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COEFFICIENTS ASSESSMENT FOR CERTAIN SUBCLASSES OF BI-UNIVALENT FUNCTIONS RELATED WITH QUASI-SUBORDINATION

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Abstract. We investigate specific new subclasses of the function class Σ of bi-univalent function defined in the open unit disc, which is connected with quasi-subordination. We find estimates on the Taylor–Maclaurin coefficients $|a_2|$ and $|a_3|$ for functions in these subclasses. Already pointed out are some documented and new implications of those findings.

1. Introduction

Let $\mathcal{A}$ denote the analytic function class in the open unit disc $\mathcal{D} = \{z \in \mathbb{C} : |z| < 1\}$ which contains the shape

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n \qquad (z \in \mathcal{D}), \tag{1.1}$$

then let $\mathcal{S}$ be the class of all univalent functions from $\mathcal{A}$ in $\mathcal{D}$. The Koebe One Quarter Theorem [7] states that the image of $\mathcal{D}$ beneath every function f from $\mathcal{S}$ contains a radius disk of $\frac{1}{4}$. This univalent function, therefore, has an inverse one f^{-1} which satisfies

$$f^{-1}(f(z)) = z,\ (z \in \mathcal{D}) \text{ and } f(f^{-1}(w)) = w, \quad \left(|w| < r_0(f),\ r_0(f) \geqslant \tfrac{1}{4}\right).$$

In fact the inverse function f^{-1} is given by

$$g(w) = f^{-1}(w) = w - a_2 w^2 + (2a_2^2 - a_3)w^3 - (5a_2^3 - 5a_2a_3 + a_4)w^4 + \cdots. \tag{1.2}$$

A function $f \in \mathcal{A}$ is said to be bi-univalent in $\mathcal{D}$ if both f and f^{-1} are univalent in $\mathcal{D}$. Let Σ denote the class of bi-univalent functions defined in the unit disc $\mathcal{D}$. Ma–Minda [11] introduction the following classes using subordination:

$$S^*(h) = \left\{ f \in \mathcal{A} : \frac{zf'(z)}{f(z)} \prec h(z) \right\},$$

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where h is an analytic function with positive real part on $\mathcal{D}$ with $h(0) = 1$, $h'(0) > 0$ which maps the unit disc $\mathcal{D}$ on a starlike area with respect to 1 and which is symmetric consider to the real axis. A function $f \in S^*(h)$ is called Ma–Minda starlike. $C(h)$ is a class of convex function $f \in \mathcal{A}$ for which

$$1 + \frac{zf''(z)}{f'(z)} \prec h(z).$$

The classes $S^*(h)$ and $C(h)$ contain various well-known subcategories of starlike and convex function as private case. The notion of subordination is propagated in 1970 by Robertson [**20**] through introducing a new notion of quasi-subordination.

For two analytic functions f and h, the function f is quasi subordination to h written as $f(z) \prec_q h(z)$ $(z \in \mathcal{D})$ in the event of an analytical function ϑ and w, with $|\vartheta(z)| \leqslant 1$, $w(0) = 0$ and $|w(z)| < 1$ such that $f(z)/\vartheta(z) \prec h(z)$, which is equivalent to $f(z) = \vartheta(z)h(w(z))$ $(z \in \mathcal{D})$. Note that if $\vartheta(z) = 1$, then $f(z) = h(w(z))$, so that $f(z) \prec h(z)$ in $\mathcal{D}$, also if $w(z) = z$, then $f(z) = \vartheta(z)h(z)$ and it is said that $f(z)$ is majorized by $h(z)$ and written as $f(z) \ll h(z)$ in $\mathcal{D}$. Hence it is perceptible that the quasi-subordination is a popularization of the usual subordination as well as majorization. The labor on quasi-subordination is very extensive and that includes some recent investigations [**2, 4, 9, 10, 12, 14, 19, 20**].

In 1967, Lewin [**10**] researched the class Σ of bi-univalent functions and gained the limit for the second coefficient a_2. Brannan and Taha [**5**] examined specific subclasses of bi-univalent functions similar to the common subclasses of univalent functions consisting of strongly starlike, starlike and convex functions. They introduced the bi-starlike function, bi-convex function classes and acquired non-sharp bounds for the first two Taylor–Maclaurin coefficients $|a_2|$ and $|a_3|$. The study and the investigation of various subclasses of the bi-univalent function class Σ was revived in recent years by Srivastava et al. [**27**] and significantly large number of continuation (see [**3, 21, 22, 23, 26, 29, 32**]) refer to Srivastava et al. [**27**]. Recently Ali et al. [**1**], Deniz [**6**], Tang et al. [**30**], Peng et al. [**16**], Ramchandran et al. [**18**], Murugusundaramoorthy et al. [**13**], Srivastava et al. [**24**], [**28**], etc., have examined and studied Ma–Minda type subclasses of bi-univalent functions class Σ. Further generalization of Ma–Minda type subclasses of class Σ have been made several authors including [**8, 15, 12, 25, 31**] using quasi-subordination. Motivated by work in [**9, 14**] on quasi-subordination, we introduce and investigate here certain new subclasses of class Σ.

Throughout this sheet, it is assumed that $h(z)$ is analytic and univalent with positive real part in $\mathcal{D}$ and let

$$h(z) = 1 + B_1 z + B_2 z^2 + \cdots, \quad (B_1 \in \mathbb{R}^+), \tag{1.3}$$

and h maps $\mathcal{D}$ onto a region starlike with respect to 1 and symmetric with respect to the real axis. Also, let $\vartheta(z)$ be an analytic function in $\mathcal{D}$ and

$$\vartheta(z) = A_0 + A_1 z + A_2 z^2 + \cdots, \quad (|\vartheta(z)| \leqslant 1,\ z \in \mathcal{D}). \tag{1.4}$$

Definition 1.1. For $0 \leqslant \delta \leqslant 1$ and $\lambda \geqslant 0$, a function $f \in \Sigma$ is said to be in the class $H_{\Sigma}^{q}(\lambda, \delta, \vartheta)$ if the following quasi-subordination hold

$$\frac{(1-\delta)f(z)+\delta z f'(z)}{z} + \lambda z f''(z) - 1 \prec_q (\vartheta(z)-1)$$
$$\frac{(1-\delta)g(w)+\delta w g'(w)}{w} + \lambda w g''(w) - 1 \prec_q (\vartheta(w)-1)$$

where the function g is the extension of f^{-1} to $\mathcal{D}$.

Definition 1.2. A function $f \in \Sigma$ is said to be in the class $M_{\Sigma}^{q}(\beta, \vartheta)$ if the following quasi-subordination hold

$$\frac{z\Psi'(z)}{\Psi(z)} - 1 \prec_q (\vartheta(z)-1), \quad \frac{w\Phi'(w)}{\Phi(w)} - 1 \prec_q (\vartheta(w)-1),$$

where $\Psi(z)$ and $\Phi(w)$ are as follows:

$$\frac{1}{\Psi(z)} = \frac{1-\beta}{f(z)} + \frac{\beta}{zf'(z)} \quad \text{and} \quad \frac{1}{\Phi(w)} = \frac{1-\beta}{g(w)} + \frac{\beta}{wg'(w)} \quad (\beta \in \mathbb{C}),$$

and Φ is the extension of Ψ^{-1} to $\mathcal{D}$.

Lemma 1.1 (See [**17**]). *Let $\mathcal{P}$ be class of all functions p analytic in U for which $Re(p(z)) > 0$ and have the form $p(z) = 1 + p_1 z + p_2 z^2 + \cdots$ for $z \in \mathcal{D}$, then $|p_i| \leqslant 2$ for each $i \in \mathbb{N}$.*

2. Main Results

Theorem 2.1. $0 \leqslant \delta \leqslant 1$ *and* $\lambda \geqslant 0$. *If $f \in \Sigma$ of the form* (1.1) *belonging to the class $H_{\Sigma}^{q}(\lambda, \delta, \vartheta)$, then*

$$|a_2| \leqslant \frac{|A_0| B_1 \sqrt{B_1}}{\sqrt{|(1+2\delta+6\lambda)A_0 B_1^2 + (1+\delta+2\lambda)^2 (B_1 - B_2)|}}, \tag{2.1}$$

$$|a_3| \leqslant \frac{|A_0|^2 B_1^2}{(1+\delta+2\lambda)^2} + \frac{|A_1| B_1}{1+2\delta+6\lambda} + \frac{|A_0| B_1}{1+2\delta+6\lambda}. \tag{2.2}$$

Proof. Let $f \in H_{\Sigma}^{q}(\lambda, \delta, \vartheta)$. In view of Definition 1.1, there exist then Schwarz functions $k(z)$, $s(w)$ and an analytic function $\vartheta(z)$ such that

$$\frac{(1-\delta)f(z)+\delta z f'(z)}{z} + \lambda z f''(z) - 1 = \vartheta(z)\left(h(k(z)) - 1\right), \tag{2.3}$$

$$\frac{(1-\delta)g(w)+\delta w g'(w)}{w} + \lambda w g''(w) - 1 = \vartheta(w)\left(h(s(w)) - 1\right). \tag{2.4}$$

Define the functions $p(z)$ and $q(w)$ by

$$p(z) = \frac{1+k(z)}{1-k(z)} = 1 + c_1 z + c_2 z^2 + \cdots \tag{2.5}$$

$$q(w) = \frac{1+s(w)}{1-s(w)} = 1 + b_1 w + b_2 w^2 + \cdots, \tag{2.6}$$

or equivalently,

$$k(z) = \frac{p(z)-1}{p(z)+1} = \frac{1}{2}\left(c_1 z + \left(c_2 - \frac{1}{2}c_1^2\right) z^2 + \cdots\right), \tag{2.7}$$

$$s(w) = \frac{q(w)-1}{q(w)+1} = \frac{1}{2}\left(b_1 w + \left(b_2 - \frac{1}{2}b_1^2\right) w^2 + \cdots\right). \tag{2.8}$$

It is clear that $p(z)$ and $q(w)$ are analytic and have positive real parts in $\mathcal{D}$. In view of (2.3), (2.4), (2.7) and (2.8) clearly

$$\frac{(1-\delta)f(z)+\delta z f'(z)}{z} + \lambda z f''(z) - 1 = \vartheta(z)\left[h\left(\frac{p(z)-1}{p(z)+1}\right) - 1\right], \tag{2.9}$$

$$\frac{(1-\delta)g(w)+\delta w g'(w)}{w} + \lambda w g''(w) - 1 = \vartheta(w)\left[h\left(\frac{q(w)-1}{q(w)+1}\right) - 1\right]. \tag{2.10}$$

The series expansions for $f(z)$ and $g(w)$ as given in (1.1) and (1.2) respectively, provide us

$$\begin{aligned}&\frac{(1-\delta)f(z)+\delta z f'(z)}{z} + \lambda z f''(z) - 1 \\ &\quad = (1+\delta)a_2 z + (1+2\delta)a_3 z^2 + 2\lambda a_2 z + 6\lambda a_3 z^2 + \cdots\end{aligned} \tag{2.11}$$

$$\begin{aligned}&\frac{(1-\delta)g(w)+\delta w g'(w)}{w} + \lambda w g''(w) - 1 \\ &\quad = -(1+\delta)a_2 w + (1+2\delta)(2a_2^2 - a_3)w^2 - 2\lambda a_2 w + 6\lambda(2a_2^2 - a_3)w^2 + \cdots .\end{aligned} \tag{2.12}$$

Using (2.5) and (2.6) together with (1.3) and (1.4)

$$\begin{aligned}&\vartheta(z)\left[h\left(\frac{p(z)-1}{p(z)+1}\right) - 1\right] \\ &\quad = \frac{1}{2}A_0 B_1 c_1 z + \left[\frac{1}{2}A_1 B_1 c_1 + \frac{1}{2}A_0 B_1\left(c_2 - \frac{c_1^2}{2}\right) + \frac{A_0 B_2 c_1^2}{4}\right] z^2 + \cdots\end{aligned} \tag{2.13}$$

$$\begin{aligned}&\vartheta(z)\left[h\left(\frac{q(w)-1}{q(w)+1}\right) - 1\right] \\ &\quad = \frac{1}{2}A_0 B_1 b_1 w + \left[\frac{1}{2}A_1 B_1 b_1 + \frac{1}{2}A_0 B_1\left(b_2 - \frac{b_1^2}{2}\right) + \frac{A_0 B_2 b_1^2}{4}\right] w^2 + \cdots .\end{aligned} \tag{2.14}$$

Now equating (2.11) and (2.13) in view of (2.9) and comparing the coefficients of z and z^2, we have

$$(1+\delta+2\lambda)a_2 = \frac{1}{2}A_0 B_1 c_1, \tag{2.15}$$

$$(1+2\delta+6\lambda)a_3 = \frac{1}{2}A_1 B_1 c_1 + \frac{1}{2}A_0 B_1\left(c_2 - \frac{c_1^2}{2}\right) + \frac{A_0 B_2 c_1^2}{4}. \tag{2.16}$$

Similarly, (2.10) given us

$$-(1+\delta+2\lambda)a_2 = \frac{1}{2}A_0B_1b_1, \tag{2.17}$$

$$(1+2\delta+6\lambda)(2a_2^2-a_3) = \frac{1}{2}A_1B_1b_1 + \frac{1}{2}A_0B_1\left(b_2-\frac{b_1^2}{2}\right) + \frac{A_0B_2b_1^2}{4}. \tag{2.18}$$

From (2.15) and (2.17) we find that

$$c_1 = -b_1 \tag{2.19}$$

and (2.16), (2.17) and (2.19) yields

$$a_2^2 = \frac{A_0^2B_1^3(b_2+c_2)}{4(1+2\delta+6\lambda)A_0B_1^2 - 4(1+\delta+2\lambda)^2(B_2-B_1)}.$$

Now further computations (2.16) to (2.18) lead to

$$a_3 = \frac{A_1B_1(c_1-b_1)}{4(1+2\delta+6\lambda)} + \frac{A_0B_1(c_2-b_2)}{4(1+2\delta+6\lambda)} + \frac{A_0^2B_1^2(c_1^2+b_1^2)}{8(1+2\delta+2\lambda)^2}.$$

Using the above results and in view of the inequalities $|c_i| \leqslant 2$ and $|b_i| \leqslant 2$ $(i = 1, 2)$ for functions with positive real part yield the requested estimate in (2.1) and (2.2). □

Remark 2.1. For $\delta = 0$, a function $f \in \Sigma$ defined in (1.1) is said to be in the class $H_\Sigma^q(\lambda, \delta, \vartheta)$ if the following conditions are satisfied

$$\frac{f(z)}{z} + \lambda z f''(z) - 1 \prec_q (\vartheta(z)-1) \quad \text{and} \quad \frac{g(w)}{w} + \lambda w g''(w) - 1 \prec_q (\vartheta(w)-1).$$

For $\delta = 0$, we have the class $H_\Sigma^q(\lambda, 0, \vartheta) = H_\Sigma^q(\lambda, \vartheta)$.

Corollary 2.1. *Let $f(z)$ given by (1.1) belong to the class $H_\Sigma^q(\lambda, \vartheta)$ then*

$$|a_2| \leqslant \frac{|A_0|B_1\sqrt{B_1}}{\sqrt{|(1+6\lambda)A_0B_1^2 + (1+2\lambda)^2(B_1-B_2)|}},$$

$$|a_3| \leqslant \frac{|A_0|^2B_1^2}{(1+2\lambda)^2} + \frac{|A_1|B_1}{1+6\lambda} + \frac{|A_0|B_1}{1+6\lambda}.$$

By putting $\delta = 1$ in Theorem 2.1, we have the following corollary.

Corollary 2.2. *Let $f(z)$ given by (1.1) to the class $H_\Sigma^q(\lambda, 1, \vartheta)$ then*

$$|a_2| \leqslant \frac{|A_0|B_1\sqrt{B_1}}{\sqrt{|3(1+2\lambda)A_0B_1^2 + 4(1+\lambda)^2(B_1-B_2)|}},$$

$$|a_3| \leqslant \frac{|A_0|^2B_1^2}{4(1+\lambda)^2} + \frac{|A_1|B_1}{3(1+2\lambda)} + \frac{|A_0|B_1}{3(1+2\lambda)}.$$

THEOREM 2.2. *If $f \in M_{\Sigma}^{q}(\beta,\vartheta)$, then*

$$|a_2| \leqslant \frac{|A_0|B_1\sqrt{B_1}}{\sqrt{|(1+\beta^2)A_0B_1^2+(1+\beta)^2(B_1-B_2)|}}$$

$$|a_3| \leqslant \frac{|A_0|^2B_1^2}{|1+\beta|^2}+\frac{|A_1|B_1}{2|1+2\beta|}+\frac{|A_0|B_1}{2|1+2\beta|}.$$

PROOF. Since $f \in M_{\Sigma}^{q}(\beta,\vartheta)$ and $\Phi=\Psi^{-1}$ then there exist analytic functions $k,s:\mathcal{D}\to\mathcal{D}$ with $k(0)=s(0)=0$ satisfying

$$\frac{z\Psi'(z)}{\Psi(z)}-1=\vartheta(z)\left(h(k(z))-1\right), \tag{2.20}$$

$$\frac{w\Phi'(w)}{\Phi(w)}-1=\vartheta(w)\left(h(s(w))-1\right). \tag{2.21}$$

For $p(z)$ and $q(w)$ as given in (2.5) and (2.6), respectively, in view of (2.20), (2.21), clearly

$$\frac{z\Psi'(z)}{\Psi(z)}-1=\vartheta(z)\left[h\left(\frac{p(z)-1}{p(z)+1}\right)-1\right], \tag{2.22}$$

$$\frac{w\Phi'(w)}{\Phi(w)}-1=\vartheta(w)\left[h\left(\frac{q(w)-1}{q(w)+1}\right)-1\right]. \tag{2.23}$$

Since

$$\frac{z\Psi'(z)}{\Psi(z)}-1=(1+\beta)a_2z+\left[2(1+2\beta)a_3+(\beta^2-4\beta-1)a_2^2\right]z^2+\cdots, \tag{2.24}$$

$$\frac{w\Phi'(w)}{\Phi(w)}-1=-(1+\beta)a_2w+\left[-2(1+2\beta)a_3+(\beta^2+4\beta+3)a_2^2\right]w^2+\cdots. \tag{2.25}$$

The right-hand sides of (2.22) and (2.23) are given by (2.13) and (2.14) respectively. Now using (2.13) and (2.22) in (2.24) and comparing the coefficients of z and z^2, we get

$$(1+\beta)a_2=\frac{1}{2}A_0B_1c_1 \tag{2.26}$$

$$2(1+2\beta)a_3+(\beta^2-4\beta-1)a_2^2=\frac{1}{2}A_1B_1c_1+\frac{1}{2}A_0B_1\left(c_2-\frac{c_1^2}{2}\right)+\frac{A_0B_2c_1^2}{4}. \tag{2.27}$$

Similarly, it follows from (2.14), (2.23) and (2.25) that

$$-(1+\beta)a_2=\frac{1}{2}A_0B_1b_1, \tag{2.28}$$

$$-2(1+2\beta)a_3+(\beta^2+4\beta+3)a_2^2=\frac{1}{2}A_1B_1b_1+\frac{1}{2}A_0B_1\left(b_2-\frac{b_1^2}{2}\right)+\frac{A_0B_2b_1^2}{4}. \tag{2.29}$$

From (2.26) and (2.28) it follows that

$$c_1=-b_1 \tag{2.30}$$

and (2.27), (2.29) and (2.30), yield

$$a_2^2=\frac{A_0^2B_1^3(b_2+c_2)}{4\left[(1+\beta^2)A_0B_1^2-(1+\beta)^2(B_2-B_1)\right]}.$$

Now further computation (2.27) to (2.29) leads to

$$a_3 = \frac{A_0^2 B_1^2 (c_1^2 + b_1^2)}{8(1+\beta)^2} + \frac{A_1 B_1 (c_1 - b_1)}{8(1+2\beta)} + \frac{A_0 B_1 (c_2 - b_2)}{8(1+2\beta)}. \qquad \square$$

REMARK 2.2. Putting $\vartheta(z) = 1$ in Theorem 2.2, we get the corresponding result given by Deniz [**6**], for $\beta = 0$ the above Theorem 2.2 reduces to the following corollary.

COROLLARY 2.3. *If $f \in M_\Sigma^q(0, \vartheta)$, then*

$$|a_2| \leqslant \frac{|A_0| B_1 \sqrt{B_1}}{\sqrt{|A_0 B_1^2 + (B_1 - B_2)|}}, \quad \textit{and} \quad |a_3| \leqslant |A_0|^2 B_1^2 + \frac{|A_1| B_1}{2} + \frac{|A_0| B_1}{2}.$$

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