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CLOSED FORMULAS FOR SPECIAL BELL POLYNOMIALS BY STIRLING NUMBERS AND ASSOCIATE STIRLING NUMBERS

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Dedicated to Professor Hari Mohan Srivastava on the occasion of his 80th birthday

Abstract. We derive two explicit formulas for two sequences of special values of the Bell polynomials of the second kind in terms of associate Stirling numbers of the second kind, give an explicit formula for associate Stirling numbers of the second kind in terms of the Stirling numbers of the second kind, and, consequently, present two explicit formulas for two sequences of special values of the Bell polynomials of the second kind in terms of the Stirling numbers of the second kind.

1. Preliminaries

The Bell polynomials of the second kind, denoted by $\mathrm{B}_{n,k}(x_1, x_2, \dots, x_{n-k+1})$ for $n \ge k \ge 0$ and variables $x_i \in \mathbb{C}$ with $i \ge 1$, can be defined by

$$\mathrm{B}_{n,k}(x_1, x_2, \dots, x_{n-k+1}) = \sum_{\substack{1 \le i \le n-k+1 \\ \ell_i \in \{0\} \cup \mathbb{N} \\ \sum_{i=1}^{n-k+1} i\ell_i = n \\ \sum_{i=1}^{n-k+1} \ell_i = k}} \frac{n!}{\prod_{i=1}^{n-k+1} \ell_i!} \prod_{i=1}^{n-k+1} \left(\frac{x_i}{i!}\right)^{\ell_i},$$

can be generated by

$$\frac{1}{k!}\left(\sum_{i=1}^{\infty} x_i \frac{t^i}{i!}\right)^k = \sum_{n=k}^{\infty} \mathrm{B}_{n,k}(x_1, x_2, \dots, x_{n-k+1}) \frac{t^n}{n!}, \tag{1.1}$$

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and can be applied in the famous Faà di Bruno formula which states that the n-th derivative of the composite function $f(h(t))$ can be computed by

$$\frac{d^n}{dt^n} f\circ h(t)=\sum_{k=0}^{n} f^{(k)}(h(t))\,\mathrm{B}_{n,k}\big(h'(t),h''(t),\dots,h^{(n-k+1)}(t)\big),$$

where f and h are both n-time differentiable functions. See [**1**, Definition 11.2 and Theorem 11.4], [**2**, p. 134, Theorem A; p. 139, Theorem C], the papers [**7**–**10**] and closely related references therein.

In analytic combinatorics [**1**, **2**], the Stirling numbers of the second kind, denoted by $S(n,k)$ for $n\ge k\ge 0$, can be computed by

$$S(n,k)=\frac{1}{k!}\sum_{\ell=0}^{k}(-1)^{k-\ell}\binom{k}{\ell}\ell^n$$

and can be generated by

$$\frac{(e^x-1)^k}{k!}=\sum_{n=k}^{\infty}S(n,k)\frac{x^n}{n!}. \tag{1.2}$$

In [**5**, p. 303, eq. (1.2)], the r-associate Stirling numbers of the second kind, denoted by $S(n,k;r)$, were defined by

$$\left(e^x-\sum_{i=0}^{r}\frac{x^i}{i!}\right)^k=\left(\sum_{i=r+1}^{\infty}\frac{x^i}{i!}\right)^k=k!\sum_{n=(r+1)k}^{\infty}S(n,k;r)\frac{x^n}{n!}. \tag{1.3}$$

2. Motivations

In [**2**, p. 135], it is given that $\mathrm{B}_{n,k}(1,1,\dots,1)=S(n,k)$. See also [**11**, Section 1.1]. In [**3**, Theorem 1], the formula

$$\mathrm{B}_{n,k}(0,1,\dots,1)=\sum_{\ell=0}^{k}(-1)^{\ell}\binom{n}{\ell}S(n-\ell,k-\ell)$$

was established. See also [**11**, Section 1.8]. In general, what is the general and closed formula of

$$\mathrm{B}_{n,k}(\underbrace{0,\dots,0}_{s-1},1,1,\dots,1)$$

for $n\ge k+s-1\ge s-1\ge 2$ in terms of the Stirling numbers of the second kind $S(n,k)$?

In [**11**, Remark 4.4], it was deduced that

$$\mathrm{B}_{n,k}(1,2,3,4,\dots,n-k+1)=\frac{n!}{(n-k)!}\sum_{\ell=0}^{k}\frac{S(n-k,\ell)}{(k-\ell)!}.$$

In [**4**, Section 2, eq. (2.2)], the closed formula

$$\mathrm{B}_{n,k}(0,2,3,\dots,n-k+1)=\frac{n!}{(n-k)!}S(n-k,k)$$

was listed. In [**4**, p. 978, eq. (2.3)], the formula

$$\mathrm{B}_{n,k}(0,0,3,4,\dots,n-k+1)=\frac{n!}{(n-k)!}S(n-k,k;1) \tag{2.1}$$

was claimed. In [**6**, Theorem 2.1], the formula 2.1 was alternatively expressed as

$$\mathrm{B}_{n,k}(0,0,3,4,\dots,n-k+1)=n!\sum_{\ell=0}^{k}\frac{(-1)^\ell}{\ell!}\frac{S(n-k-\ell,k-\ell)}{(n-k-\ell)!}$$

for $n\ge k+2\ge 2$. As did in [**6**, Remark 2.1], we ask again a question: what is the general and closed formula of

$$\mathrm{B}_{n,k}(\underbrace{0,\dots,0}_{s-1},s,s+1,\dots,n-k+1)$$

for $n\ge k+s-1\ge s-1\ge 3$ in terms of the Stirling numbers of the second kind $S(n,k)$?

From (1.1) and (1.3), it follows that

$$\begin{aligned}\sum_{n=k}^{\infty}\mathrm{B}_{n,k}(\underbrace{0,\dots,0}_{s-1},1,1,\dots,1)\frac{t^n}{n!}&=\frac{1}{k!}\left(\sum_{i=s}^{\infty}\frac{t^i}{i!}\right)^k\\&=\frac{1}{k!}\left(e^t-\sum_{i=0}^{s-1}\frac{t^i}{i!}\right)^k=\sum_{n=sk}^{\infty}S(n,k;s-1)\frac{x^n}{n!}.\end{aligned}$$

This means that

$$\mathrm{B}_{n,k}(\underbrace{0,\dots,0}_{s-1},1,1,\dots,1)=S(n,k;s-1). \tag{2.2}$$

From (1.1) and (1.3), it follows that

$$\begin{aligned}\sum_{n=k}^{\infty}\mathrm{B}_{n,k}(\underbrace{0,\dots,0}_{s-1},s,s+1,\dots,n-k+1)\frac{t^n}{n!}&=\frac{1}{k!}\left[\sum_{i=s}^{\infty}\frac{t^i}{(i-1)!}\right]^k\\&=\frac{t^k}{k!}\left(e^t-\sum_{i=0}^{s-2}\frac{t^i}{i!}\right)^k=t^k\sum_{n=(s-1)k}^{\infty}S(n,k;s-2)\frac{t^n}{n!}\\=\sum_{n=(s-1)k}^{\infty}S(n,k;s-2)\frac{t^{n+k}}{n!}&=\sum_{n=sk}^{\infty}\frac{n!}{(n-k)!}S(n-k,k;s-2)\frac{t^n}{n!}.\end{aligned}$$

This means that

$$\mathrm{B}_{n,k}(\underbrace{0,\dots,0}_{s-1},s,s+1,\dots,n-k+1)=\frac{n!}{(n-k)!}S(n-k,k;s-2). \tag{2.3}$$

The equations (2.2) and (2.3) tell us that, in order to give solutions to the above two problems, it is sufficient to represent the associate Stirling numbers of the second kind $S(n,k;r)$ in terms of the Stirling numbers of the second kind $S(n,k)$.

3. An explicit formula of associate Stirling numbers

In this section, we establish an explicit formula for the r-associate Stirling numbers of the second kind $S(n,k;r)$ in terms of the Stirling numbers of the second kind $S(n,k)$.

THEOREM 3.1. *For integers $n \ge k \ge 0$ and $r \ge 0$, the r-associate Stirling numbers of the second kind $S(n,k;r)$ can be computed by*

$$
\begin{aligned}
(3.1)\quad & S(n,k;r) \\
&= n! \sum_{\substack{m-q+\sum_{i=1}^{r} i j_i = n-k \\ j_1+j_2+\dots+j_r=q \\ 0\le j_i\le q, 1\le i\le r}} \sum_{\substack{j_1+j_2+\dots+j_r=q \\ 0\le j_i\le q, 1\le i\le r}} \frac{(-1)^q}{\prod_{i=1}^{r}[(i!)^{j_i} j_i!]} \frac{S(m+k-q,k-q)}{(m+k-q)!}.
\end{aligned}
$$

PROOF. From (1.2) and (1.3), it follows that

$$
\begin{aligned}
\sum_{n=(r+1)k}^{\infty} S(n,k;r)\frac{x^n}{n!} &= \frac{1}{k!}\left[(e^x-1)-\sum_{i=1}^{r}\frac{x^i}{i!}\right]^k \\
&= \frac{1}{k!}\sum_{\ell=0}^{k}\binom{k}{\ell}(e^x-1)^{\ell}(-1)^{k-\ell}\left(\sum_{i=1}^{r}\frac{x^i}{i!}\right)^{k-\ell} \\
&= \frac{1}{k!}\sum_{\ell=0}^{k}\binom{k}{\ell}\frac{(e^x-1)^{\ell}}{\ell!}\ell!(-1)^{k-\ell}\left(\sum_{i=1}^{r}\frac{x^i}{i!}\right)^{k-\ell} \\
&= \sum_{\ell=0}^{k}\frac{(-1)^{k-\ell}}{(k-\ell)!}\left[\sum_{m=\ell}^{\infty}S(m,\ell)\frac{x^m}{m!}\right]\left(\sum_{i=1}^{r}\frac{x^i}{i!}\right)^{k-\ell} \\
&= \sum_{\ell=0}^{k}\frac{(-1)^{k-\ell}}{(k-\ell)!}\left[\sum_{m=0}^{\infty}S(m+\ell,\ell)\frac{x^{m+\ell}}{(m+\ell)!}\right]\left(\sum_{i=1}^{r}\frac{x^i}{i!}\right)^{k-\ell} \\
&= \sum_{m=0}^{\infty}\sum_{\ell=0}^{k}\frac{(-1)^{k-\ell}}{(k-\ell)!}\frac{S(m+\ell,\ell)}{(m+\ell)!}x^{m+\ell}\left(\sum_{i=1}^{r}\frac{x^i}{i!}\right)^{k-\ell} \\
&= \sum_{m=0}^{\infty}\sum_{\ell=0}^{k}\frac{(-1)^{k-\ell}}{(k-\ell)!}\frac{S(m+\ell,\ell)}{(m+\ell)!}x^{m+\ell}\sum_{\substack{j_1+j_2+\dots+j_r=k-\ell \\ 0\le j_i\le k-\ell, 1\le i\le r}}\binom{k-\ell}{j_1,j_2,\dots,j_r}\prod_{i=1}^{r}\left(\frac{x^i}{i!}\right)^{j_i} \\
&= \sum_{m=0}^{\infty}\sum_{\ell=0}^{k}\frac{(-1)^{k-\ell}}{(k-\ell)!}\frac{S(m+\ell,\ell)}{(m+\ell)!}x^{m+\ell}\sum_{\substack{j_1+j_2+\dots+j_r=k-\ell \\ 0\le j_i\le k-\ell, 1\le i\le r}}\binom{k-\ell}{j_1,j_2,\dots,j_r}\frac{x^{\sum_{i=1}^{r} i j_i}}{\prod_{i=1}^{r}(i!)^{j_i}} \\
&= \sum_{m=0}^{\infty}\sum_{\ell=0}^{k}\frac{(-1)^{k-\ell}}{(k-\ell)!}\frac{S(m+\ell,\ell)}{(m+\ell)!}\sum_{\substack{j_1+j_2+\dots+j_r=k-\ell \\ 0\le j_i\le k-\ell, 1\le i\le r}}\binom{k-\ell}{j_1,j_2,\dots,j_r}\frac{x^{m+\ell+\sum_{i=1}^{r} i j_i}}{\prod_{i=1}^{r}(i!)^{j_i}}
\end{aligned}
$$

$$
\begin{aligned}
&= \sum_{m=0}^{\infty} \sum_{q=0}^{k} \frac{(-1)^q}{q!} \frac{S(m+k-q,k-q)}{(m+k-q)!} \\
&\quad \times \sum_{\substack{j_1+j_2+\dots+j_r=q\\ 0\le j_i\le q, 1\le i\le r}} \binom{q}{j_1,j_2,\dots,j_r} \frac{x^{m+k-q+\sum_{i=1}^{r} ij_i}}{\prod_{i=1}^{r}(i!)^{j_i}} \\
&= \sum_{n=k}^{\infty} \sum_{\substack{m-q+\sum_{i=1}^{r} ij_i=n-k\\ j_1+j_2+\dots+j_r=q\\ 0\le j_i\le q, 1\le i\le r}} \frac{(-1)^q}{q!} \frac{S(m+k-q,k-q)}{(m+k-q)!} \\
&\quad \times \sum_{\substack{j_1+j_2+\dots+j_r=q\\ 0\le j_i\le q, 1\le i\le r}} \binom{q}{j_1,j_2,\dots,j_r} \frac{x^n}{\prod_{i=1}^{r}(i!)^{j_i}}.
\end{aligned}
$$

This means that the formula (3.1) is thus proved. □

THEOREM 3.2. *For integers $n \ge k \ge 0$ and $s \ge 1$, the Bell polynomials of the second kind $\mathrm{B}_{n,k}$ satisfy*

$$
\begin{aligned}
&\mathrm{B}_{n,k}(\underbrace{0,\dots,0}_{s-1},1,1,\dots,1) \\
&\quad = n! \sum_{\substack{m-q+\sum_{i=1}^{s-1} ij_i=n-k\\ j_1+j_2+\dots+j_{s-1}=q\\ 0\le j_i\le q, 1\le i\le s-1}} \sum_{\substack{j_1+j_2+\dots+j_{s-1}=q\\ 0\le j_i\le q, 1\le i\le s-1}} \frac{(-1)^q}{\prod_{i=1}^{s-1}[(i!)^{j_i} j_i!]} \frac{S(m+k-q,k-q)}{(m+k-q)!}
\end{aligned}
$$

$$
\begin{aligned}
&\mathrm{B}_{n,k}(\underbrace{0,\dots,0}_{s-1},s,s+1,\dots,n-k+1) \\
&\quad = n! \sum_{\substack{m-q+\sum_{i=1}^{s-2} ij_i=n-2k\\ j_1+j_2+\dots+j_{s-2}=q\\ 0\le j_i\le q, 1\le i\le s-2}} \sum_{\substack{j_1+j_2+\dots+j_{s-2}=q\\ 0\le j_i\le q, 1\le i\le s-2}} \frac{(-1)^q}{\prod_{i=1}^{s-2}[(i!)^{j_i} j_i!]} \frac{S(m+k-q,k-q)}{(m+k-q)!},
\end{aligned}
$$

where the empty product is understood to be 1*, while the empty sum is understood to be* 0*.*

PROOF. This follows from substituting the formula (3.1) into (2.2) and (2.3). □

4. A remark

Motivated by Section 2 in this paper, one can naturally pose another problem: what is the general and closed formula of

$$
\mathrm{B}_{n,k}(\ell,\ell+1,\dots,\ell+n-k)
$$

for $n \ge k \ge 0$ and $\ell \ge 1$ in terms of the Stirling numbers of the second kind $S(n,k)$ or other closed quantities? In fact, more general solutions to this problem have been established in Theorem 3.1 and Remark 4.3 in [**11**].

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