

A FAMILY OF THETA-FUNCTION IDENTITIES BASED UPON R_α, R_β AND R_m -FUNCTIONS RELATED TO JACOBI'S TRIPLE-PRODUCT IDENTITY

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ABSTRACT. We establish a set of two new relationships involving R_α, R_β and R_m -functions, which are based on Jacobi's famous triple-product identity. We, also provide answer for an open problem of Srivastava, Srivastava, Chaudhary and Uddin, which suggest to find an inter-relationships between R_α, R_β and $R_m (m \in \mathbb{N})$, q -product identities and continued-fraction identities.

1. Introduction and Definitions

Recently, Srivastava et al. [28] coined an open problem, which state as find an inter-relationships between R_α, R_β and $R_m (m \in \mathbb{N})$, q -product identities and continued-fraction identities. In this article we establish the relationships between R_α, R_β and $R_m (m \in \mathbb{N})$, q -product identities.

Throughout this article, we denote by \mathbb{N}, \mathbb{Z} , and \mathbb{C} the set of positive integers, the set of integers and the set of complex numbers, respectively. We also let

$$\mathbb{N}_0 := \mathbb{N} \cup \{0\} = \{0, 1, 2, \dots\}$$

and recall the following q -notations (see, for example, [5], [7, Chapter 3, Section 3.2.1], [29, Chapter 6] and [30, pp. 346 et seq.]). The q -shifted factorial $(a; q)_n$ is defined (for $|q| < 1$) by

$$(a; q)_n := \begin{cases} 1 & (n = 0) \\ \prod_{k=0}^{n-1} (1 - aq^k) & (n \in \mathbb{N}), \end{cases}$$

where $a, q \in \mathbb{C}$ and it is assumed *tacitly* that $a \neq q^{-m}$ ($m \in \mathbb{N}_0$). We also write

$$(1.1) \quad (a; q)_\infty := \prod_{k=0}^{\infty} (1 - aq^k) = \prod_{k=1}^{\infty} (1 - aq^{k-1}) \quad (a, q \in \mathbb{C}; |q| < 1).$$

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It should be noted that, when $a \neq 0$ and $|q| \geq 1$, the infinite product in the equation (1.1) diverges. So, whenever $(a; q)_\infty$ is involved in a given formula, the constraint $|q| < 1$ will be *tacitly* assumed to be satisfied.

The following notations are also frequently used in our investigation:

$$(a_1, a_2, \dots, a_m; q)_n := (a_1; q)_n (a_2; q)_n \cdots (a_m; q)_n,$$

$$(a_1, a_2, \dots, a_m; q)_\infty := (a_1; q)_\infty (a_2; q)_\infty \cdots (a_m; q)_\infty.$$

Ramanujan (see [20] and [21]) defined the general theta function $f(a, b)$ as follows (see, for details, [8, p. 31, (18.1)] and [25]):

$$(1.2) \quad f(a, b) = 1 + \sum_{n=1}^{\infty} (ab)^{\frac{n(n-1)}{2}} (a^n + b^n) = \sum_{n=-\infty}^{\infty} a^{\frac{n(n+1)}{2}} b^{\frac{n(n-1)}{2}} = f(b, a)$$

We find from this last equation (1.2) that

$$(1.3) \quad f(a, b) = a^{\frac{n(n+1)}{2}} b^{\frac{n(n-1)}{2}} f(a(ab)^n, b(ab)^{-n}) = f(b, a) \quad (n \in \mathbb{Z}).$$

In fact, Ramanujan (see [20] and [21]) also rediscovered Jacobi's famous triple-product identity which, in Ramanujan's notation, is given by (see [8, p. 35, Entry 19]):

$$(1.4) \quad f(a, b) = (-a; ab)_\infty (-b; ab)_\infty (ab; ab)_\infty$$

or, equivalently, by (see [18])

$$(1.5) \quad \sum_{n=-\infty}^{\infty} q^{n^2} z^n = \prod_{n=1}^{\infty} (1 - q^{2n})(1 + zq^{2n-1}) \left(1 + \frac{1}{z} q^{2n-1}\right)$$

$$= (q^2; q^2)_\infty (-zq; q^2)_\infty \left(-\frac{q}{z}; q^2\right)_\infty, \quad (|q| < 1; z \neq 0).$$

Note 1. Historically speaking, q -series identity (1.4) or its above-mentioned equivalent form was first proved by Carl Friedrich Gauss (1777–1855).

Note 2. Equation (1.5) holds true as stated only if n is any integer. Moreover, in case n is not an integer, result (1.5) is only approximately true (see, for details, [20, Vol. 2, Chapter XVI, p. 193, Entry 18 (iv)]).

Several q -series identities, which emerge naturally from Jacobi's triple-product identity (1.4), are worthy of note here (see, for details, [8, pp. 36–37, Entry 22]):

$$(1.6) \quad \varphi(q) := \sum_{n=-\infty}^{\infty} q^{n^2} = 1 + 2 \sum_{n=1}^{\infty} q^{n^2} = \{(-q; q^2)_\infty\}^2 (q^2; q^2)_\infty$$

$$= \frac{(-q; q^2)_\infty (q^2; q^2)_\infty}{(q; q^2)_\infty (-q^2; q^2)_\infty};$$

$$(1.7) \quad \psi(q) := f(q, q^3) = \sum_{n=0}^{\infty} q^{\frac{n(n+1)}{2}} = \frac{(q^2; q^2)_\infty}{(q; q^2)_\infty};$$

$$(1.8) \quad f(-q) = f(-q, -q^2) = \sum_{n=-\infty}^{\infty} (-1)^n q^{\frac{n(3n-1)}{2}}$$

$$= \sum_{n=0}^{\infty} (-1)^n q^{\frac{n(3n-1)}{2}} + \sum_{n=1}^{\infty} (-1)^n q^{\frac{n(3n+1)}{2}} = (q; q)_{\infty}.$$

Equation (1.8) is known as Euler’s *pentagonal number theorem*, which states as

THEOREM 1.1 (Euler’s Pentagonal Number Theorem). *The number of partitions of a given positive integer n into distinct parts is equal to the number of partitions of n into odd parts.*

Remarkably, the following q -series identity:

$$(-q; q)_{\infty} = \frac{1}{(q; q^2)_{\infty}} = \frac{1}{\chi(-q)}$$

provides the analytic equivalent form of Euler’s famous theorem (see, for details, [5] and [7]).

By introducing the general family $R(s, t, l, u, v, w)$, Andrews et al. [6] investigated a number of interesting double-summation hypergeometric q -series representations for several families of partitions and further explored the role of double series in combinatorial-partition identities:

$$R(s, t, l, u, v, w) := \sum_{n=0}^{\infty} q^{s\binom{n}{2}+tn} r(l, u, v, w; n),$$

where

$$r(l, u, v, w : n) := \sum_{j=0}^{\lfloor \frac{n}{u} \rfloor} (-1)^j \frac{q^{uv\binom{j}{2}+(w-ul)j}}{(q; q)_{n-uj} (q^{uv}; q^{uv})_j}.$$

We also recall the following interesting special cases of $R(s, t, l, u, v, w)$:

$$\begin{aligned} R(2, 1, 1, 1, 2, 2) &= (-q; q^2)_{\infty}, \\ R(2, 2, 1, 1, 2, 2) &= (-q^2; q^2)_{\infty}, \\ R(m, m, 1, 1, 1, 2) &= \frac{(q^{2m}; q^{2m})_{\infty}}{(q^m; q^{2m})_{\infty}}. \end{aligned}$$

Recently, Srivastava et al. [28] have introduced the following three notations:

$$(1.9) \quad R_{\alpha} = R(2, 1, 1, 1, 2, 2), \quad R_{\beta} = R(2, 2, 1, 1, 2, 2), \quad R_m = R(m, m, 1, 1, 1, 2)$$

where ($m \in \mathbb{N}$), for multivariate R -functions, which we shall use for computation of our main results in Section 2.

Ever since the year 2015, several new advancements and generalizations of the existing results were made in regard to combinatorial partition-theoretic identities (see, for example, [14] to [15] ; [25] to [27]). In particular, Chaudhary et al. generalized several known results on character formulas (see [13]), Ramanujan’s modular equations of degrees 3, 7 and 9 (see [10] and [11]), and so on, by using combinatorial partition-theoretic identities. An interesting recent investigation on the subject of combinatorial partition-theoretic identities by Hahn et al. [17] is also worth mentioning in this connection.

Each of the following preliminary results will be needed for the demonstration of our main results here (see [4, Theorem 5.1], [8, Entry 51, p.204], [19, Theorem 3.1]). If

$$(1.10) \quad P = \frac{\phi(-q)}{\phi(-q^3)}, \quad Q = \frac{\phi(-q^6)}{\phi(-q^{18})}, \quad T = \frac{\psi(q^6)}{\psi(q^8)}, \quad u = \frac{\phi^2(q)}{q^{\frac{1}{2}}\phi^2(q^3)}, \quad v = \frac{\phi^2(q^6)}{q^3\phi^2(q^{18})},$$

then

$$(1.11) \quad \begin{aligned} & \frac{Q^6}{P^6} - 21 \left(3 \frac{P^2}{Q^2} + 2 \frac{Q^2}{P^2} \right) - 3 \left(P^2 Q^2 - \frac{153}{P^2 Q^2} \right) + \left(Q^6 - \frac{81}{Q^6} \right) \left(1 + \frac{3}{P^4} \right) \\ & - 3 \left(7Q^2 - \frac{27}{Q^2} \right) - \frac{18}{P^4} \left(5Q^2 - \frac{18}{Q^2} \right) + \frac{81}{Q^6} \left(P^2 - \frac{5}{P^2} \right) \\ & + 3P^2 \left(Q^2 - \frac{3}{Q^2} \right) + \left(P^6 Q^2 + 3 \frac{Q^6}{P^2} \right) = 0; \end{aligned}$$

$$(1.12) \quad \begin{aligned} & 9TQ^2P^6 - 27Pu + 27P^5u - 81TQ^2P^2 - 9uQ^4P^3u \\ & + Q^4P^7uv + TQ^6v - TQ^6P^4v = 0 \end{aligned}$$

2. Main Results

In this section, we establish a set of two new relationships involving R_α, R_β and R_m -functions, q -product identities and continued-fraction identities. Also answer an open problem of Srivastava et al. [28], which state as find inter-relationships between R_α, R_β and $R_m (m \in \mathbb{N})$, q -product identities and continued-fraction identities. In this article we establish the relationships between R_α, R_β and $R_m (m \in \mathbb{N})$, q -product identities.

THEOREM 2.1. *Each of the following relationships holds true:*

$$(2.1) \quad \begin{aligned} & 72 \left\{ \frac{(q; q)_\infty (q^{18}; q^{18})_\infty (-q^3, -q^6; q^6)_\infty (-q^6, -q^{12}; q^{12})_\infty}{R_\alpha R_\beta (q^3; q^3)_\infty (q^6; q^6)_\infty (-q^{18}, -q^{36}; q^{36})_\infty} \right\}^2 \\ & + 21 \left\{ \frac{(q^6; q^6)_\infty (-q^{18}, -q^{36}; q^{36})_\infty}{(q^{18}; q^{18})_\infty (-q^6, -q^{12}; q^{12})_\infty} \right\}^2 + 81 \left\{ \frac{(q^{18}, q^{18})_\infty (-q^6, -q^{12}; q^{12})_\infty}{(q^6; q^6)_\infty (-q^{18}, -q^{36}; q^{36})_\infty} \right\}^6 \\ & + 42 \left\{ \frac{R_\alpha R_\beta (q^3; q^3)_\infty (q^6; q^6)_\infty (-q^{18}, -q^{36}; q^{36})_\infty}{(q; q)_\infty (q^{18}; q^{18})_\infty (-q^3, -q^6; q^6)_\infty (-q^6, -q^{12}; q^{12})_\infty} \right\}^2 \\ & + 90 \left\{ \frac{\{R_\alpha R_\beta (q^3; q^3)_\infty\}^2 (q^6; q^6)_\infty (-q^{18}, -q^{36}; q^{36})_\infty}{\{(q; q)_\infty (-q^3, -q^6; q^6)_\infty\}^2 (q^{18}; q^{18})_\infty (-q^6, -q^{12}; q^{12})_\infty} \right\}^2 \\ & + 405 \left\{ \frac{R_\alpha R_\beta (q^3; q^3)_\infty}{(q; q)_\infty (-q^3, -q^6; q^6)_\infty} \right\}^2 \left\{ \frac{(q^{18}; q^{18})_\infty (-q^6, -q^{12}; q^{12})_\infty}{(q^6; q^6)_\infty (-q^{18}, -q^{36}; q^{36})_\infty} \right\}^6 \\ & + 243 \frac{\{R_\alpha R_\beta (q^3; q^3)_\infty\}^4 \{(q^{18}; q^{18})_\infty (-q^6, -q^{12}; q^{12})_\infty\}^6}{\{(q; q)_\infty (-q^3, -q^6; q^6)_\infty\}^4 \{(q^6; q^6)_\infty (-q^{18}, -q^{36}; q^{36})_\infty\}^6} \\ & = \left\{ \frac{R_\alpha R_\beta (q^3; q^3)_\infty (q^6; q^6)_\infty (-q^{18}, -q^{36}; q^{36})_\infty}{(q; q)_\infty (q^{18}; q^{18})_\infty (-q^3, -q^6; q^6)_\infty (-q^6, -q^{12}; q^{12})_\infty} \right\}^6 \end{aligned}$$

$$\begin{aligned}
& +459 \left\{ \frac{R_\alpha R_\beta (q^3; q^3)_\infty (q^{18}; q^{18})_\infty (-q^6, -q^{12}; q^{12})_\infty}{(q; q)_\infty (q^6; q^6)_\infty (-q^3, -q^6; q^6)_\infty (-q^{18}, -q^{36}; q^{36})_\infty} \right\}^2 \\
& + \left\{ \frac{(q^6; q^6)_\infty (-q^{18}, -q^{36}; q^{36})_\infty}{(q^{18}; q^{18})_\infty (-q^6, -q^{12}; q^{12})_\infty} \right\}^6 + \left\{ \frac{9(q^{18}; q^{18})_\infty (-q^6, -q^{12}; q^{12})_\infty}{(q^6; q^6)_\infty (-q^{18}, -q^{36}; q^{36})_\infty} \right\}^2 \\
& + 3 \frac{\{R_\alpha R_\beta (q^3; q^3)_\infty\}^4 \{(q^6; q^6)_\infty (-q^{18}, -q^{36}; q^{36})_\infty\}^6}{\{(q; q)_\infty (-q^3, -q^6; q^6)_\infty\}^4 \{(q^{18}; q^{18})_\infty (-q^6, -q^{12}; q^{12})_\infty\}^6} \\
& + 324 \left\{ \frac{\{R_\alpha R_\beta (q^3; q^3)_\infty\}^2 (q^{18}; q^{18})_\infty (-q^6, -q^{12}; q^{12})_\infty}{\{(q; q)_\infty (-q^3, -q^6; q^6)_\infty\}^2 (q^6; q^6)_\infty (-q^{18}, -q^{36}; q^{36})_\infty} \right\}^2 \\
& + \left\{ \frac{9(q; q)_\infty (-q^3, -q^6; q^6)_\infty \{(q^{18}; q^{18})_\infty (-q^6, -q^{12}; q^{12})_\infty\}^3}{R_\alpha R_\beta (q^3; q^3)_\infty \{(q^6; q^6)_\infty (-q^{18}, -q^{36}; q^{36})_\infty\}^3} \right\}^2 \\
& + \left\{ \frac{(q^6; q^6)_\infty (-q^{18}, -q^{36}; q^{36})_\infty}{(q^{18}; q^{18})_\infty (-q^6, -q^{12}; q^{12})_\infty} \right\}^2 \left\{ \frac{(q; q)_\infty (-q^3, -q^6; q^6)_\infty}{R_\alpha R_\beta (q^3; q^3)_\infty} \right\}^6 \\
& + 3 \left\{ \frac{R_\alpha R_\beta (q^3; q^3)_\infty}{(q; q)_\infty (-q^3, -q^6; q^6)_\infty} \right\}^2 \left\{ \frac{(q^6; q^6)_\infty (-q^{18}, -q^{36}; q^{36})_\infty}{(q^{18}; q^{18})_\infty (-q^6, -q^{12}; q^{12})_\infty} \right\}^6
\end{aligned}$$

Equation (2.1) gives inter-relationships between R_α , R_β , and q -products identities and

$$\begin{aligned}
(2.2) \quad & \frac{R_6}{R_{18}} \left\{ \frac{3R_3(-q^{18}, -q^{36}; q^{36})_\infty}{(q^6; q^6)_\infty (q^{18}; q^{18})_\infty (-q^6, -q^{12}; q^{12})_\infty (q^3; q^3)_\infty^2} \right\}^2 \\
& \cdot \left\{ \frac{(q; q)_\infty (-q^3, -q^6; q^6)_\infty}{R_\alpha R_\beta} \right\}^6 + \frac{27}{q^{\frac{1}{2}}} \left\{ \frac{(q; q)_\infty (-q^3, -q^6; q^6)_\infty}{R_\alpha R_\beta (q^3; q^3)_\infty} \right\}^5 \left\{ \frac{R_1}{R_3} \right\}^2 \\
& + \frac{1}{q^{\frac{7}{2}}} \left\{ \frac{R_3 R_1 R_6}{R_{18}} \right\}^2 \frac{1}{\{(q^3; q^3)_\infty\}^3} \cdot \left\{ \frac{(-q^{18}, -q^{36}; q^{36})_\infty}{(q^6; q^6)_\infty (q^{18}; q^{18})_\infty (-q^6, -q^{12}; q^{12})_\infty} \right\}^4 \\
& \cdot \left\{ \frac{(q; q)_\infty (-q^3, -q^6; q^6)_\infty}{R_\alpha R_\beta} \right\}^7 + \frac{1}{q^3} \left\{ \frac{R_6}{R_{18}} \right\}^3 \left\{ \frac{(q^6; q^6)_\infty (-q^{18}, -q^{36}; q^{36})_\infty}{(q^{18}; q^{18})_\infty (-q^6, -q^{12}; q^{12})_\infty} \right\}^6 \\
& = \frac{R_6}{R_{18}} \left\{ \frac{9R_3}{R_\alpha R_\beta} \frac{(q; q)_\infty (-q^3, -q^6; q^6)_\infty (-q^{18}, -q^{36}; q^{36})_\infty}{(q^6; q^6)_\infty (q^{18}; q^{18})_\infty (-q^6, -q^{12}; q^{12})_\infty} \right\}^2 \\
& \frac{27(q; q)_\infty (-q^3, -q^6; q^6)_\infty}{q^{\frac{1}{2}} R_\alpha R_\beta (q^3; q^3)_\infty} \left\{ \frac{R_1}{R_3} \right\}^2 + \frac{R_3 (q^6; q^{12})_\infty}{q^{\frac{7}{2}} (q^{12}; q^{12})_\infty} \left\{ \frac{3R_1}{R_{18} (q^6; q^6)_\infty} \right\}^2 \\
& \cdot \left\{ \frac{R_6 (q; q)_\infty (-q^3, -q^6; q^6)_\infty}{R_\alpha R_\beta} \right\}^3 \cdot \left\{ \frac{(-q^{18}, -q^{36}; q^{36})_\infty}{(q^{18}; q^{18})_\infty (-q^6, -q^{12}; q^{12})_\infty} \right\}^4 \\
& + \frac{1}{q^3} \left\{ \frac{(q^6; q^{12})_\infty}{(q^6; q^6)_\infty (q^{12}; q^{12})_\infty} \right\}^2 \left\{ \frac{R_3 (q; q)_\infty (-q^3, -q^6; q^6)_\infty}{R_\alpha R_\beta} \right\}^4 \\
& \cdot \frac{\{R_6\}^5}{\{R_{18}\}^3} \left\{ \frac{(-q^{18}, -q^{36}; q^{36})_\infty}{(-q^6, -q^{12}; q^{12})_\infty (q^{18}; q^{18})_\infty} \right\}^6.
\end{aligned}$$

Equation (2.2) gives inter-relationships between $R_1, R_3, R_6, R_{18}, R_\alpha, R_\beta$, and q -product identities.

It is assumed that each member of the assertions (2.1) and (2.2) exists.

PROOF. First of all, in order to prove assertion (2.1), apply identities (1.6) and (1.7), under the given precondition of result (1.11); and further using (1.9), we obtain

$$(2.3) \quad \frac{Q^6}{P^6} = \left\{ \frac{R_\alpha R_\beta(q^3; q^3)_\infty (q^6; q^6)_\infty (-q^{18}, -q^{18}; q^{36})_\infty}{(q; q)_\infty (q^{18}; q^{18})_\infty (-q^3, -q^6; q^6)_\infty (-q^6, -q^{12}; q^{12})_\infty} \right\}^6,$$

$$(2.4) \quad -21 \left(3 \frac{P^2}{Q^2} + 2 \frac{Q^2}{P^2} \right) \\ = -63 \left\{ \frac{(q; q)_\infty (q^{18}; q^{18})_\infty (-q^3, -q^6; q^6)_\infty (-q^6, -q^{12}; q^{12})_\infty}{R_\alpha R_\alpha(q^3; q^3)_\infty (q^6; q^6)_\infty (-q^{18}, -q^{36}; q^{36})_\infty} \right\}^2 \\ - 42 \left\{ \frac{R_\alpha R_\beta(q^3; q^3)_\infty (q^6; q^6)_\infty (-q^{18}, -q^{36}; q^{36})_\infty}{(q; q)_\infty (q^{18}; q^{18})_\infty (-q^3, -q^6; q^6)_\infty (-q^6, -q^{12}; q^{12})_\infty} \right\}^2,$$

$$(2.5) \quad -3 \left(P^2 Q^2 - \frac{153}{P^2 Q^2} \right) \\ = -3 \left\{ \frac{(q; q)_\infty (q^6; q^6)_\infty (-q^3, -q^6; q^6)_\infty (-q^{18}, -q^{36}; q^{36})_\infty}{R_\alpha R_\beta(q^3; q^3)_\infty (q^{18}; q^{18})_\infty (-q^6, -q^{12}; q^{12})_\infty} \right\}^2 \\ + 459 \left\{ \frac{R_\alpha R_\beta(q^3; q^3)_\infty (q^{18}; q^{18})_\infty (-q^6, -q^{12}; q^{12})_\infty}{(q; q)_\infty (q^6; q^6)_\infty (-q^3, -q^6; q^6)_\infty (-q^{18}, -q^{36}; q^{36})_\infty} \right\}^2,$$

$$(2.6) \quad \left(Q^6 - \frac{81}{Q^6} \right) \left(1 + \frac{3}{P^4} \right) = \left\{ \frac{(q^6; q^6)_\infty (-q^{18}, -q^{36}; q^{36})_\infty}{(q^{18}; q^{18})_\infty (-q^6, -q^{12}; q^{12})_\infty} \right\}^6 \\ + \frac{3 \cdot \{R_\alpha R_\beta(q^3; q^3)_\infty\}^4 \{(q^6; q^6)_\infty (-q^{18}, -q^{36}; q^{36})_\infty\}^6}{\{(q; q)_\infty (-q^3, -q^6; q^6)_\infty\}^4 \{(q^{18}; q^{18})_\infty (-q^6, -q^{12}; q^{12})_\infty\}^6} \\ - 81 \cdot \left\{ \frac{(q^{18}, q^{18})_\infty (-q^6, -q^{12}; q^{12})_\infty}{(q^6; q^6)_\infty (-q^{18}, -q^{36}; q^{36})_\infty} \right\}^6 \\ - \frac{243 \cdot \{R_\alpha R_\beta(q^3; q^3)_\infty\}^4 \{(q^{18}; q^{18})_\infty (-q^6, -q^{12}; q^{12})_\infty\}^6}{\{(q; q)_\infty (-q^3, -q^6; q^6)_\infty\}^4 \{(q^6; q^6)_\infty (-q^{18}, -q^{36}; q^{36})_\infty\}^6},$$

$$(2.7) \quad -3 \left(7Q^2 - \frac{27}{Q^2} \right) \\ = -21 \left\{ \frac{(q^6; q^6)_\infty (-q^{18}, -q^{36}; q^{36})_\infty}{(q^{18}; q^{18})_\infty (-q^6, -q^{12}; q^{12})_\infty} \right\}^2 + \left\{ \frac{9 \cdot (q^{18}; q^{18})_\infty (-q^6, -q^{12}; q^{12})_\infty}{(q^6; q^6)_\infty (-q^{18}, -q^{36}; q^{36})_\infty} \right\}^2,$$

$$(2.8) \quad -\frac{18}{P^4} \left(5Q^2 - \frac{18}{Q^2} \right) \\ = -90 \left\{ \frac{\{R_\alpha R_\beta(q^3; q^3)_\infty\}^2 (q^6; q^6)_\infty (-q^{18}, -q^{36}; q^{36})_\infty}{\{(q; q)_\infty (-q^3, -q^6; q^6)_\infty\}^2 (q^{18}; q^{18})_\infty (-q^6, -q^{12}; q^{12})_\infty} \right\}^2$$

$$\begin{aligned}
& + 324 \left\{ \frac{\{R_\alpha R_\beta(q^3; q^3)_\infty\}^2 (q^{18}; q^{18})_\infty (-q^6, -q^{12}; q^{12})_\infty}{\{(q; q)_\infty (-q^3, -q^6; q^6)_\infty\}^2 (q^6; q^6)_\infty (-q^{18}, -q^{36}; q^{36})_\infty} \right\}^2 \\
(2.9) \quad & \frac{81}{Q^6} \left(P^2 - \frac{5}{P^2} \right) \\
& = \left\{ \frac{9(q; q)_\infty (-q^3, -q^6; q^6)_\infty \{(q^{18}; q^{18})_\infty (-q^6, -q^{12}; q^{12})_\infty\}^3}{R_\alpha R_\beta(q^3; q^3)_\infty \{(q^6; q^6)_\infty (-q^{18}, -q^{36}; q^{36})_\infty\}^3} \right\}^2 \\
& - 405 \left\{ \frac{R_\alpha R_\beta(q^3; q^3)_\infty}{(q; q)_\infty (-q^3, -q^6; q^6)_\infty} \right\}^2 \left\{ \frac{(q^{18}; q^{18})_\infty (-q^6, -q^{12}; q^{12})_\infty}{(q^6; q^6)_\infty (-q^{18}, -q^{36}; q^{36})_\infty} \right\}^6,
\end{aligned}$$

$$\begin{aligned}
(2.10) \quad & 3P^2 \left(Q^2 - \frac{3}{Q^2} \right) \\
& = 3 \left\{ \frac{(q; q)_\infty (q^6; q^6)_\infty (-q^3, -q^6; q^6)_\infty (-q^{18}, -q^{36}; q^{36})_\infty}{R_\alpha R_\beta(q^3; q^3)_\infty (q^{18}; q^{18})_\infty (-q^6, -q^{12}; q^{12})_\infty} \right\}^2 \\
& - \left\{ \frac{3(q; q)_\infty (q^{18}; q^{18})_\infty (-q^3, -q^6; q^6)_\infty (-q^6, -q^{12}; q^{12})_\infty}{R_\alpha R_\beta(q^3; q^3)_\infty (q^6; q^6)_\infty (-q^{18}, -q^{36}; q^{36})_\infty} \right\}^2,
\end{aligned}$$

$$\begin{aligned}
(2.11) \quad & P^6 Q^2 + 3 \cdot \frac{Q^6}{P^2} \\
& = \left\{ \frac{(q^6; q^6)_\infty (-q^{18}, -q^{36}; q^{36})_\infty}{(q^{18}; q^{18})_\infty (-q^6, -q^{12}; q^{12})_\infty} \right\}^2 \left\{ \frac{(q; q)_\infty (-q^3, -q^6; q^6)_\infty}{R_\alpha R_\beta(q^3; q^3)_\infty} \right\}^6 \\
& + 3 \left\{ \frac{R_\alpha R_\beta(q^3; q^3)_\infty}{(q; q)_\infty (-q^3, -q^6; q^6)_\infty} \right\}^2 \left\{ \frac{(q^6; q^6)_\infty (-q^{18}, -q^{36}; q^{36})_\infty}{(q^{18}; q^{18})_\infty (-q^6, -q^{12}; q^{12})_\infty} \right\}^6.
\end{aligned}$$

Now, upon substituting from these last results (2.3) to (2.11) into (1.11), if we rearrange the terms and use some algebraic manipulations, we are led to the first assertion (2.1) of Theorem 2.1.

Finally, we proceed to prove identity (2.2), apply identities (1.6) and (1.7), under the given precondition of result (1.12); and further using (1.9), we obtain:

$$\begin{aligned}
(2.12) \quad 9TQ^2P^6 = & \frac{9R_6}{R_{18}} \cdot \left\{ \frac{R_3(-q^{18}, -q^{36}; q^{36})_\infty}{(q^6; q^6)_\infty (q^{18}; q^{18})_\infty (-q^6, -q^{12}; q^{12})_\infty} \right\}^2 \\
& \cdot \left\{ \frac{(q; q)_\infty (-q^3, -q^6; q^6)_\infty}{R_\alpha R_\beta} \right\}^6 \left\{ \frac{1}{(q^3; q^3)_\infty} \right\}^4.
\end{aligned}$$

$$(2.13) \quad -27Pu = -\frac{27(q; q)_\infty (-q^3, -q^6; q^6)_\infty}{q^{\frac{1}{2}} R_\alpha R_\beta(q^3; q^3)_\infty} \left\{ \frac{R_1}{R_3} \right\}^2.$$

$$(2.14) \quad 27P^5u = \frac{27}{q^{\frac{1}{2}}} \left\{ \frac{(q; q)_\infty (-q^3, -q^6; q^6)_\infty}{R_\alpha R_\beta(q^3; q^3)_\infty} \right\}^5 \left\{ \frac{R_1}{R_3} \right\}^2.$$

$$(2.15) \quad -81TQ^2P^2 = -\frac{R_6}{R_{18}} \left\{ \frac{9R_3}{R_\alpha R_\beta} \right\}^2 \cdot \left\{ \frac{(q; q)_\infty (-q^3, -q^6; q^6)_\infty (-q^{18}, -q^{36}; q^{36})_\infty}{(q^6; q^6)_\infty (q^{18}; q^{18})_\infty (-q^6, -q^{12}; q^{12})_\infty} \right\}^2,$$

$$(2.16) \quad -9vQ^4P^3u = -\frac{R_3(q^6; q^{12})_\infty}{q^{\frac{7}{2}}(q^{12}; q^{12})_\infty} \left\{ \frac{3R_1}{R_{18}(q^6; q^6)_\infty} \right\}^2 \\ \cdot \left\{ \frac{R_6(q; q)_\infty (-q^3, -q^6; q^6)_\infty}{R_\alpha R_\beta} \right\}^3 \left\{ \frac{(-q^{18}, -q^{36}; q^{36})_\infty}{(q^{18}; q^{18})_\infty (-q^6, -q^{12}; q^{12})_\infty} \right\}^4.$$

$$(2.17) \quad Q^4P^7uv = \frac{1}{q^{\frac{7}{2}}} \frac{1}{\{(q^3; q^3)_\infty\}^3} \left\{ \frac{R_3 R_1 R_6}{R_{18}} \right\}^2 \\ \cdot \left\{ \frac{(-q^{18}, -q^{36}; q^{36})_\infty}{(q^6; q^6)_\infty (q^{18}; q^{18})_\infty (-q^6, -q^{12}; q^{12})_\infty} \right\}^4 \cdot \left\{ \frac{(q; q)_\infty (-q^3, -q^6; q^6)_\infty}{R_\alpha R_\beta} \right\}^7.$$

$$(2.18) \quad TQ^6v = \frac{1}{q^3} \left\{ \frac{R_6}{R_{18}} \right\}^3 \left\{ \frac{(q^6; q^6)_\infty (-q^{18}, -q^{36}; q^{36})_\infty}{(q^{18}; q^{18})_\infty (-q^6, -q^{12}; q^{12})_\infty} \right\}^6.$$

$$(2.19) \quad -TQ^6P^4v = -\frac{1}{q^3} \left\{ \frac{(q^6; q^{12})_\infty}{(q^6; q^6)_\infty (q^{12}; q^{12})_\infty} \right\}^2 \left\{ \frac{R_3(q; q)_\infty (-q^3, -q^6; q^6)_\infty}{R_\alpha R_\beta} \right\}^4 \\ \cdot \frac{\{R_6\}^5}{\{R_{18}\}^3} \left\{ \frac{(-q^{18}, -q^{36}; q^{36})_\infty}{(-q^6, -q^{12}; q^{12})_\infty (q^{18}; q^{18})_\infty} \right\}^6.$$

Now, upon substituting from these last results (2.12) to (2.19) into (1.12), if we rearrange the terms and use some algebraic manipulations, we are led to assertion (2.2). \square

3. Concluding Remarks and Observations

The present investigation was motivated by several recent developments dealing essentially with theta-function identities and combinatorial partition-theoretic identities. Here, we have established a family of three presumably new theta-function identities which depict the inter-relationships that exist among R_α, R_β and derivatives of R_m -functions. We have also considered several closely-related identities such as (for example) q -product identities and Jacobi's triple-product identities. A view to further motivating researches involving theta-function identities and combinatorial partition-theoretic identities, we have chosen to indicate rather briefly a number of recent developments on the subject-matter of this article. In a recently-published review-cum-expository review article, in addition to applying the q -analysis to geometric function theory of complex analysis, Srivastava [24] pointed out the fact that the results for the q -analogues can easily (and possibly trivially) be translated into the corresponding results for the (p, q) -analogues (with $0 < |q| < p \leq 1$) by applying some obvious parametric and argument variations, the additional parameter p being *redundant*. Of course, this exposition and observation of Srivastava [24, p. 340] would apply also to the results which we have considered

in our present investigation for $|q| < 1$. The list of citations, which we have included in this article, is believed to be potentially useful for indicating some of the directions for further researches and related developments on the subject-matter which we have dealt with here. In particular, the recent works by Adiga et al. (see [1]–[3]), Cao et al. [9], Chaudhary et al. (see [10] to [16]), Hahn et al. [17], and Srivastava et al. (see [26], [31]–[33]) are worth mentioning here.

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