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A CLASS OF HARMONIC FUNCTIONS ASSOCIATED WITH A GENERALIZED DIFFERENTIAL OPERATOR

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Abstract. We introduce a new class of harmonic univalent functions by using a generalized differential operator and investigate some of its geometric properties, like, coefficient estimates, extreme points and inclusion relations. Finally, we show that this class is invariant under Bernandi–Libera–Livingston integral for harmonic functions.

1. Introduction

Let $\mathcal{A}$ denote the class of analytic functions f, in the open unit disk $\mathbb{E} = \{z \in \mathbb{C} : |z| < 1\}$, normalized by $f(0) = f'(0) - 1 = 0$ thus functions in the class $\mathcal{A}$ has the form $f(z) = z + \sum_{k=2}^{\infty} a_k z^k$. Let S be the subclass of $\mathcal{A}$ whose functions are univalent in $\mathbb{E}$ and further S^* and K are subclasses of S whose functions map $\mathbb{E}$ on to starlike and convex domains, respectively. For $\mu \geqslant 0$ and $\nu \geqslant 0$, let $D^m(\mu,\nu) : \mathcal{A} \to \mathcal{A}$ be a linear differential operator defined by

$$D^m(\mu,\nu)f(z) = z + \sum_{k=2}^{\infty} \phi_k^m(\mu,\nu) a_k z^k$$

where, $m \in \mathbb{N} \cup \{0\}$ and $\phi_{k+1}^m(\mu,\nu) = [(k\mu+1)(k\nu+1)]^m$. Setting $\mu = 0$ and $\nu = 1$ the operator $D^m(\mu,\nu)$ becomes the well known Salagean differential operator. In the past three decades, many researchers, using $D^m(\mu,\nu)$ for particular choices of μ, ν, defined various subclasses of $\mathcal{A}$ whose functions satisfy certain geometric properties. In particular,

(i) Fournier and Ruscheweyh [**6**] defined the class $P(\beta)$, where

$$P(\beta) = \Big\{ f \in \mathcal{A} : \operatorname{Re} \frac{D^1(0,1)f(z)}{z} > \beta \Big\}.$$

(ii) Kim and Rønning [**11**] defined the class $P_\alpha(\beta)$, where

$$P_\alpha(\beta) = \Big\{ f \in \mathcal{A} : \operatorname{Re} \frac{D^1(0,\alpha)f(z)}{z} > \beta \Big\}.$$

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(iii) Ponnusamy and Rønning [**14**] defined the class $R_\gamma(\beta)$, where

$$R_\gamma(\beta) = \Big\{ f \in \mathcal{A} : \operatorname{Re} \frac{D^1(1,\gamma)f(z)}{z} > \beta \Big\}.$$

(iv) For $\mu \geqslant 0$, $\nu \geqslant 0$ and $m = 1$, define $\alpha \geqslant 0$ and $\gamma \geqslant 0$ such that $\alpha - \gamma = \mu + \nu$ and $\gamma = \mu\nu$. Ali et al. [**1**] defined the class $W_\beta(\alpha,\gamma)$, where

$$W_\beta(\alpha,\gamma) = \Big\{ f \in \mathcal{A} : \operatorname{Re} \frac{D^1(\mu,\nu)f(z)}{z} > \beta \Big\}.$$

The authors applied the Duality theory to prove the starlikeness of the linear integral transform over functions f in these classes.

We apply this operator on complex valued harmonic functions $f = h + \overline{g}$ and explore its various geometric properties. Here, we call h the analytic part and g the co-analytic part of f. Let $\mathcal{H}$ be the class of complex valued harmonic mappings, $f = h + \overline{g}$, defined in $\mathbb{E}$ and normalized by $h(0) = g(0) = h'(0) - 1 = 0$. Such mappings have the following power series representation

$$f(z) = z + \sum_{k=2}^{\infty} a_k z^k + \overline{\sum_{k=1}^{\infty} b_k z^k}. \tag{1.1}$$

A necessary and sufficient condition for $f \in \mathcal{H}$ to be locally univalent and sense-preserving in $\mathbb{E}$ is that $|h'(z)| > |g'(z)|$ for all $z \in \mathbb{E}$ (see [**13**]). Clunie and Sheil-Small, in 1984, studied the harmonic mappings as a generalization of analytic functions and defined analogues classes, see [**5**]. They defined the class S_H, of univalent and sense-preserving harmonic mappings, as a generalization of the class S and explored geometric properties of S_H including its subclasses. Denote by S_H^* and K_H the analogues subclasses of S^* and K, respectively. After the induction of Clunie and Sheil-Small's paper intensive research has been done in this area and researchers defined various subclasses of S_H by using some linear operators, see ([**2**–**4**, **7**–**10**, **12**, **15**–**20**]) and references therein.

In this paper, we define a new subclass $S_H^m(\mu,\nu,\lambda,b)$ of S_H consisting of harmonic functions $f = h + \overline{g}$ of the form (1.1) and satisfying

$$\operatorname{Re}\left[\frac{D^{m+1}(\mu,\nu)f(z)}{\lambda z + (1-\lambda)D^m(\mu,\nu)f(z)}\right] \geqslant b, \tag{1.2}$$

here $D^m(\mu,\nu)f(z) = D^m(\mu,\nu)h(z) + (-1)^m\overline{D^m(\mu,\nu)g(z)}$, $0 \leqslant \lambda \leqslant 1$ and $0 \leqslant b < 1$. Let $\overline{S}_H$ denote the subclass of S_H consisting of functions of the type $f_m = h + \overline{g}_m$, where

$$h(z) = z - \sum_{k=2}^{\infty} |a_k| z^k, \quad g_m(z) = (-1)^m \sum_{k=1}^{\infty} |b_k| z^k. \tag{1.3}$$

Let $\overline{S_H^m}(\mu,\nu,\lambda,b)$ be the subclass of $\overline{S}_H$ whose functions satisfy condition (1.2). In subsequent sections, we obtain a sufficient condition for harmonic functions $f = h + \overline{g}$ given by (1.1) to be in $S_H^m(\mu,\nu,\lambda,b)$ and then we prove that this condition is also necessary for the functions in the class $\overline{S_H^m}(\mu,\nu,\lambda,b)$. We investigate extreme points, covering theorem, convolution properties, convex combinations and Bernandi–Libera–Livingston integral for the functions in the class $\overline{S_H^m}(\mu,\nu,\lambda,b)$.

2. Characterization Properties

THEOREM 2.1. *For $\mu,\nu \geqslant 0$, $m \in \mathbb{N}\cup\{0\}$, $0 \leqslant \lambda \leqslant 1$, and $0 \leqslant b < 1$, let a function $f = h + \overline{g}$ be given by (1.1) and satisfy*

$$\sum_{k=2}^{\infty} \Phi(k,m,\mu,\nu,\lambda,b)|a_k| + \sum_{k=1}^{\infty} \Psi(k,m,\mu,\nu,\lambda,b)|b_k| \leqslant (1-b), \tag{2.1}$$

where

$$\Phi(k,m,\mu,\nu,\lambda,b) = \phi_k^m(\mu,\nu)\left(\phi_k^1(\mu,\nu) - b(1-\lambda)\right),$$
$$\Psi(k,m,\mu,\nu,\lambda,b) = \phi_k^m(\mu,\nu)\left(\phi_k^1(\mu,\nu) + b(1-\lambda)\right)$$

with $\phi_{k+1}^m(\mu,\nu) = \left[(k\mu+1)(k\nu+1)\right]^m$; then $f \in S_H^m(\mu,\nu,\lambda,b)$.

PROOF. First we shall prove that f is sense-preserving and univalent in $\mathbb{E}$. Since $|z|<1$, we have

$$\begin{aligned} |h'(z)| &\geqslant 1 - \sum_{k=2}^{\infty} k|a_k||z|^{k-1} > 1 - \sum_{k=2}^{\infty} k|a_k| \\ &> 1 - \sum_{k=2}^{\infty} \frac{\Phi(k,m,\mu,\nu,\lambda,b)}{1-b}|a_k| \\ &\geqslant \sum_{k=1}^{\infty} \frac{\Psi(k,m,\mu,\nu,\lambda,b)}{1-b}|b_k|, \quad \text{in view of (2.1)} \\ &> \sum_{k=1}^{\infty} k|b_k||z|^{k-1} \geqslant |g'(z)|. \end{aligned}$$

It gives that f is sense-preserving in $\mathbb{E}$. Next we prove the univalence of f in $\mathbb{E}$. For $z_1 \neq z_2 \in \mathbb{E}$,

$$\begin{aligned} \left|\frac{f(z_1)-f(z_2)}{h(z_1)-h(z_2)}\right| &\geqslant 1 - \left|\frac{g(z_1)-g(z_2)}{h(z_1)-h(z_2)}\right| > 1 - \frac{\sum_{k=1}^{\infty} k|b_k|}{1-\sum_{k=2}^{\infty} k|a_k|} \\ &\geqslant 1 - \frac{\sum_{k=1}^{\infty} \frac{\Psi(k,m,\mu,\nu,\lambda,b)}{1-b}|b_k|}{1-\sum_{k=2}^{\infty} \frac{\Phi(k,m,\mu,\nu,\lambda,b)}{1-b}|a_k|} \geqslant 0. \end{aligned}$$

Hence in view of condition (2.1), f is univalent in $\mathbb{E}$. Next we shall show that $\operatorname{Re}\left(\frac{D^{m+1}(\mu,\nu)f(z)}{\lambda z+(1-\lambda)D^m(\mu,\nu)f(z)}\right) \geqslant b$. We know that $\operatorname{Re}(w) \geqslant \alpha$ if and only if $|1-\alpha+w| \geqslant |1+\alpha-w|$. So, it suffices to show that

$$\begin{aligned} &\left|\lambda(1-b)z + (1-\lambda)(1-b)D^m(\mu,\nu)f(z) + D^{m+1}(\mu,\nu)f(z)\right| \\ &\qquad - \left|\lambda(1+b)z + (1-\lambda)(1+b)D^m(\mu,\nu)f(z) - D^{m+1}(\mu,\nu)f(z)\right| \geqslant 0. \end{aligned}$$

Now,

$$\begin{aligned}
&\left|\lambda(1-b)z+(1-\lambda)(1-b)D^m(\mu,\nu)f(z)+D^{m+1}(\mu,\nu)f(z)\right| \\
&\qquad -\left|\lambda(1+b)z+(1-\lambda)(1+b)D^m(\mu,\nu)f(z)-D^{m+1}(\mu,\nu)f(z)\right| \\
=&\Bigg|(2-b)z+\sum_{k=2}^{\infty}\left[((k-1)\mu+1)\left((k-1)\nu+1\right)\right]^m\left[((k-1)\mu+1)\left((k-1)\nu+1\right)\right. \\
&+\left.(1-\lambda)(1-b)\right]a_kz^k-(-1)^m\sum_{k=1}^{\infty}\left[((k-1)\mu+1)\left((k-1)\nu+1\right)\right]^m \\
&\left[((k-1)\mu+1)\left((k-1)\nu+1\right)-(1-\lambda)(1-b)\right]\overline{b_k}\overline{z}^k\Big| \\
&-\Bigg|bz-\sum_{k=2}^{\infty}\left[((k-1)\mu+1)\left((k-1)\nu+1\right)\right]^m\left[((k-1)\mu+1)\left((k-1)\nu+1\right)\right. \\
&-\left.(1-\lambda)(1+b)\right]a_kz^k+(-1)^m\sum_{k=1}^{\infty}\left[((k-1)\mu+1)\left((k-1)\nu+1\right)\right]^m \\
&\left[((k-1)\mu+1)\left((k-1)\nu+1\right)+(1-\lambda)(1+b)\right]\overline{b_k}\overline{z}^k\Big| \\
\geqslant&2|z|\left[(1-b)-\sum_{k=2}^{\infty}\Phi(k,m,\mu,\nu,\lambda,b)|a_k||z|^{k-1}-\sum_{k=1}^{\infty}\Psi(k,m,\mu,\nu,\lambda,b)|b_k||z|^{k-1}\right] \\
\geqslant&0,\quad \text{in view of (2.1).}
\end{aligned}$$

□

THEOREM 2.2. *Let* $f_m = h+\overline{g}_m$ *be given by* (1.3). *Then* $f_m \in \overline{S_H^m}(\mu,\nu,\lambda,b)$ *if and only if* f_m *satisfy condition* (2.1).

PROOF. Since $\overline{S_H^m}(\mu,\nu,\lambda,b) \subset S_H^m(\mu,\nu,\lambda,b)$ therefore, 'if' part can be proved from Theorem 2.1, thus we prove 'only if' part. Let $f_m = h+\overline{g_m} \in \overline{S_H^m}(\mu,\nu,\lambda,b)$, thus

$$\operatorname{Re}\left[\frac{D^{m+1}(\mu,\nu)f_m(z)}{\lambda z+(1-\lambda)D^m(\mu,\nu)f_m(z)}-b\right]\geqslant 0.$$

After applying the differential operator $D^m(\mu,\nu)$, the above inequality reduces to,

$$\operatorname{Re}\left[\frac{z(1-b)-\sum_{k=2}^{\infty}\phi_k^m(\mu,\nu)\left(\phi_k^1(\mu,\nu)-b(1-\lambda)\right)|a_k|z^k}{\lambda z+(1-\lambda)D^m(\mu,\nu)f_m(z)}\right.$$
$$\left.-\frac{(-1)^{2m}\sum_{k=1}^{\infty}\phi_k^m(\mu,\nu)\left(\phi_k^1(\mu,\nu)+b(1-\lambda)\right)|b_k|\overline{z}^k}{\lambda z+(1-\lambda)D^m(\mu,\nu)f_m(z)}\right]\geqslant 0.$$

The above condition holds for all values of z, $|z|=r<1$. Choosing the values of z on positive real axis, where $0\leqslant z=r<1$, we have

(2.2)
$$\operatorname{Re}\left[\frac{(1-b)\sum_{k=2}^{\infty}\Phi(k,m,\mu,\nu,\lambda,b)|a_k|r^{k-1}-\sum_{k=1}^{\infty}\Psi(k,m,\mu,\nu,\lambda,b)|b_k|r^{k-1}}{1-(1-\lambda)\sum_{k=2}^{\infty}\phi_k^m(\mu,\nu)|a_k|r^{k-1}+(-1)^{2m}(1-\lambda)\sum_{k=1}^{\infty}\phi_k^m(\mu,\nu)|b_k|r^{k-1}}\right]\geqslant 0.$$

In case condition (2.1) does not hold, then the numerator of the left-hand side of (2.2) becomes negative for r sufficiently close to 1. This leads to a contradiction. □

THEOREM 2.3. *If* $f \in \overline{S_H^m}(\mu,\nu,\lambda,b)$, *then* $f \in S_H^*$.

PROOF. A function $f_m = h + \overline{g_m} \in \overline{S_H^m}(\mu,\nu,\lambda,b)$ maps the unit disk $\mathbb{E}$ onto a starlike domain if and only if for $z \in \mathbb{E}$,

$$\operatorname{Re}\left\{\frac{zh'(z) - \overline{zg'_m(z)}}{h(z) + \overline{g_m(z)}}\right\} > 0. \tag{2.3}$$

We know that $\operatorname{Re}(w) > 0$ if and only if $|1+w| > |1-w|$. Therefore to prove (2.3), it suffices to show that

$$|h(z) + \overline{g_m(z)} + zh'(z) - \overline{zg'_m(z)}| - |h(z) + \overline{g_m(z)} - zh'(z) + \overline{zg'_m(z)}| > 0.$$

Since

$$\begin{aligned}
&|h(z) + \overline{g_m(z)} + zh'(z) - \overline{zg'_m(z)}| - |h(z) + \overline{g_m(z)} - zh'(z) + \overline{zg'_m(z)}|\\
=&\Big|z - \sum_{k=2}^{\infty}|a_k|z^k + (-1)^m\sum_{k=1}^{\infty}|b_k|\overline{z}^k + z(1 - \sum_{k=2}^{\infty}k|a_k|z^{k-1}) - (-1)^m\overline{z}\sum_{k=1}^{\infty}k|b_k|\overline{z}^{k-1}\Big|\\
&- \Big|z - \sum_{k=2}^{\infty}|a_k|z^k + (-1)^m\sum_{k=1}^{\infty}|b_k|\overline{z}^k - z\Big(1 - \sum_{k=2}^{\infty}k|a_k|z^{k-1}\Big) + (-1)^m\overline{z}\sum_{k=1}^{\infty}k|b_k|\overline{z}^{k-1}\Big|\\
=&\Big|2z - \sum_{k=2}^{\infty}(k+1)|a_k|z^k - (-1)^m\sum_{k=1}^{\infty}(k-1)|b_k|\overline{z}^k\Big|\\
&\qquad - \Big|\sum_{k=2}^{\infty}(k-1)|a_k|z^k + (-1)^m\sum_{k=1}^{\infty}(k+1)|b_k|\overline{z}^k\Big|\\
\geqslant& 2|z|\Big[1 - \sum_{k=2}^{\infty}k|a_k||z|^{k-1} - (-1)^m\sum_{k=1}^{\infty}k|b_k||z|^{k-1}\Big]\\
>& 2|z|\Big[1 - \sum_{k=2}^{\infty}k|a_k| - \sum_{k=1}^{\infty}k|b_k|\Big]\\
\geqslant& 2|z|\Big[1 - \sum_{k=2}^{\infty}\frac{\Phi(k,m,\mu,\nu,\lambda,b)}{(1-b)}|a_k| - \sum_{k=1}^{\infty}\frac{\Psi(k,m,\mu,\nu,\lambda,b)}{(1-b)}|b_k|\Big] > 0.
\end{aligned}$$

In view of condition (2.1). Hence it completes the proof. □

Next we establish that the family $\overline{S_H^m}(\mu,\nu,\lambda,b)$ is non-empty, that is we prove that there exist harmonic functions which are members of the class $\overline{S_H^m}(\mu,\nu,\lambda,b)$ and moreover these functions are the extremum for the family $\overline{S_H^m}(\mu,\nu,\lambda,b)$. In the next result we obtain the extreme points for the class $\overline{S_H^m}(\mu,\nu,\lambda,b)$.

THEOREM 2.4. *If* $f_m = h + \overline{g_m}$, *then* $f_m \in \overline{S_H^m}(\mu,\nu,\lambda,b)$ *if and only if* $f_m(z) = \sum_{k=1}^{\infty}(X_k h_k(z) + Y_k g_{m_k}(z))$, *where* $h_1(z) = z$ *and*

$$h_k(z) = z - \frac{(1-b)}{\Phi(k,m,\mu,\nu,\lambda,b)}z^k, \quad k = 2,3,4\ldots$$

$$g_{m_k}(z) = z + (-1)^m \frac{(1-b)}{\Psi(k,m,\mu,\nu,\lambda,b)} z^k, \quad k = 1,2,3,\ldots$$

and $\sum_{k=1}^{\infty}(X_k + Y_k) = 1$*,* $X_k \geqslant 0$*,* $Y_k \geqslant 0$*. In particular, the extreme points of* $\overline{S_H^m}(\mu,\nu,\lambda,b)$ *are* $\{h_k\}$ *and* $\{g_{m_k}\}$*.*

Proof. Suppose

$$\begin{aligned}
f_m(z) &= \sum_{k=1}^{\infty}(X_k h_k + Y_k g_{m_k}) \\
&= \sum_{k=1}^{\infty}(X_k + Y_k)\, z - \sum_{k=2}^{\infty} \frac{(1-b)}{\Phi(k,m,\mu,\nu,\lambda,b)} X_k z^k + \sum_{k=1}^{\infty} \frac{(-1)^m(1-b)}{\Psi(k,m,\mu,\nu,\lambda,b)} Y_k \overline{z^k} \\
&= z - \sum_{k=2}^{\infty} \frac{(1-b)}{\Phi(k,m,\mu,\nu,\lambda,b)} X_k z^k + (-1)^m \sum_{k=1}^{\infty} \frac{(1-b)}{\Psi(k,m,\mu,\nu,\lambda,b)} Y_k \overline{z^k}
\end{aligned}$$

Then

$$\begin{aligned}
&\sum_{k=2}^{\infty} \frac{\Phi(k,m,\mu,\nu,\lambda,b)}{(1-b)}|a_k| + \sum_{k=1}^{\infty} \frac{\Psi(k,m,\mu,\nu,\lambda,b)}{(1-b)}|b_k| \\
&\quad = \sum_{k=2} \frac{\Phi(k,m,\mu,\nu,\lambda,b)}{(1-b)} \left(\frac{1-b}{\Phi(k,m,\mu,\nu,\lambda,b)} X_k \right) \\
&\qquad + \sum_{k=1}^{\infty} \frac{\Psi(k,m,\mu,\nu,\lambda,b)}{1-b} \left(\frac{1-b}{\Psi(k,m,\mu,\nu,\lambda,b)} Y_k \right) \\
&\quad = \sum_{k=2}^{\infty} X_k + \sum_{k=1}^{\infty} Y_k = 1 - X_1 \leqslant 1.
\end{aligned}$$

Therefore, $f_m(z) \in \overline{S_H^m}(\mu,\nu,\lambda,b)$. Conversely, if $f_m(z) \in \overline{S_H^m}(\mu,\nu,\lambda,b)$, let

$$\begin{aligned}
(1-b)X_k &= \Phi(k,m,\mu,\nu,\lambda,b)|a_k| \qquad (k = 2,3,\ldots), \\
(1-b)Y_k &= \Psi(k,m,\mu,\nu,\lambda,b)|b_k| \qquad (k = 1,2,3\ldots)
\end{aligned}$$

and $X_1 = 1 - \sum_{k=2}^{\infty} X_k - \sum_{k=1}^{\infty} Y_k$. Then the required representation is

$$\begin{aligned}
f_m(z) &= z - \sum_{k=2}^{\infty} |a_k| z^k + (-1)^m \sum_{k=1}^{\infty} |b_k| \overline{z}^k \\
&= z - \sum_{k=2}^{\infty} \frac{(1-b)}{\Phi(k,m,\mu,\nu,\lambda,b)} X_k z^k + (-1)^m \sum_{k=1}^{\infty} \frac{(1-b)}{\Psi(k,m,\mu,\nu,\lambda,b)} Y_k \overline{z}^k \\
&= z - \sum_{k=2}^{\infty} \left[z - h_k(z)\right] X_k + (-1)^m \sum_{k=1}^{\infty} \left[z - g_{m_k}(z)\right] Y_k \\
&= \left[1 - \sum_{k=2}^{\infty} X_k - \sum_{k=1}^{\infty} Y_k\right] z + \sum_{k=2}^{\infty} h_k(z) X_k + \sum_{k=1}^{\infty} g_{m_k}(z) Y_k = \sum_{k=1}^{\infty}(X_k h_k + Y_k g_{m_k}). \ \square
\end{aligned}$$

Our next theorem is on the distortion bounds for functions in $\overline{S_H^m}(\mu,\nu,\lambda,b)$, which yields a covering result for this family.

THEOREM 2.5. *If* $F_m = H + G_m \in \overline{S_H^m}(\mu,\nu,\lambda,b)$*, given by*

$$F_m(z) = z - \sum_{k=2}^{\infty} |A_k| z^k + (-1)^m \sum_{k=1}^{\infty} |B_k| \overline{z}^k,$$

and $|z| = r < 1$*, then*

(2.4)

$$(1-|B_1|)r - \frac{1}{\Phi(2,m,\mu,\nu,\lambda,b)} M \leqslant |F_m(z)| \leqslant (1+|B_1|)r + \frac{1}{\Phi(2,m,\mu,\nu,\lambda,b)} M$$

where $M = [(1-b) - \Psi(1,m,\mu,\nu,\lambda,b)|B_1|]\, r^2$.

PROOF. First, We shall prove the left hand side of inequality (2.4). Let $F_m \in \overline{S_H^m}(\mu,\nu,\lambda,b)$; then

$$\begin{aligned}
|F_m(z)| &= \left| z + |B_1|\overline{z} - \sum_{k=2}^{\infty} \left(|A_k| z^k - |B_k| \overline{z}^k\right) \right| \\
&\geqslant (1-|B_1|)\, r - \sum_{k=2}^{\infty} (|A_k| + |B_k|)\, r^k \\
&\geqslant (1-|B_1|)\, r - \sum_{k=2}^{\infty} (|A_k| + |B_k|)\, r^2 \\
&= (1-|B_1|)\, r - \frac{(1-b)}{\Phi(2,m,\mu,\nu,\lambda,b)} \sum_{k=2}^{\infty} \frac{\Phi(2,m,\mu,\nu,\lambda,b)}{1-b} (|A_k| + |B_k|)\, r^2 \\
&\geqslant (1-|B_1|)\, r - \frac{(1-b)}{\Phi(2,m,\mu,\nu,\lambda,b)} \sum_{k=2}^{\infty} \left[\frac{\Phi(k,m,\mu,\nu,\lambda,b)}{1-b} |A_k| \right. \\
&\qquad \left. + \frac{\Psi(k,m,\mu,\nu,\lambda,b)}{1-b} |B_k| \right] r^2 \\
&\geqslant (1-B_1) r - \frac{1-b}{\Phi(2,m,\mu,\nu,\lambda,b)} \times \left(1 - \frac{\Psi(1,m,\mu,\nu,\lambda,b)}{1-b} |B_1| \right) | r^2 \\
&\geqslant (1-B_1) r - \frac{1}{\Phi(2,m,\mu,\nu,\lambda,b)} \left((1-b) - \Psi(1,m,\mu,\nu,\lambda,b)|B_1|\right) r^2.
\end{aligned}$$

By similar arguments, we can easily prove the right-hand inequality in (2.4). □

3. Inclusion Properties

In this section, we show that the class $\overline{S_H^m}(\mu,\nu,\lambda,b)$ is closed under convolution. For harmonic functions

$$f_m(z) = z - \sum_{k=2}^{\infty} |a_k| z^k + (-1)^m \sum_{k=1}^{\infty} |b_k| \overline{z}^k,$$

$$F_m(z) = z - \sum_{k=2}^{\infty} |A_k| z^k + (-1)^m \sum_{k=1}^{\infty} |B_k| \overline{z}^k,$$

their convolution $f_m * F_m$ is given by

$$(f_m * F_m)(z) = f_m(z) * F_m(z) = z - \sum_{k=2}^{\infty} |a_k A_k| z^k + (-1)^m \sum_{k=1}^{\infty} |b_k B_k| \overline{z}^k.$$

THEOREM 3.1. *For* $0 \leqslant b' \leqslant b'' < 1$, *let* $f_m \in \overline{S_H^m}(\mu, \nu, \lambda, b')$ *and* $F_m \in \overline{S_H^m}(\mu, \nu, \lambda, b'')$; *then* $f_m * F_m \in \overline{S_H^m}(\mu, \nu, \lambda, b'') \subset \overline{S_H^m}(\mu, \nu, \lambda, b')$.

PROOF. We need to show that the coefficients of $f_m * F_m$ satisfy condition (2.1). For $F_m \in \overline{S_H^m}(\mu, \nu, \lambda, b'')$, note that $|A_k| < 1$ and $|B_k| < 1$. Now for the convolution function $f_m * F_m$, we have

$$\begin{aligned}
&\sum_{k=2}^{\infty} \frac{\Phi(k, m, \mu, \nu, \lambda, b'')}{(1 - b'')} |a_k A_k| + \sum_{k=1}^{\infty} \frac{\Psi(k, m, \mu, \nu, \lambda, b'')}{(1 - b'')} |b_k B_k| \\
&\quad = \sum_{k=2}^{\infty} \frac{\Phi(k, m, \mu, \nu, \lambda, b'')}{(1 - b'')} |a_k| |A_k| + \sum_{k=1}^{\infty} \frac{\Psi(k, m, \mu, \nu, \lambda, b'')}{(1 - b'')} |b_k| |B_k| \\
&\qquad \leqslant \sum_{k=2} \frac{\Phi(k, m, \mu, \nu, \lambda, b')}{(1 - b')} |a_k| + \sum_{k=1}^{\infty} \frac{\Psi(k, m, \mu, \nu, \lambda, b')}{(1 - b')} |b_k| \leqslant 1.
\end{aligned}$$

Therefore, $f_m * F_m \in \overline{S_H^m}(\mu, \nu, \lambda, b'') \subset \overline{S_H^m}(\mu, \nu, \lambda, b')$. □

Here, we prove that the class $\overline{S_H^m}(\mu, \nu, \lambda, b)$ is closed under the convex combinations of its members. Let the function $f_{m,i}(z)$ be defined for $i = 1, 2, \dots$ by

$$f_{m,i}(z) = z - \sum_{k=2}^{\infty} |a_{k,i}| z^k + (-1)^m \sum_{k=1}^{\infty} |b_{k,i}| \overline{z}^k. \tag{3.1}$$

THEOREM 3.2. *Let the functions* $f_{m,i}(z)$, *defined by* (3.1) *be in* $\overline{S_H^m}(\mu, \nu, \lambda, b)$, *for every* $i = 1, 2, 3 \dots$ *Then the functions* $c(z)$ *defined by*

$$c(z) = \sum_{i=1}^{\infty} t_i f_{m,i}(z), \quad 0 \leqslant t_i \leqslant 1$$

are also in the class $\overline{S_H^m}(\mu, \nu, \lambda, b)$, *where* $\sum_{k=1}^{\infty} t_i = 1$.

PROOF. By the definition of $c(z)$, we have

$$c(z) = z - \sum_{k=2}^{\infty} \left(\sum_{i=1}^{\infty} t_i |a_{k,i}| \right) z^k + (-1)^m \sum_{k=1}^{\infty} \left(\sum_{i=1}^{\infty} t_i |b_{k,i}| \right) \overline{z}^k.$$

Further, since $f_{m,i}(z)$ are in $\overline{S_H^m}(\mu, \nu, \lambda, b)$ for every $i = 1, 2, 3 \dots$, by Theorem 2.2, we have

$$\sum_{k=2}^{\infty} \Phi(k, m, \mu, \nu, \lambda, b) \left(\sum_{i=1}^{\infty} t_i |a_{k,i}| \right) + \sum_{k=1}^{\infty} \Psi(k, m, \mu, \nu, \lambda, b) \left(\sum_{i=1}^{\infty} t_i |b_{k,i}| \right)$$

$$= \sum_{i=1}^{\infty} \left(\sum_{k=2}^{\infty} \Phi(k,m,\mu,\nu,\lambda,b)|a_{k,i}| + \sum_{k=1}^{\infty} \Psi(k,m,\mu,\nu,\lambda,b)|b_{k,i}| \right) t_i$$

$$\leqslant (1-b) \sum_{i=1}^{\infty} t_i = 1-b,$$

thus the required result follows in the view of condition (2.1). □

Finally, we prove that the class $\overline{S_H^m}(\mu,\nu,\lambda,b)$ is closed under the generalized Bernardi–Libera–Livingston integral operator $\mathfrak{L}_c$. For an analytic function f, its generalized Bernardi–Libera–Livingston integral operator is defined by

$$\mathfrak{L}_c[f(z)] = \frac{c+1}{z^c} \int_0^z t^{c-1} f(t)\, dt, \quad (c > -1),$$

whereas for harmonic functions $f = h + \overline{g}$ it is defined by

$$\mathfrak{L}_c[f(z)] = \frac{c+1}{z^c} \int_0^z t^{c-1} h(t)\, dt + \overline{\frac{c+1}{z^c} \int_0^z t^{c-1} g(t)\, dt}, \quad (c > -1). \tag{3.2}$$

THEOREM 3.3. *If* $f_m \in \overline{S_H^m}(\mu,\nu,\lambda,b)$, *then* $\mathfrak{L}_c[f_m(z)] \in \overline{S_H^m}(\mu,\nu,\lambda,b)$.

PROOF. By definition of $\mathfrak{L}_c[f_m(z)]$ given in (3.2), it follows that

$$\begin{aligned}\mathfrak{L}_c[f_m(z)] &= \frac{c+1}{z^c} \int_0^z t^{c-1} \left(t - \sum_{k=2}^{\infty} |a_k| t^k + (-1)^m \sum_{k=1}^{\infty} |b_k| \overline{t}^k \right) dt \\ &= z - \sum_{k=2}^{\infty} \left(\frac{c+1}{c+k} \right) |a_k| z^k + (-1)^m \sum_{k=1}^{\infty} \left(\frac{c+1}{c+k} \right) |b_k| \overline{z}^k \\ &= z - \sum_{k=2}^{\infty} A_k z^k + (-1)^m \sum_{k=1}^{\infty} B_k \overline{z}^k.\end{aligned}$$

Here $A_k = \left(\frac{c+1}{c+k}\right)|a_k|$, $B_k = \left(\frac{c+1}{c+k}\right)|b_k|$. Hence,

$$\begin{aligned}&\sum_{k=2}^{\infty} \Phi(k,m,\mu,\nu,\lambda,b) \left(\frac{c+1}{c+k} \right) |a_k| + \sum_{k=1}^{\infty} \Psi(k,m,\mu,\nu,\lambda,b) \left(\frac{c+1}{c+k} \right) |b_k| \\ &\qquad \leqslant \sum_{k=2}^{\infty} \Phi(k,m,\mu,\nu,\lambda,b)|a_k| + \sum_{k=1}^{\infty} \Psi(k,m,\mu,\nu,\lambda,b)|b_k| \leqslant 1-b.\end{aligned}$$

Therefore, by Theorem 2.2, we have $L_c[f_m(z)] \in \overline{S_H^m}(\mu,\nu,\lambda,b)$. □

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