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# SOME SOMOS'S THETA FUNCTION IDENTITIES OF LEVEL 6 AND APPLICATION TO PARTITIONS

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*Dedicated to Prof. Hari M. Srivastava on his* $80^{th}$ *birthday*

Abstract. M. Somos discovered around 6200 theta function identities using PARI/GP scripts without offering the proof. He runs PARI/GP scripts and it works as a sophisticated programmable calculator. These identities highly resemble those of Ramanujan's identities. Here we prove a few theta-function identities of level 6 discovered by Somos by using modular equations of degree 3 given by Ramanujan and further we extract some interesting combinatorial interpretations of colored partitions.

## 1. Introduction

All through the paper, we assume $|q| < 1$ and employ the standard notation

$$(x;q)_\infty := \prod_{n=0}^{\infty}(1 - xq^n).$$

Ramanujan's theta function $\mathfrak{f}(x,y)$ is defined as

$$\mathfrak{f}(x,y) := \sum_{n=-\infty}^{\infty} x^{n(n+1)/2}y^{n(n-1)/2}, \quad |xy| < 1.$$

The function $\mathfrak{f}(x,y)$ enjoys the well-known triple-product identity of Jacobi [**5**, p. 35] given by

$$\mathfrak{f}(x,y) = (-x;xy)_\infty(-y;xy)_\infty(xy;xy)_\infty.$$

The important special cases of $\mathfrak{f}(x,y)$ [**5**, p. 36] are as follows:

$$\psi(q) := \mathfrak{f}(q,q^3) = \sum_{n=0}^{\infty} q^{\frac{n(n+1)}{2}} = \frac{(q^2;q^2)_\infty}{(q;q^2)_\infty},$$

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$$\varphi(q) := \mathfrak{f}(q, q) = \sum_{n=-\infty}^{\infty} q^{n^2} = (-q; q^2)_\infty^2 (q^2; q^2)_\infty,$$

$$f(-q) := \mathfrak{f}(-q, -q^2) = \sum_{n=-\infty}^{\infty} (-1)^n q^{n(3n-1)/2} = (q; q)_\infty.$$

Observe that, if $q = e^{2\pi i\tau}$ then $f(-q) = e^{-\pi i\tau/12}\eta(\tau)$, where $\eta(\tau)$ is the Dedekind eta-function defined as

$$\eta(\tau) := q^{1/24} \prod_{n=1}^{\infty} (1 - q^n), \quad \mathrm{Im}(\tau) > 0,$$

where $\tau$ is a complex number. After Ramanujan, we define

$$\chi(q) := (-q; q^2)_\infty.$$

For our convenience from now onwards, we write $f(-q^n) = f_n$. A theta function identity which relates $f_1, f_2, f_n$ and $f_{2n}$ is called the theta function identity of level $2n$. Ramanujan documented many modular equations which involve quotients of the function $f_1$ at different arguments. For example [**6**, p. 206], [**10**], if

$$P := \frac{f_1}{q^{1/6} f_5} \quad \text{and} \quad Q := \frac{f_2}{q^{1/3} f_{10}}$$

then

$$PQ + \frac{5}{PQ} = \left(\frac{Q}{P}\right)^3 + \left(\frac{P}{Q}\right)^3.$$

Berndt [**5**,**6**] proved similar types of the identities recorded by Ramanujan and used it to evaluate continued fractions, Weber class invariants and many more. After the publication of [**5**, **6**], many mathematicians discovered similar identities. For a wonderful work one can see [**1**–**4**, **8**, **12**–**14**, **19**, **20**]. Motivated by the above work, Michael Somos [**11**] presented thousands of theta function identities of various levels with the aid of computer and he has not offered any analytical proof. He used PARI/GP script to generate several Dedekind eta-function identities by using computer. He runs PARI/GP scripts and it works as a sophisticated programmable calculator. These identities highly resemble those of Ramanujan's identities. Further, Yattunan [**22**] has proved some of these identities of level 4, 6 and 8. Vasuki and Veeresha [**21**] proved all the twenty four identities of level fourteen and Srivatsa Kumar and Veeresha [**15**] obtained partition identities for the same. Srivatsa Kumar and Anu Radha [**16**] have given a proof of few Somos's identities of levels 10. Srivatsa Kumar et al. [**17**, **18**] have proved theta function identities of level 6 and 8 which are analogous to Ramanujan's theta function identities.

Our aim is to prove some of these new theta-function identities of level 6 conjectured by Somos by using modular equation of degree 3 given by Ramanujan and to establish certain partition identities for them. Now we list four Somos's identities of level 6:

$$f_1^8 f_3 f_6^4 + 8q f_1^3 f_2 f_6^9 - f_2^4 f_3^9 = 0, \tag{1.1}$$

$$f_1 f_2^3 f_3^9 + q f_1^4 f_6^9 - f_2^8 f_3^4 f_6 = 0, \tag{1.2}$$

$$f_1^9 f_6^4 + 8f_2^9 f_3^3 f_6 - 9f_1 f_2^4 f_3^8 = 0, \tag{1.3}$$

$$f_1^9 f_3 f_6^3 + 9q f_1^4 f_2 f_6^8 - f_2^9 f_3^4 = 0. \tag{1.4}$$

The above listed identities of level 6 contain the arguments in $f(-q)$, $f(-q^2)$, $f(-q^3)$ and $f(-q^6)$ namely $-q$, $-q^2$, $-q^3$ and $-q^6$ all have exponents dividing 6, which is thus equal to the 'level' of the identity 6. Before concluding this Section we define a modular equation as defined by Ramanujan. The natural generalization of the Gauss hypergeometric function ${}_2F_1$ is called the generalized hypergeometric series ${}_pF_q$ defined by

$$\begin{aligned} {}_rF_s\left[\begin{matrix}\alpha_1, \alpha_2, \dots, \alpha_r \\ \beta_1, \beta_2, \dots, \beta_s\end{matrix}; \ z\right] &= \sum_{k=0}^{\infty} \frac{(\alpha_1)_k (\alpha_2)_k \cdots (\alpha_r)_k}{(\beta_1)_k (\beta_2)_k \cdots (\beta_s)_k k!} z^k, \quad |z| < 1, \\ &= {}_rF_s(\alpha_1, \alpha_2, \dots, \alpha_r; \beta_1, \beta_2, \dots, \beta_s; z), \end{aligned}$$

where $(\lambda_n)$ is the Pochhammer symbol defined as

$$(\lambda)_0 := 1 \quad \text{and} \quad (\lambda)_n := \lambda(\lambda+1)\cdots(\lambda+n-1).$$

For a wonderful work on hypergeometric functions one may refer [**7**]. A modular equation [**5**] of degree $n$ is an equation relating $\alpha$ and $\beta$ that is induced by

$$n\frac{{}_2F_1\left(\frac{1}{2}, \frac{1}{2}; 1; 1-\alpha\right)}{{}_2F_1\left(\frac{1}{2}, \frac{1}{2}; 1; \alpha\right)} = \frac{{}_2F_1\left(\frac{1}{2}, \frac{1}{2}; 1; 1-\beta\right)}{{}_2F_1\left(\frac{1}{2}, \frac{1}{2}; 1; \beta\right)},$$

where

$${}_2F_1(a, b; c; z) := \sum_{n=0}^{\infty} \frac{(a)_n (b)_n}{(c)_n n!} z^n, \quad |z| < 1,$$

Then, we say that $\beta$ is of degree $n$ over $\alpha$ and call the ratio $m := \frac{z_1}{z_n}$ the multiplier, where

$$z_1 = {}_2F_1\Big(\frac{1}{2}, \frac{1}{2}; 1; \alpha\Big) \quad \text{and} \quad z_n = {}_2F_1\Big(\frac{1}{2}, \frac{1}{2}; 1; \beta\Big).$$

## 2. Proof of (1.1)–(1.4)

Proof of (1.1). Ramanujan [**9**, p. 238] and [**5**, pp. 230–238, Entry 13(ix) and (xiv)] documented the following modular equation of degree 3. If $\beta$ has degree 3 over $\alpha$, we have

$$P := \{16\alpha\beta(1-\alpha)(1-\beta)\}^{1/8} \quad \text{and} \quad Q := \left\{\frac{\beta(1-\beta)}{\alpha(1-\alpha)}\right\}^{1/4},$$

then

$$Q + \frac{1}{Q} + 2\sqrt{2}\Big(P - \frac{1}{P}\Big) = 0. \tag{2.1}$$

From [**5**, pp. 122–124, Entry 10(i) and Entry 12(v)], for $q = e^{-y}$, we have

$$\varphi(q) = \sqrt{z}, \tag{2.2}$$

$$\chi(q) = 2^{1/6}\left\{\frac{x(1-x)}{q}\right\}^{-1/24} \tag{2.3}$$

for $x \neq 1$. On transforming (2.1) using (2.3), we obtain

$$\left(\frac{u}{v}\right)^6 + \left(\frac{v}{u}\right)^6 = (uv)^3 - \frac{8}{(uv)^3}, \tag{2.4}$$

where $u := u(q) = q^{-1/24}\chi(q)$ and $v := v(q) = q^{-1/8}\chi(q^3)$. On multiplying (2.4) throughout by $4(uv)^{-9}$, we obtain

$$\frac{4}{u^3v^{15}} + \frac{4}{u^{15}v^3} - \frac{4}{u^6v^6} + \frac{32}{u^{12}v^{12}} = 0,$$

which is equivalent to

$$3\Big(1 - 4\frac{u^3}{v^9}\Big) - \frac{u^8}{v^8}\Big(\frac{8}{u^5v} + \frac{v^8}{u^8}\Big)\Big(4\frac{v^3}{u^9} - 1\Big) = 0. \tag{2.5}$$

Also from [**5**, pp. 230–238, Entry 13(ix) and (xiv)], if $\beta$ has degree 3 over $\alpha$, we have

$$m = \frac{1 - 2\left(\frac{\beta^3(1-\beta)^3}{\alpha(1-\alpha)}\right)^{1/8}}{1 - 2(\alpha\beta)^{1/4}} \quad \text{and} \quad \frac{3}{m} = \frac{2\left(\frac{\alpha^3(1-\alpha)^3}{\beta(1-\beta)}\right)^{1/8} - 1}{1 - 2(\alpha\beta)^{1/4}},$$

which gives

$$\frac{m^2}{3} = \frac{1 - 2\left(\frac{\beta^3(1-\beta)^3}{\alpha(1-\alpha)}\right)^{1/8}}{2\left(\frac{\alpha^3(1-\alpha)^3}{\beta(1-\beta)}\right)^{1/8} - 1}. \tag{2.6}$$

On transforming (2.6) into theta function from (2.2) and (2.3), we have

$$\frac{\varphi^4(q)}{3\varphi^4(q^3)} = \frac{1 - 4\frac{u^3}{v^9}}{4\frac{v^3}{u^9} - 1}. \tag{2.7}$$

Employing (2.7) in (2.5), we find that

$$1 - \frac{u^8}{v^8}\Big(\frac{8}{u^5v} + \frac{v^8}{u^8}\Big)\frac{\varphi^4(q^3)}{\varphi^4(q)} = 0. \tag{2.8}$$

By using $q$-identities, one can easily see that

$$\varphi(q) = \frac{f_2^5}{f_1^2f_4^2}, \quad \chi(q) = \frac{f_2^2}{f_1f_4} \quad \text{and} \quad \chi(-q) = \frac{f_1}{f_2}. \tag{2.9}$$

From (2.9), we observe that

$$\frac{\varphi(q)}{\varphi(q^3)} = q^{1/6}\frac{u^2}{v^2}\frac{f_2}{f_6}. \tag{2.10}$$

Using (2.10) in (2.8), we obtain

$$1 - q^{2/3}\Big(\frac{8}{u^5v} + \frac{v^8}{u^8}\Big)\Big(\frac{f_6}{f_2}\Big)^4 = 0.$$

Now, letting $q \to -q$ in the above, rewriting $u(-q)$ and $v(-q)$ in terms of $f_n$ by using (2.9) and then multiplying throughout by $f_1^8f_3f_6^4$, we deduce the result. □

PROOF OF (1.2). On multiplying (2.4) throughout by $(v/u)^6$, we obtain

$$8\frac{v^3}{u^9}+1+\frac{v^{12}}{u^{12}}-\frac{v^9}{u^3}=0,$$

which is equivalent to

$$\Big(1-\frac{v^9}{u^3}\Big)\Big(4\frac{v^3}{u^9}-1\Big)+3\frac{v^{12}}{u^{12}}\Big(1-4\frac{u^3}{v^9}\Big)=0.$$

Using (2.7) in the above, we have

$$\Big(1-\frac{v^9}{u^3}\Big)+\frac{v^{12}}{u^{12}}\frac{\varphi^4(q)}{\varphi^4(q^3)}=0.$$

Employing (2.10) in the above, we obtain

$$1-\frac{v^9}{u^3}+\frac{1}{q^{2/3}}\frac{v^4}{u^4}\Big(\frac{f_2}{f_6}\Big)^4=0.$$

On letting $q\to -q$ in the above, rewriting $u(-q)$ and $v(-q)$ in terms of $f_n$ by using (2.9) and then multiplying throughout by $qf_1^4f_6^9$, we complete the proof. □

PROOF OF (1.3). On multiplying (2.4) by $4(uv)^{-9}$ and then grouping the terms, we have

$$\Big(1-4\frac{u^3}{v^9}\Big)\Big(1+8\frac{v^3}{u^9}\Big)-3\Big(4\frac{v^3}{u^9}-1\Big)=0.$$

Using (2.7) in the above, we have

$$1+8\frac{v^3}{u^9}-9\frac{\varphi^4(q^3)}{\varphi^4(q)}=0.$$

Using (2.10) in the above, we observe that

$$1+8\frac{v^3}{u^9}-9q^{2/3}\frac{v^8}{u^8}\Big(\frac{f_6}{f_2}\Big)^4=0.$$

Now on letting $q\to -q$ in the above, rewriting $u(-q)$ and $v(-q)$ in terms of $f_n$ by using (2.9) and then multiplying throughout by $f_1^9f_6^4$, we complete the proof. □

PROOF OF (1.4). On multiplying (2.4) by $(u/v)^6$ and then grouping the terms, we obtain

$$\Big(1-4\frac{u^3}{v^9}\Big)\Big(\frac{u^9}{v^3}-1\Big)-3\frac{u^{12}}{v^{12}}\Big(4\frac{v^3}{u^9}-1\Big)=0.$$

Using (2.7) in the above, we have

$$\frac{u^9}{v^3}-1-9\frac{u^{12}}{v^{12}}\frac{\varphi^4(q^3)}{\varphi^4(q)}=0.$$

Employing (2.10) in the above, we have

$$\frac{u^9}{v^3}-1-9q^{2/3}\frac{u^4}{v^4}\Big(\frac{f_6}{f_2}\Big)^4=0.$$

On letting $q\to -q$ in the above, rewriting $u(-q)$ and $v(-q)$ in terms of $f_n$ by using (2.9) and then multiplying throughout by $f_2^9f_3^4$, we complete the proof. □

## 3. Application to colored partitions

Somos's identities have wide application in colored partitions. In this Section we demonstrate this by giving combinatorial interpretations for (1.1). Similarly, we can establish the partition identities for (1.2)–(1.4). For convenience, we use the notation

$$(x_1, x_2, \ldots, x_n; q)_\infty = \prod_{i=1}^{n} (x_i; q)_\infty,$$

and define

$$(q^{\pm a}; q^b)_\infty := (q^a, q^{b-a}; q^b)_\infty,$$

where $a < b$, and $a, b \in \mathbb{Z}^+$. As an example of this, $(q^{\pm 3}; q^8)_\infty$ implies $(q^3, q^5; q^8)_\infty = (q^3; q^8)_\infty (q^5; q^8)_\infty$. Now we define colored partition as defined in the literature. "A positive integer $n$ has $l$ colors if there are $l$ copies of $n$ available colors and all of them are viewed as distinct objects. Partitions of a positive integer into parts with colors are called colored partitions".

Example 3.1. If 3 colors are assigned to 1, then all possible colored partitions of 2 are 2, $1_r + 1_r$, $1_g + 1_g$, $1_y + 1_y$, $1_r + 1_g$, $1_r + 1_y$, and $1_g + 1_y$, here we used the notations $r$ for 'red', $g$ for 'green' and $y$ for 'yellow' to distinguish three colors of 1.

Also, the total partitions of $n$ is given by the generating function

$$\frac{1}{(q^\alpha; q^\beta)_\infty^\gamma} = \sum_{n=0}^{\infty} p(n) q^n,$$

where each and every parts are congruent to $\alpha$ *modulo* $\beta$ with $\gamma$ number of colors with $p(0) = 1$.

Theorem 3.1. *Let $A(n)$ be the total number of partitions of $n$ being divided into parts congruent to $\pm 2$ or $+3$ modulo 6 with 4 and 8 colors respectively. Let $B(n)$ indicate the number of partitions of $n$ being split into parts congruent to $\pm 1, \pm 2$ or $+3$ modulo 6 with 5, 8 and 6 colors respectively. Let $C(n)$ be taken to represent the number of partitions of $n$ into several parts congruent to $\pm 1$, $\pm 2$ or $+3$ modulo 6 with 8 colors each. Then the following identity holds true:*

$$A(n) + 8B(n-1) - C(n) = 0, \quad n \geqslant 1.$$

Proof. On dividing (1.1) by $f_1^8 f_2^4 f_3^9 f_6^9$ and then rewriting each term to the common base $q^6$, we have

$$\frac{1}{(q_4^2, q_8^3, q_4^4; q^6)_\infty} + \frac{8q}{(q_5^1, q_8^2, q_6^3, q_8^4, q_5^5; q^6)_\infty} - \frac{1}{(q_8^1, q_8^2, q_8^3, q_8^4, q_8^5; q^6)_\infty} = 0.$$

Since $(q^{a\pm}; q^b)_\infty = (q^a, q^{b-a}; q^b)_\infty$, the above equation reduces to

$$\frac{1}{(q_4^{2\pm}, q_8^{3+}; q^6)_\infty} + \frac{8q}{(q_5^{1\pm}, q_8^{2\pm}, q_6^{3+}; q^6)_\infty} - \frac{1}{(q_8^{1\pm}, q_8^{2\pm}, q_8^{3+}; q^6)_\infty} = 0.$$

We observe that the above identity generates $A(n)$, $B(n)$ and $C(n)$ as generating functions and hence we have

$$\sum_{n=0}^{\infty} A(n)q^n + 8q\sum_{n=0}^{\infty} B(n)q^n - \sum_{n=0}^{\infty} C(n)q^n = 0$$

where we set the values $A(0) = B(0) = C(0) = 1$. Now on extracting the terms of $q^n$ in the above, we obtain the result. □

The following table verifies the partitions for $n = 2$.

Table 1.

| | |
|---|---|
| $A(2) = 4$ | $2_r, 2_y, 2_b, 2_g$ |
| $B(1) = 5$ | $1_r, 1_y, 1_b, 1_g, 1_m$ |
| $C(2) = 44$ | $1_r + 1_r, 1_g + 1_g, 1_y + 1_y, 1_o + 1_o, 1_p + 1_p, 1_b + 1_b, 1_m + 1_m, 1_v + 1_v,$ $1_r + 1_g, 1_r + 1_y, 1_r + 1_o, 1_r + 1_p, 1_r + 1_b, 1_r + 1_m, 1_r + 1_v, 1_g + 1_y,$ $1_g + 1_o, 1_g + 1_p, 1_g + 1_b, 1_g + 1_m, 1_g + 1_v, 1_y + 1_o, 1_y + 1_p, 1_y + 1_b,$ $1_y + 1_m, 1_y + 1_r, 1_o + 1_p, 1_o + 1_b, 1_o + 1_m, 1_o + 1_v, 1_p + 1_b, 1_p + 1_m,$ $1_p + 1_v, 1_b + 1_m, 1_b + 1_v, 1_m + 1_r, 2_r, 2_g, 2_y, 2_o, 2_p, 2_b, 2_m, 2_r$ |

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