 = \sin^2 \xi, \quad \sigma_2 = \cos^2 \xi, \quad \sigma_3 = \frac{\cos^2 \xi}{\Omega_1\Omega_2}, \quad \Omega_1 = 1 - x \sin^2 \xi, \quad \Omega_2 = 1 - z \sin^2 \xi;$$

and

$$(2.7) \quad \frac{\Gamma(b_1)\Gamma(c_1 - b_1)}{\Gamma(c_1)} \sqrt{\frac{\pi}{2p}} H_{C,p,\nu}(b_1, b_2, b_3; c_1; x, y, z)$$

$$= (1 + \lambda)^{b_1-\frac{1}{2}} \int_0^1 \frac{\xi^{b_1-\frac{3}{2}} (1 - \xi)^{c_1-b_1-\frac{3}{2}}}{(1 + \lambda\xi)^{c_1-b_2-b_3-1}} \Omega_1^{-b_2} \Omega_2^{-b_3}$$

$$\times K_{\nu+\frac{1}{2}}\left(\frac{p}{\sigma_1\sigma_2}\right) {}_2F_1\left(\begin{matrix} b_2, b_3 \\ c_1 - b_1 \end{matrix}; \sigma_3 y\right) d\xi,$$

where, with $\lambda > -1$,

$$\sigma_1 = \frac{(1 + \lambda)\xi}{1 + \lambda\xi}, \quad \sigma_2 = \frac{1 - \xi}{1 + \lambda\xi}, \quad \sigma_3 = \frac{(1 - \xi)(1 + \lambda\xi)}{\Omega_1\Omega_2},$$

$$\Omega_1 = 1 + \lambda\xi - (1 + \lambda)x\xi, \quad \Omega_2 = 1 + \lambda\xi - (1 + \lambda)z\xi.$$

Integral representations (2.4)–(2.7) can be proved directly by using the following transformations

$$(2.4) : \quad t = \frac{\xi}{1 + \xi}, \quad \frac{dt}{d\xi} = \frac{1}{(1 + \xi)^2}$$

$$(2.5) : \quad t = \frac{(\beta - \gamma)(\xi - \alpha)}{(\beta - \alpha)(\xi - \gamma)}, \quad \frac{dt}{d\xi} = \frac{(\beta - \gamma)(\alpha - \gamma)}{(\beta - \alpha)(\xi - \gamma)^2},$$

$$(2.6) : \quad t = \sin^2 \xi, \quad \frac{dt}{d\xi} = 2 \sin \xi \cos \xi$$

$$(2.7) : \quad t = \frac{(1 + \lambda)\xi}{1 + \lambda\xi}, \quad \frac{dt}{d\xi} = \frac{(1 + \lambda)}{(1 + \lambda\xi)^2},$$

in turn in (2.3) to obtain the right-hand side of each result.

Finally, use of the integral representation [13, p. 388]

$${}_2F_1\left(\begin{matrix} b_1, b_2 \\ c_1 \end{matrix}; z\right) = \frac{1}{B(b_2, c_1 - b_2)} \int_0^1 \frac{t^{b_2-1}(1-t)^{c_1-b_2-1}}{(1-zt)^{b_1}} dt, \quad (|\arg(1-z)| < \pi)$$

for $\operatorname{Re}(c_1) > \operatorname{Re}(b_2) > 0$, shows that

$$\begin{aligned} & H_{C,p,\nu}(b_1, b_2, b_3; c_1; x, y, z) \\ &= \frac{\Gamma(c_1)}{\Gamma(b_1)\Gamma(b_2)\Gamma(c_1-b_1-b_2)} \sqrt{\frac{2p}{\pi}} \int_0^1 \int_0^1 s^{b_2-1} t^{b_1-\frac{3}{2}} (1-s)^{c_1-b_1-b_2-1} (1-t)^{c_1-b_1-\frac{3}{2}} \\ & \quad \times \frac{(1-xt)^{b_3-b_2}}{\{(1-xt)(1-zt) - ys(1-t)\}^{b_3}} K_{\nu+\frac{1}{2}}\left(\frac{p}{t(1-t)}\right) ds dt \end{aligned}$$

provided, in addition, $\operatorname{Re}(c_1 - b_1 - b_2) > 0$.

3. The Mellin transform for $H_{C,p,\nu}(\cdot)$

The Mellin transform of a locally integrable function $f(x)$ on $(0, \infty)$ is given by (see, for example, [12, p. 193, Section 2.1])

$$(3.1) \quad \Phi(s) = \mathcal{M}\{f(x)\}(s) = \int_0^\infty x^{s-1} f(x) dx,$$

which defines an analytic function in its strip of analyticity $a < \operatorname{Re}(s) < b$. The inverse Mellin transform of the above function (3.1) is defined by

$$f(x) = \mathcal{M}^{-1}\{\Phi(s)\} = \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} x^{-s} \Phi(s) ds \quad (a < c < b).$$

THEOREM 3.1. *The following Mellin transform of the extended Srivastava triple hypergeometric function $H_{C,p,\nu}(\cdot)$ holds true:*

$$(3.2) \quad \begin{aligned} \mathcal{M}\{H_{C,p,\nu}(b_1, b_2, b_3; c_1; x, y, z)\}(s) &= \int_0^\infty p^{s-1} H_{C,p,\nu}(b_1, b_2, b_3; c_1; x, y, z) dp, \\ &= \frac{2^{s-1}}{\sqrt{\pi}} \Gamma\left(\frac{s-\nu}{2}\right) \Gamma\left(\frac{s+\nu+1}{2}\right) H_C^{(s)}(b_1, b_2, b_3; c_1; x, y, z), \end{aligned}$$

where $\operatorname{Re}(s) > \nu > 0$, $c_1 \in \mathbf{C} \setminus \mathbf{Z}_0^-$ and $H_C^{(s)}(\cdot)$ is defined in (1.5).

PROOF. Substituting the extended Srivastava function (2.1) into the integral on the left-hand side of (3.2) and changing the order of integration (by the uniform convergence of the integral), we obtain

$$\begin{aligned} \mathcal{M}\{H_{C,p,\nu}(b_1, b_2, b_3; c_1; x, y, z)\}(s) &= \sum_{m,n,k=0}^\infty \frac{(b_2)_{m+n} (b_3)_{n+k}}{(c_1)_n B(b_1, c_1+n-b_1)} \frac{x^m y^n z^k}{m! n! k!} \\ & \quad \times \left\{ \int_0^\infty p^{s-1} B_{p,\nu}(b_1+m+k, c_1+n-b_1) dp \right\}. \end{aligned}$$

Use of the extended Beta function (1.6) then shows that

$$\mathcal{M}\{H_{C,p,\nu}(b_1, b_2, b_3; c_1; x, y, z)\}(s)$$

$$= \sqrt{\frac{2}{\pi}} \sum_{m,n,k=0}^{\infty} \frac{(b_2)_{m+n}(b_3)_{n+k}}{(c_1)_n B(b_1, c_1 + n - b_1)} \frac{x^m y^n z^k}{m! n! k!} \\ \times \int_0^1 t^{b_1+m+k-\frac{3}{2}} (1-t)^{c_1+n-b_1-\frac{3}{2}} \left\{ \int_0^{\infty} p^{s-\frac{1}{2}} K_{\nu+\frac{1}{2}} \left(\frac{p}{t(1-t)} \right) dp \right\} dt.$$

If we apply the result [13, (10.43.19)]

$$\int_0^{\infty} w^{s-\frac{1}{2}} K_{\alpha+\frac{1}{2}}(w) dw = 2^{s-\frac{3}{2}} \Gamma\left(\frac{s-\alpha}{2}\right) \Gamma\left(\frac{s+\alpha+1}{2}\right) \quad (|\operatorname{Re}(\alpha)| < \operatorname{Re}(s))$$

followed by the substitution $w = p/(t(1-t))$, we obtain

$$\mathcal{M}\{H_{C,p,\nu}(b_1, b_2, b_3; c_1; x, y, z)\}(s) = \frac{2^{s-1}}{\sqrt{\pi}} \Gamma\left(\frac{s-\nu}{2}\right) \Gamma\left(\frac{s+\nu+1}{2}\right) \\ \times \sum_{m,n,k=0}^{\infty} \frac{(b_2)_{m+n}(b_3)_{n+k}}{(c_1)_n B(b_1, c_1 + n - b_1)} \frac{x^m y^n z^k}{m! n! k!} \\ \times \left\{ \int_0^1 t^{b_1+m+k+s-1} (1-t)^{c_1+n+s-b_1-1} dt \right\}.$$

Evaluation of the integral in terms of the classical Beta function, then finally yields

$$\Phi(s) = \mathcal{M}\{H_{C,p,\nu}(b_1, b_2, b_3; c_1; x, y, z)\}(s) = \frac{2^{s-1}}{\sqrt{\pi}} \Gamma\left(\frac{s-\nu}{2}\right) \Gamma\left(\frac{s+\nu+1}{2}\right) \\ \times \sum_{m,n,k=0}^{\infty} \frac{(b_2)_{m+n}(b_3)_{n+k}}{(c_1)_n} \frac{B(b_1 + s + m + k, c_1 + s + n - b_1)}{B(b_1, c_1 + n - b_1)} \frac{x^m y^n z^k}{m! n! k!}.$$

Identifying the above sum as $H_C^{(s)}(b_1, b_2, b_3; c_1; x, y, z)$ defined in (1.5), we obtain the right-hand side of (3.2). \square

COROLLARY 3.1. *The following inverse Mellin formula for $H_{C,p,\nu}(\cdot)$ holds:*

$$H_{C,p,\nu}(b_1, b_2, b_3; c_1; x, y, z) = \mathcal{M}^{-1}\{\Phi(s)\} = \\ \frac{\pi^{-3/2}}{4i} \int_{c-i\infty}^{c+i\infty} \left(\frac{2}{p}\right)^s \Gamma\left(\frac{s-\nu}{2}\right) \Gamma\left(\frac{s+\nu+1}{2}\right) H_C^{(s)}(b_1, b_2, b_3; c_1; x, y, z) ds,$$

where $c > \nu$.

4. A differentiation formula for $H_{C,p,\nu}(\cdot)$

THEOREM 4.1. *The following derivative formula for $H_{C,p,\nu}(\cdot)$ holds:*

$$(4.1) \quad \frac{\partial^{M+N+K}}{\partial x^M \partial y^N \partial z^K} H_{C,p,\nu}(b_1, b_2, b_3; c_1; x, y, z) = \frac{(b_1)_{M+K} (b_2)_{M+N} (b_3)_{N+K}}{(c_1)_{M+N+K}} \\ \times H_{C,p,\nu}(b_1 + M + K, b_2 + M + N, b_3 + N + K; c_1 + M + N + K; x, y, z),$$

where $M, N, K \in \mathbf{N}_0$.

PROOF. If we differentiate partially the series for $\mathcal{H} \equiv H_{C,p,\nu}(b_1, b_2, b_3; c_1; x, y, z)$ in (2.1) with respect to x we obtain

$$\frac{\partial \mathcal{H}}{\partial x} = \sum_{m=1}^{\infty} \sum_{n,k=0}^{\infty} \frac{(b_2)_{m+n}(b_3)_{n+k}}{(c_1)_n} \frac{B_{p,\nu}(b_1+m+k, c_1+n-b_1)}{B(b_1, c_1+n-b_1)} \frac{x^{m-1}}{(m-1)!} \frac{y^n}{n!} \frac{z^k}{k!}.$$

Making use of the fact that

$$(4.2) \quad B(b_1, c_1+n-b_1) = \frac{(c_1+n)}{b_1} B(b_1+1, c_1+n-b_1)$$

and $(\lambda)_{m+n} = (\lambda)_m(\lambda+m)_n$, we have upon setting $m \rightarrow m+1$

$$(4.3) \quad \begin{aligned} \frac{\partial \mathcal{H}}{\partial x} &= \frac{b_1 b_2}{c_1} \sum_{m,n,k=0}^{\infty} \frac{(b_2+1)_{m+n}(b_3)_{n+k}}{(c_1+1)_n} \frac{B_{p,\nu}(b_1+1+m+k, c_1+n-b_1)}{B(b_1+1, c_1+n-b_1)} \frac{x^m}{m!} \frac{y^n}{n!} \frac{z^k}{k!} \\ &= \frac{b_1 b_2}{c_1} H_{C,p,\nu}(b_1+1, b_2+1, b_3; c_1+1; x, y, z). \end{aligned}$$

Repeated application of (4.3) then yields for $M = 1, 2, \dots$

$$\frac{\partial^M \mathcal{H}}{\partial x^M} = \frac{(b_1)_M (b_2)_M}{(c_1)_M} H_{C,p,\nu}(b_1+M, b_2+M, b_3; c_1+M; x, y, z).$$

A similar reasoning shows that

$$(4.4) \quad \begin{aligned} \frac{\partial^{M+1} \mathcal{H}}{\partial x^M \partial y} &= \frac{(b_1)_M (b_2)_M}{(c_1)_M} \sum_{m,k=0}^{\infty} \sum_{n=1}^{\infty} \frac{(b_2+M)_{m+n}(b_3)_{n+k}}{(c_1+M)_n} \\ &\quad \times \frac{B_{p,\nu}(b_1+M+m+k, c_1+M+n+k)}{B(b_1+M, c_1+M)} \frac{x^m}{m!} \frac{y^{n-1}}{(n-1)!} \frac{z^k}{k!} \\ &= \frac{(b_1)_M (b_2)_{M+1} b_3}{(c_1)_{M+1}} H_{C,p,\nu}(b_1+M, b_2+M+1, b_3+1; c_1+M+1; x, y, z) \end{aligned}$$

upon putting $n \rightarrow n+1$ and using the property of the Beta function in (1.3). Repeated differentiation of (4.4) N times with respect to y then produces

$$\frac{\partial^{M+N} \mathcal{H}}{\partial x^M \partial y^N} = \frac{(b_1)_M (b_2)_{M+N} (b_3)_N}{(c_1)_{M+N}} H_{C,p,\nu}(b_1+M, b_2+M+N, b_3+N; c_1+M+N; x, y, z).$$

Application of the same procedure (making use of (4.2)) to deal with differentiation with respect to z then yields the result stated in (4.1) \square

5. An upper bound for $H_{C,p,\nu}(\cdot)$

THEOREM 5.1. *Let the parameters $c_1 > 0$, $b_j > 0$ ($1 \leq j \leq 3$) with $c_1 - b_1 > 0$ and the variables x, y, z be complex. Then the following bound for $H_{C,p,\nu}(\cdot)$ holds:*

$$(5.1) \quad \begin{aligned} &|H_{C,p,\nu}(b_1, b_2, b_3; c_1; x, y, z)| \\ &< \frac{2^\nu |p|^{\nu+1}}{\sqrt{\pi} (\operatorname{Re}(p))^{2\nu+1}} \Gamma(\nu+1/2) H_C^{(\nu)}(b_1, b_2, b_3; c_1; |x|, |y|, |z|), \end{aligned}$$

where $\operatorname{Re}(p) > 0$, $\nu > 0$ and $H_C^{(\nu)}(\cdot)$ is defined in (1.5)

The integral representation of the extension $H_{B,p,\nu}(\cdot)$ in (2.3) is associated with the modified Bessel function of the second kind, for which we have the following expression [13, (10.32.8)]

$$K_{\nu+\frac{1}{2}}(z) = \frac{\sqrt{\pi}\left(\frac{1}{2}z\right)^{\nu+\frac{1}{2}}}{\Gamma(\nu+1)} \int_1^\infty e^{-zt}(t^2-1)^\nu dt, \quad (\nu > -1, \operatorname{Re}(z) > 0).$$

In our problem we have $\nu > 0, \operatorname{Re}(z) > 0$. Further, we let $x = \operatorname{Re}(z)$, so that

$$(5.2) \quad |K_{\nu+\frac{1}{2}}(z)| \leq \frac{\sqrt{\pi}\left(\frac{1}{2}|z|\right)^{\nu+\frac{1}{2}}}{\Gamma(\nu+1)} \left| \int_1^\infty e^{-zt}(t^2-1)^\nu dt \right| < \frac{\sqrt{\pi}\left(\frac{1}{2}|z|\right)^{\nu+\frac{1}{2}}}{\Gamma(\nu+1)} \int_0^1 t^{2\nu} e^{-xt} dt \\ = \frac{\sqrt{\pi}\left(\frac{1}{2}|z|\right)^{\nu+\frac{1}{2}}}{\Gamma(\nu+1)} \frac{\Gamma(2\nu+1, x)}{x^{2\nu+1}},$$

where $\Gamma(a, z)$ is the upper incomplete gamma function [13, (8.2.2)]. We can simplify (5.2) by making use of the simple inequality $\Gamma(2\nu+1, x) < \Gamma(2\nu+1)$ to find

$$(5.3) \quad |K_{\nu+\frac{1}{2}}(z)| < \frac{\sqrt{\pi}\left(\frac{1}{2}|z|\right)^{\nu+\frac{1}{2}}}{\Gamma(\nu+1)} \frac{\Gamma(2\nu+1)}{x^{2\nu+1}} = \frac{1}{2} \left(\frac{2|z|}{x^2}\right)^{\nu+\frac{1}{2}} \Gamma\left(\nu + \frac{1}{2}\right),$$

upon use of the duplication formula for the gamma function.

PROOF. : Setting $z = p/(t(1-t))$, where $t \in (0, 1)$ and $\operatorname{Re}(p) > 0$, in (5.3) we obtain

$$\left| K_{\nu+\frac{1}{2}}\left(\frac{p}{t(1-t)}\right) \right| < \frac{1}{2} \left(\frac{2|p|t(1-t)}{(\operatorname{Re}(p))^2}\right)^{\nu+\frac{1}{2}} \Gamma\left(\nu + \frac{1}{2}\right).$$

We shall assume that the parameters $c_1 > 0, b_j > 0$ ($1 \leq j \leq 3$), with $c_1 - b_1 > 0$. Then, from (2.3),

$$|H_{C,p,\nu}(b_1, b_2, b_3; c_1; x, y, z)| \leq \frac{2^\nu |p|^{\nu+1} \Gamma\left(\nu + \frac{1}{2}\right)}{\sqrt{\pi} (\operatorname{Re}(p))^{2\nu+1}} \sum_{m,n,k=0}^\infty \frac{(b_2)_{m+n} (b_3)_{n+k}}{(c_1)_n B(b_1, c_1 - b_1 + n)} \\ \times \frac{|x|^m}{m!} \frac{|y|^n}{n!} \frac{|z|^k}{k!} \int_0^1 t^{b_1+\nu+m+k-1} (1-t)^{c_1-b_1+\nu+n-1} dt \\ < \frac{2^\nu |p|^{\nu+1} \Gamma\left(\nu + \frac{1}{2}\right)}{\sqrt{\pi} (\operatorname{Re}(p))^{2\nu+1}} \sum_{m,n,k=0}^\infty \frac{(b_2)_{m+n} (b_3)_{n+k}}{(c_1)_n} \\ \times \frac{B(b_1+\nu+m+k, c_1-b_1+\nu+n)}{B(b_1, c_1-b_1+n)} \frac{|x|^m}{m!} \frac{|y|^n}{n!} \frac{|z|^k}{k!}$$

which is the result stated in (5.1). \square

6. Recursion formulas for $H_{C,p,\nu}(\cdot)$

In this section, we obtain two recursion formulas for the extended Srivastava function $H_{C,p,\nu}(\cdot)$. The first formula gives recursions with respect to the numerator parameters b_2 and b_3 , and the second a recursion with respect to the denominator parameter c_1 .

THEOREM 6.1. *The following recursions for $H_{C,p,\nu}(\cdot)$ with respect to the numerator parameters b_2 and b_3 hold:*

$$(6.1) \quad H_{C,p,\nu}(b_1, b_2 + 1, b_3; c_1; x, y, z) = H_{C,p,\nu}(b_1, b_2, b_3; c_1; x, y, z) \\ + \frac{xb_1}{c_1} H_{C,p,\nu}(b_1 + 1, b_2 + 1, b_3; c_1 + 1; x, y, z) \\ + \frac{yb_3}{c_1} H_{C,p,\nu}(b_1, b_2 + 1, b_3 + 1; c_1 + 1; x, y, z),$$

$$(6.2) \quad H_{C,p,\nu}(b_1, b_2, b_3 + 1; c_1; x, y, z) = H_{C,p,\nu}(b_1, b_2, b_3; c_1; x, y, z) \\ + \frac{yb_2}{c_1} H_{C,p,\nu}(b_1, b_2 + 1, b_3 + 1; c_1 + 1; x, y, z) \\ + \frac{zb_1}{c_1} H_{C,p,\nu}(b_1 + 1, b_2, b_3 + 1; c_1 + 1; x, y, z).$$

PROOF. From (2.1) and the result $(b_2 + 1)_{m+n} = (b_2)_{m+n}(1 + m/b_2 + n/b_2)$, we obtain

$$(6.3) \quad H_{C,p,\nu}(b_1, b_2 + 1, b_3; c_1; x, y, z) \\ = \sum_{m,n,k=0}^{\infty} \frac{(b_2 + 1)_{m+n} (b_3)_{n+k}}{(c_1)_n} \frac{B_{p,\nu}(b_1 + m + k, c_1 + n - b_1)}{B(b_1, c_1 + n - b_1)} \frac{x^m y^n z^k}{m! n! k!} \\ = H_{C,p,\nu}(b_1, b_2, b_3; c_1; x, y, z) \\ + \frac{x}{b_2} \sum_{m=1}^{\infty} \sum_{n=0}^{\infty} \sum_{k=0}^{\infty} \frac{(b_2)_{m+n} (b_3)_{n+k}}{(c_1)_n} \frac{B_{p,\nu}(b_1 + m + k, c_1 + n - b_1)}{B(b_1, c_1 + n - b_1)} \frac{x^{m-1} y^n z^k}{(m-1)! n! k!} \\ + \frac{y}{b_2} \sum_{m=0}^{\infty} \sum_{n=1}^{\infty} \sum_{k=0}^{\infty} \frac{(b_2)_{m+n} (b_3)_{n+k}}{(c_1)_n} \frac{B_{p,\nu}(b_1 + m + k, c_1 + n - b_1)}{B(b_1, c_1 + n - b_1)} \frac{x^m y^{n-1} z^k}{m! (n-1)! k!}.$$

Consider the first sum in (6.3) which we denote by S . Put $m \rightarrow m + 1$ and use the identity $(a)_{n+1} = a(a + 1)_n$ to find

$$S = \frac{x}{b_2} \sum_{m,n,k=0}^{\infty} \frac{(b_2)_{m+n+1} (b_3)_{n+k}}{(c_1)_n} \frac{B_{p,\nu}(b_1 + 1 + m + k, c_1 + n - b_1)}{B(b_1, c_1 + n - b_1)} \frac{x^m y^n z^k}{m! n! k!} \\ = x \sum_{m,n,k=0}^{\infty} \frac{(b_2 + 1)_{m+n} (b_3)_{n+k}}{(c_1)_n} \frac{B_{p,\nu}(b_1 + 1 + m + k, c_1 + n - b_1)}{B(b_1, c_1 + n - b_1)} \frac{x^m y^n z^k}{m! n! k!}.$$

Using (4.2), we then obtain

$$(6.4) \quad S = \frac{xb_1}{c_1} \sum_{m,n,k=0}^{\infty} \frac{(b_2 + 1)_{m+n} (b_3)_{n+k}}{(c_1 + 1)_n} \frac{B_{p,\nu}(b_1 + 1 + m + k, c_1 + n - b_1)}{B(b_1 + 1, c_1 + n - b_1)} \frac{x^m y^n z^k}{m! n! k!} \\ = \frac{xb_1}{c_1} H_{C,p,\nu}(b_1 + 1, b_2 + 1, b_3; c_1 + 1; x, y, z).$$

Proceeding in a similar manner for the second series in (6.3) with $n \rightarrow n + 1$, we find that this sum can be expressed as

$$(6.5) \quad \frac{yb_3}{c_1} H_{C,p,\nu}(b_1, b_2 + 1, b_3 + 1; c_1 + 1; x, y, z).$$

Combination of (6.4) and (6.5) with (6.3) then produces the result stated in (6.1). The proof of (6.2) can be established in a similar manner. \square

COROLLARY 6.1. *From (6.1) and (6.2) the following recursions hold:*

$$\begin{aligned} H_{C,p,\nu}(b_1, b_2 + N, b_3; c_1; x, y, z) &= H_{C,p,\nu}(b_1, b_2, b_3; c_1; x, y, z) \\ &+ \frac{xb_1}{c_1} \sum_{\ell=1}^N H_{C,p,\nu}(b_1 + 1, b_2 + \ell, b_3; c_1 + 1; x, y, z) \\ &+ \frac{yb_3}{c_1} \sum_{\ell=1}^N H_{C,p,\nu}(b_1, b_2 + \ell, b_3 + 1; c_1 + 1; x, y, z), \end{aligned}$$

$$\begin{aligned} H_{C,p,\nu}(b_1, b_2, b_3 + N; c_1; x, y, z) &= H_{C,p,\nu}(b_1, b_2, b_3; c_1; x, y, z) \\ &+ \frac{yb_2}{c_1} \sum_{\ell=1}^N H_{C,p,\nu}(b_1, b_2 + 1, b_3 + \ell; c_1 + 1; x, y, z) \\ &+ \frac{zb_1}{c_1} \sum_{\ell=1}^N H_{C,p,\nu}(b_1 + 1, b_2, b_3 + \ell; c_1 + 1; x, y, z) \end{aligned}$$

for positive integer N .

THEOREM 6.2. *The following 3-term recursion for $H_{C,p,\nu}(\cdot)$ with respect to the denominator parameter c_1 holds:*

$$(6.6) \quad \begin{aligned} H_{C,p,\nu}(b_1, b_2, b_3; c_1; x, y, z) &= H_{C,p,\nu}(b_1, b_2, b_3; c_1 + 1; x, y, z) \\ &+ \frac{yb_2b_3}{c_1(c_1 + 1)} H_{C,p,\nu}(b_1, b_2 + 1, b_3 + 1; c_1 + 2; x, y, z). \end{aligned}$$

PROOF. Consider the case when c_1 is reduced by 1, namely

$$H \equiv H_{C,p,\nu}(b_1, b_2, b_3; c_1 - 1; x, y, z)$$

and use $(c_1 - 1)_n = (c_1)_n / \{1 + \frac{n}{c_1 - 1}\}$. Then

$$\begin{aligned} H &= \sum_{m,n,k=0}^{\infty} \frac{(b_2)_{m+n} (b_3)_{n+k}}{(c_1 - 1)_n} \frac{B_{p,\nu}(b_1 + m + k, c_1 + n - b_1)}{B(b_1, c_1 + n - b_1)} \frac{x^m y^n z^k}{m! n! k!} \\ &= \sum_{m,n,k=0}^{\infty} \frac{(b_2)_{m+n} (b_3)_{n+k}}{(c_1)_n} \frac{B_{p,\nu}(b_1 + m + k, c_1 + n - b_1)}{B(b_1, c_1 + n - b_1)} \left(1 + \frac{n}{c_1 - 1}\right) \frac{x^m y^n z^k}{m! n! k!} \\ &= H_{C,p,\nu}(b_1, b_2, b_3; c_1; x, y, z) \end{aligned}$$

$$+ \frac{y}{c_1 - 1} \sum_{m=1}^{\infty} \sum_{n,k=0}^{\infty} \frac{(b_2)_{m+n} (b_3)_{n+k}}{(c_1)_n} \frac{B_{p,\nu}(b_1+m+k, c_1+n-b_1)}{B(b_1, c_1+n-b_1)} \frac{x^m}{m!} \frac{y^{n-1}}{(n-1)!} \frac{z^k}{k!}.$$

Putting $n \rightarrow n+1$ in the above sum, we obtain

$$\begin{aligned} H_{C,p,\nu}(b_1, b_2, b_3; c_1 - 1; x, y, z) &= H_{C,p,\nu}(b_1, b_2, b_3; c_1; x, y, z) \\ &+ \frac{yb_2b_3}{c_1(c_1 - 1)} \sum_{m,n,k=0}^{\infty} \frac{(b_2+1)_{m+n} (b_3+1)_{n+k}}{(c_1+1)_n} \frac{B_{p,\nu}(b_1+m+k, c_1+1+n-b_1)}{B(b_1, c_1+1+n-b_1)} \frac{x^m}{m!} \frac{y^n}{n!} \frac{z^k}{k!} \\ &= H_{C,p,\nu}(b_1, b_2, b_3; c_1; x, y, z) + \frac{yb_2b_3}{c_1(c_1 - 1)} H_{C,p,\nu}(b_1, b_2+1, b_3+1; c_1+1; x, y, z). \end{aligned}$$

Replacement of c_1 by c_1+1 then yields the result stated in (6.6). \square

7. Concluding remarks

We have introduced the (p, ν) -extension of Srivastava's triple hypergeometric function given by $H_{C,p,\nu}(\cdot)$ in (2.1). We have given some integral representations of this function that involve the modified Bessel function of the second kind and a Gauss hypergeometric function. We have also established some properties of the function $H_{C,p,\nu}(\cdot)$, namely the Mellin transform, a differential formula, a bounded inequality and some recursion relations.

Acknowledgement. One of the authors (S. A. Dar) was financially supported from the University Grants Commission of India. The authors are grateful to the reviewers for their remarks which improved the earlier version of the paper.

References

1. M. A. Chaudhry, A. Qadir, M. Rafique, S. M. Zubair, *Extension of Euler's beta function*, J. Comput. Appl. Math. **78**(1) (1997) 19–32.
2. M. A. Chaudhry, A. Qadir, H. M. Srivastava, R. B. Paris, *Extended hypergeometric and confluent hypergeometric functions*, Appl. Math. Comput. **159**(2) (2004), 589–602.
3. A. Çetinkaya, M. I. Yağbasan, I. O. Kıymaz, *The extended Srivastava's triple hypergeometric functions and their integral representations*, J. Nonlinear Sci. Appl. **9** (2016), 4860–4866.
4. J. Choi, A. Hasanov, H. M. Srivastava, M. Turaev, *Integral representations for Srivastava's triple hypergeometric functions*, Taiwanese J. Math. **15**(6) (2011), 2751–2762.
5. J. Choi, A. Hasanov, M. Turaev, *Integral representations for Srivastava's triple hypergeometric functions H_A* , Honam Math. J. **34**(1) (2012), 113–124.
6. ———, *Integral representations for Srivastava's triple hypergeometric functions H_B* , J. Korean Soc. Math. Educ., Ser. B, Pure Appl. Math. **19**(2) (2012), 137–145.
7. ———, *Integral representations for Srivastava's triple hypergeometric functions H_C* , Honam Math. J. **34**(4) (2012), 473–482.
8. J. Choi, A. K. Rathie, R. K. Parmar, *Extension of extended Beta, hypergeometric function and confluent hypergeometric function*, Honam Math. J. **36**(2) (2014), 357–385.
9. S. A. Dar, R. B. Paris, *A (p, ν) -extension of the Appell function $F_1(\cdot)$ and its properties*, J. Comput. Appl. Math. **358** (2019), 12–19.
10. ———, *A (p, ν) -extension of Srivastava's triple hypergeometric function H_B and its properties*, J. Comput. Appl. Math. **348** (2019), 237–245.
11. P. W. Karlsson, *Regions of convergences for hypergeometric series in three variables*, Math. Scand. **48** (1974), 241–248.

12. F. Oberhettinger, *Tables of Mellin Transforms*, Springer-Verlag, Berlin, Heidelberg, New York, 1974.
13. F. W. J. Olver, D. W. Lozier, R. F. Boisvert, C. W. Clark (eds.), *NIST Handbook of Mathematical Functions*, Cambridge University Press, Cambridge, 2010.
14. M. A. Özarlan, *Some remarks on extended hypergeometric, extended confluent hypergeometric and extended Appell's functions*, J. Comput. Anal. Appl. **14**(6) (2012), 1148–1153.
15. R. K. Parmar, P. Chopra, R. B. Paris, *On an extension of extended Beta and hypergeometric functions*, J. Class. Anal. **11** (2017), 91–106. [arXiv:1502.06200].
16. R. K. Parmar, T. K. Pogány, *Extended Srivastava's triple hypergeometric $H_{A,p,q}$ function and related bounding inequalities*. J. Contemp. Math. Anal., Armen. Acad. Sci. **52**(6) (2017), 276–287.
17. L. J. Slater, *Generalized Hypergeometric Functions*, Cambridge Univ. Press, Cambridge 1966.
18. H. M. Srivastava, *Hypergeometric functions of three variables*, Ganita **15** (1964), 97–108.
19. ———, *Some integrals representing triple hypergeometric functions*, Rend. Circ. Mat. Palermo (2) **16** (1967), 99–115.
20. H. M. Srivastava, H. L. Manocha, *A Treatise on Generating Functions*, Halsted Press (Ellis Horwood, Chichester) Wiley, New York, Chichester, Brisbane and Toronto, 1984.
21. H. M. Srivastava, P. W. Karlsson, *Multiple Gaussian Hypergeometric Series*, Halsted Press (Ellis Horwood Limited, Chichester), Wiley, New York, Chichester, Brisbane and Toronto, 1985.

Department of Applied Sciences and Humanities
Faculty of Engineering and Technology
Jamia Millia Islamia
New Delhi
India
showkatjmi34@gmail.com

(Received 27 06 2020)

Division of Computing and Mathematics
Abertay University,
Dundee
UK
r.paris@abertay.ac.uk