

**FORMULAS INVOLVING SUMS OF POWERS,
SPECIAL NUMBERS AND POLYNOMIALS
ARISING FROM p -ADIC INTEGRALS,
TRIGONOMETRIC AND GENERATING FUNCTIONS**

Neslihan Kilar and Yilmaz Simsek

ABSTRACT. The formula for the sums of powers of positive integers, given by Faulhaber in 1631, is proven by using trigonometric identities and some properties of the Bernoulli polynomials. Using trigonometric functions identities and generating functions for some well-known special numbers and polynomials, many novel formulas and relations including alternating sums of powers of positive integers, the Bernoulli polynomials and numbers, the Euler polynomials and numbers, the Fubini numbers, the Stirling numbers, the tangent numbers are also given. Moreover, by applying the Riemann integral and p -adic integrals involving the fermionic p -adic integral and the Volkenborn integral, some new identities and combinatorial sums related to the aforementioned numbers and polynomials are derived. Furthermore, we serve up some revealing and historical remarks and observations on the results of this paper.

1. Introduction

Centuries ago, many mathematicians put the sums of powers of positive integers or the alternating sums of powers of positive integers at the center of their work. Numerous different techniques have been given to find formulas containing these sums and at the same time continue to be explored in new techniques for these sums. The well-known novel formulas for these sums are given as follows:

$$(1.1) \quad \sum_{j=1}^{n-1} j^m = \frac{B_{m+1}(n) - B_{m+1}}{m+1},$$

$$(1.2) \quad \sum_{j=1}^{n-1} (-1)^j j^m = \frac{(-1)^{n+1} E_m(n) + E_m}{2},$$

2010 *Mathematics Subject Classification*: 05A15; 11B68; 11B73; 11S80; 26A42; 33B10.

Key words and phrases: Bernoulli numbers and polynomials, Euler numbers and polynomials, Fubini numbers, Tangent numbers, p -adic integrals, Riemann integral, Trigonometric functions, Generating functions.

where $m \in \mathbb{N}$ and $n \in \mathbb{N}$ with $n > 1$, $B_{m+1}(n)$, B_{m+1} , and $E_{m+1}(n)$, E_{m+1} denote the Bernoulli polynomials and numbers, and the Euler polynomials and numbers, respectively (cf. [1, 2, 4–10, 13, 15, 21, 22, 24, 27–30, 35, 38, 42–44]).

In this paper, not only Faulhaber formulas involving sums of powers, but also generating functions for special numbers and polynomials are handled in an interesting way in the direction of historical development. Sums of powers of positive integers: $1^k + 2^k + \cdots + m^k$ goes back to Johann Faulhaber, the famous German mathematician and engineer who lived between 1580 and 1635. By using arithmetic and algebraic operations, Faulhaber [10], in whose book (*Academia algebrae*, Augspurg 1631), gave novel computation formulas for these sums (for detail, see also [30] and the references therein).

Faulhaber's formulas that give the sum of powers of positive integers have inspired the work of many famous mathematicians after 1631. In 1713, James Bernoulli, in whose book (*Ars conjectandi*, Basel 1713), mentioned the name of Faulhaber with his formulas including the sum of powers of positive integers. In 1834, a meticulous proof of Faulhaber's assertion was first given by Jacobi [15]. Later on, many mathematicians such as Riordan (1968) [35], Tits (1923) [44], Edwards (1986) [9], Gessel and Viennot (1989) [11], Kim et al. [21, 24, 27, 28], Knuth [30], Simsek [39], Simsek et al. [42] and others have given Faulhaber different proofs of Faulhaber's formulas with different methods.

We now give motivation of this paper. The following formula that gives the sums of powers of positive integers is one of the important results given in this study. In this approach, we use a different technique applying the trigonometric function identity and Bernoulli polynomial properties:

THEOREM 1.1. *Let $k \in \mathbb{N}$ and $n \in \mathbb{N}$ with $n > 1$. Then we have*

$$(1.3) \quad \sum_{j=1}^{n-1} j^{2k} = \frac{B_{2k+1}(n)}{2k+1}.$$

In addition to formula (1.3), we also derive a formula for alternating sums of powers of positive integers involving the Euler numbers and polynomials and the tangent numbers. Apart from these, new novel formulas and relations, containing many different special numbers and polynomials such as the Bernoulli numbers and polynomials, the Euler numbers and polynomials, the Stirling numbers, the Fubini numbers, the tangent numbers, and other special numbers and polynomials, are given with the help of the identities of trigonometric functions, the Euler formula, and functional equations of the generating functions for special numbers and polynomials. These new results have the potential to be used in different disciplines.

1.1. Some preliminaries. Here, we give some well-known generating functions for special numbers and polynomials, formulas and notations.

We firstly give the following notations for the set of numbers:

Let \mathbb{N} , \mathbb{R} and \mathbb{C} denote the set of natural numbers, the set of real numbers and the set of complex numbers, respectively, and let $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$.

The well-known Euler formula is given by $e^{ix} = \cos(x) + i \sin(x)$, where $i^2 = -1$. The Bernoulli polynomials, $B_n(s)$, are defined by

$$(1.4) \quad K(t, s) = \frac{te^{st}}{e^t - 1} = \sum_{n=0}^{\infty} B_n(s) \frac{t^n}{n!},$$

where $|t| < 2\pi$ (cf. [1-44] and the references therein). Substituting $s = 0$ into (1.4), we have $B_n(0) = B_n$, where B_n denotes the Bernoulli numbers (cf. [1-44] and the references therein).

By using (1.4), we have the relation $K(t, n) - K(t, 0) = t \sum_{j=0}^{n-1} e^{jt}$, where $n \in \mathbb{N}$. After some elementary calculations in the above equation, one can easily arrive at (1.1).

The Euler polynomials of the first kind, $E_n(s)$, are defined by

$$(1.5) \quad G(t, s) = \frac{2e^{st}}{e^t + 1} = \sum_{n=0}^{\infty} E_n(s) \frac{t^n}{n!},$$

where $|t| < \pi$ (cf. [1-44] and the references therein). Substituting $s = 0$ into (1.5), we have $E_n(0) = E_n$, where E_n denotes the Euler numbers of the first kind (cf. [1-44] and the references therein). By using (1.5), we have the relation:

$$(-1)^{n+1}G(t, n) + G(t, 0) = 2 \sum_{j=0}^{n-1} (-1)^j e^{jt},$$

where $n \in \mathbb{N}$. After some elementary calculations in the above equation, one can easily arrive at (1.2).

The Euler numbers of the second kind, E_n^* , are defined by

$$(1.6) \quad G_{E2}(t) = \frac{2e^t}{e^{2t} + 1} = \sum_{n=0}^{\infty} E_n^* \frac{t^n}{n!},$$

where $|t| < \frac{\pi}{2}$ (cf. [1, 38, 40, 43]). Using (1.5) and (1.6), one has the following well-known relations: $E_m^* = \sum_{j=0}^m \binom{m}{j} 2^j E_j$, and $E_n^* = 2^n E_n(\frac{1}{2})$, (cf. [1, 38, 40]).

The Stirling numbers of the second kind, $S_2(n, m)$, are defined by

$$(1.7) \quad G_{S2}(t, m) = \frac{(e^t - 1)^m}{m!} = \sum_{n=0}^{\infty} S_2(n, m) \frac{t^n}{n!},$$

(cf. [1-44] and the references therein). These numbers satisfy the relations: $S_2(0, 0) = 1$, $S_2(n, 1) = S_2(n, n) = 1$, $S_2(n, 0) = 0$ if $n > 0$ and $S_2(n, m) = 0$ if $m > n$ (cf. [1-44] and the references therein).

The Fubini numbers, $w_g(n)$, are defined by

$$(1.8) \quad G_w(t) = \frac{1}{2 - e^t} = \sum_{n=0}^{\infty} w_g(n) \frac{t^n}{n!},$$

(cf. [6, 12, 17]). Combing (1.7) with (1.8), one has

$$(1.9) \quad w_g(n) = \sum_{j=0}^n j! S_2(n, j),$$

(cf. [6, 12, 17]).

The polynomials $C_n(x, y)$ and $S_n(x, y)$ are defined respectively by

$$(1.10) \quad G_C(t, x, y) = e^{xt} \cos(yt) = \sum_{n=0}^{\infty} C_n(x, y) \frac{t^n}{n!},$$

$$(1.11) \quad G_S(t, x, y) = e^{xt} \sin(yt) = \sum_{n=0}^{\infty} S_n(x, y) \frac{t^n}{n!},$$

(cf. [19, 20, 31, 32] and the references therein). Using (1.10) and (1.11), the explicit formulas for the polynomials $C_n(x, y)$ and $S_n(x, y)$ are given by:

$$(1.12) \quad C_n(x, y) = \sum_{j=0}^{\lfloor n/2 \rfloor} (-1)^j \binom{n}{2j} y^{2j} x^{n-2j},$$

$$(1.13) \quad S_n(x, y) = \sum_{j=0}^{\lfloor (n-1)/2 \rfloor} (-1)^j \binom{n}{2j+1} y^{2j+1} x^{n-2j-1},$$

(cf. [19, 20, 31, 32] and the references therein).

The tangent numbers (zag numbers) are defined by

$$(1.14) \quad F(t) = \frac{e^{2t} - 1}{e^{2t} + 1} = \sum_{n=0}^{\infty} (-1)^n T_{2n+1} \frac{t^{2n+1}}{(2n+1)!},$$

(cf. [1, 6, 21, 34, 38]). Using (1.14), the relations among the Bernoulli numbers, the Euler numbers and the tangent numbers are given, respectively, by:

$$(1.15) \quad T_{2n+1} = (-1)^{n+1} \left(\frac{2^{2n+1} - 2^{4n+3}}{n+1} \right) B_{2n+2},$$

$$(1.16) \quad T_{2n+1} = (-1)^{n+1} 2^{2n+1} E_{2n+1},$$

(cf. [1, 6, 21, 34, 38]). In [34], Qi gave many identities for the tangent numbers. One of them is given as follows:

$$(1.17) \quad T_{2n+1} = \lim_{x \rightarrow 0} \frac{d^{2n+1}}{dx^{2n+1}} \{\tan(x)\}.$$

This paper is organized as follows: In Section 2, by using trigonometric identities and generating functions, formulas involving sums of powers of positive integers and the Bernoulli polynomials and the Euler polynomials are given. In Section 3, by the aid of trigonometric functions, the Euler formula, and functional equations of the generating functions for special numbers and polynomials, some formulas and relations associated with the Bernoulli polynomials, the Euler numbers of the second kind, the polynomials $C_n(x, y)$, the polynomials $S_n(x, y)$ are derived. In Section 4, by using functional equations of the generating functions, some relations

and identities including the Fubini numbers, the Stirling numbers, the tangent numbers are given. Finally, in Section 5, by applying the Riemann integral, the fermionic p -adic integral and the Volkenborn integral to the results obtained in the previous sections, many formulas and combinatorial sums formulas including the Bernoulli numbers and polynomials, the Euler numbers are presented.

2. Sums of powers of positive integers: an approach to trigonometric identities and generating functions

In this section, applying the trigonometric identities and some properties of the Bernoulli numbers and polynomials, the Euler numbers and polynomials of the first kind, we give sums of powers of positive integers and alternating sums of powers of positive integers. Using these formulas, we also derive some interesting results for these numbers and polynomials.

THEOREM 2.1. *Let $m \in \mathbb{N}$ with $m > 1$ and $n \in \mathbb{N}$. Then we have*

$$(2.1) \quad \sum_{j=1}^n j^{m-1} - \sum_{j=1}^n (-1)^m j^{m-1} = \frac{B_m(n+1) - B_m(-n)}{m}.$$

PROOF. By using the following well-known identities

$$\frac{\sin((2n+1)t)}{\sin(t)} = 1 + 2 \sum_{j=1}^n \cos(2jt)$$

(cf. [14, p. 20]), we obtain

$$(2.2) \quad \frac{e^{2it(n+1)}}{e^{2it} - 1} - \frac{e^{-2int}}{e^{2it} - 1} = 1 + \sum_{j=1}^n (e^{2jit} + e^{-2jit}).$$

Combining (2.2) with (1.4), after some elementary calculations, we have the following relation:

$$\begin{aligned} & \sum_{m=2}^{\infty} B_m(n+1) \frac{(2it)^m}{m!} - \sum_{m=2}^{\infty} B_m(-n) \frac{(2it)^m}{m!} \\ &= \sum_{m=2}^{\infty} \left(\sum_{j=1}^n (mj^{m-1} + m(-j)^{m-1}) \right) \frac{(2it)^m}{m!}. \end{aligned}$$

Comparing the coefficients of $\frac{(2it)^m}{m!}$ on both sides of the above equation, we arrive at the desired result. □

Substituting $m = 2k$ ($k \in \mathbb{N}_0$) into equation (2.1), we have the following result:

COROLLARY 2.1. *Let $k \in \mathbb{N}_0$. Then we have $B_{2k}(n+1) = B_{2k}(-n)$.*

PROOF OF THEOREM 1.1. Combining the following well-known identities

$$(2.3) \quad B_m(n+1) = mn^{m-1} + B_m(n), \quad B_m(n) = (-1)^m B_m(-n) - mn^{m-1},$$

with equation (2.1), we obtain

$$B_m(n) + mn^{m-1} + (-1)^{m+1} (B_m(n) + mn^{m-1}) = m \sum_{j=1}^n (j^{m-1} + (-j)^{m-1}).$$

Substituting $m = 2k + 1$ ($k \in \mathbb{N}$) into the above equation, we get

$$B_{2k+1}(n) + (2k + 1)n^{2k} = (2k + 1) \sum_{j=1}^n j^{2k}.$$

After some elementary calculations in the aforementioned equation, we arrive at the desired result. \square

Putting the following well-known results in Theorem 1.1

$$B_{2k+1}(n) = \sum_{v=0}^{2k+1} \binom{2k+1}{v} n^{2k+1-v} B_v,$$

and $B_{2k+1} = 0$ ($k \in \mathbb{N}$), we have the following corollary:

COROLLARY 2.2. *Let $k \in \mathbb{N}$ and $n \in \mathbb{N}$ with $n > 1$. Then we have*

$$\begin{aligned} \sum_{j=1}^{n-1} j^{2k} &= \frac{1}{2k+1} n^{2k+1} - \frac{1}{2} n^{2k} + \frac{k}{6} n^{2k-1} - \frac{2k^3 - 3k^2 + k}{180} n^{2k-3} + \dots \\ &+ \frac{\binom{2k+1}{2k-2} B_{2k-2}}{2k+1} n^3 + \frac{\binom{2k+1}{2k} B_{2k}}{2k+1} n. \end{aligned}$$

When $k = 1$, Corollary 2.2 reduces to the following well-known result:

$$\sum_{j=1}^{n-1} j^2 = \frac{1}{3} n^3 - \frac{1}{2} n^2 + \frac{1}{6} n = \frac{n(n-1)(2n-1)}{6}.$$

THEOREM 2.2. *Let $m \in \mathbb{N}$ with $m > 2$. Then we have*

$$\begin{aligned} (2.4) \quad \sum_{v=0}^{\lfloor (m-1)/2 \rfloor} (-1)^v \binom{m}{2v+1} 4^{-v} T_{2v+1} \sum_{j=1}^n (j^{m-1-2v} - (-j)^{m-1-2v}) \\ = E_m(n+1) + E_m(-n). \end{aligned}$$

PROOF. By using the following well-known equation

$$\frac{\cos((2n+1)t)}{\cos(t)} = 1 - \frac{2 \sin t}{\cos t} \sum_{j=1}^n \sin(2jt)$$

(cf. [33, p. 342]), we have

$$(2.5) \quad \frac{e^{2i(n+1)t}}{e^{2it} + 1} + \frac{e^{-2int}}{e^{2it} + 1} = 1 + \frac{e^{2it} - 1}{e^{2it} + 1} \sum_{j=1}^n (e^{2jit} - e^{-2jit}).$$

Combining (2.5) with (1.5) and (1.14), after some elementary calculations, we get

$$\begin{aligned} & \sum_{m=3}^{\infty} 2^m E_m(n+1) \frac{(it)^m}{m!} + \sum_{m=3}^{\infty} 2^m E_m(-n) \frac{(it)^m}{m!} \\ &= \sum_{m=3}^{\infty} \sum_{v=0}^{[(m-1)/2]} (-1)^v \binom{m}{2v+1} 2^{m-2v} T_{2v+1} \sum_{j=1}^n \frac{j^{m-1-2v} - (-j)^{m-1-2v}}{m!} (it)^m. \end{aligned}$$

Comparing the coefficients of $\frac{(it)^m}{m!}$ on both sides of the aforementioned equation, we arrive at the desired result. \square

Substituting $m = 2k + 1$ ($k \in \mathbb{N}_0$) into (2.4), we have the following result:

COROLLARY 2.3. *Let $k \in \mathbb{N}_0$. Then we have $E_{2k+1}(-n) = -E_{2k+1}(n+1)$.*

Putting $m = 2k$ ($k \in \mathbb{N}$) in (2.4), we get

$$E_{2k}(n+1) + E_{2k}(-n) = \sum_{v=0}^{[(2k-1)/2]} (-1)^v \binom{2k}{2v+1} 2^{-2v+1} T_{2v+1} \sum_{j=1}^n j^{2(k-v)-1}.$$

Combining the above equation with (1.1), we obtain

$$\begin{aligned} (2.6) \quad & \sum_{v=0}^{[(2k-1)/2]} (-1)^v \binom{2k}{2v+1} T_{2v+1} \frac{B_{2(k-v)}(n+1) - B_{2(k-v)}}{4^v(k-v)} \\ &= E_{2k}(n+1) + E_{2k}(-n). \end{aligned}$$

Combining (1.16) with (2.6), we arrive at the following theorem:

THEOREM 2.3. *Let $k \in \mathbb{N}$. Then we have*

$$\sum_{v=0}^{[(2k-1)/2]} \binom{2k}{2v+1} \frac{(B_{2(k-v)} - B_{2(k-v)}(n+1)) E_{2v+1}}{k-v} = \frac{E_{2k}(n+1) + E_{2k}(-n)}{2}.$$

3. Identities and relations derive from trigonometric functions identities and functional equations of the generating functions

In this section, applying trigonometric functions identities and functional equations of the generating functions for special numbers and polynomials, we derive many new identities and relations involving the Bernoulli polynomials, the Euler numbers of the second kind, the polynomials $C_n(x, y)$, the polynomials $S_n(x, y)$.

In order to give these new identities and relations, we need the following well-known trigonometric identities:

$$(3.1) \quad \prod_{k=1}^m \cos\left(\frac{x}{2^k}\right) = \frac{\sin(x)}{2^m \sin\left(\frac{x}{2^m}\right)},$$

where $m \in \mathbb{N}$ (cf. [13, Eq. (1.18)]), and

$$(3.2) \quad t \csc(t) = \sum_{n=0}^{\infty} (-1)^n 2^{2n} B_{2n} \left(\frac{1}{2}\right) \frac{t^{2n}}{(2n)!}$$

(cf. [21, Eq. (2.24)]). Using the Euler formula, the De Moivre formula, and the binomial theorem, the following formulas are easily found:

$$(3.3) \quad \sin^{2n-1}(x) = \frac{(-1)^{n-1}}{2^{2n-2}} \sum_{j=0}^{n-1} (-1)^j \binom{2n-1}{j} \sin((2n-1-2j)x),$$

$$(3.4) \quad \sin^{2n}(x) = \frac{1}{2^{2n}} \binom{2n}{n} + \frac{(-1)^n}{2^{2n-1}} \sum_{j=0}^{n-1} (-1)^j \binom{2n}{j} \cos((2n-2j)x)$$

(cf. [16, p. 130]).

THEOREM 3.1. *Let $n \in \mathbb{N}$ and $m \in \mathbb{N}_0$. Then we have*

$$\begin{aligned} & \sum_{j_1+\dots+j_{n-1}=m} N_{j_1, j_2, \dots, j_{n-1}}^m C_{j_{n-1}} \left(x, \frac{y}{2^n}\right) \\ & \quad \times C_{j_{n-2}} \left(x, \frac{y}{2^{n-1}}\right) \cdots C_{m-j_1-\dots-j_{n-1}} \left(x, \frac{y}{2}\right) \\ & = \sum_{j=0}^{[(m+1)/2]} (-1)^j \binom{m+1}{2j} \frac{y^{2j-1} 2^{2j-2nj}}{m+1} B_{2j} \left(\frac{1}{2}\right) S_{m+1-2j}(xn, y), \end{aligned}$$

where

$$\begin{aligned} \sum_{j_1+\dots+j_{n-1}=m} & = \sum_{j_{n-1}=0}^m \sum_{j_{n-2}=0}^{m-j_{n-1}} \cdots \sum_{j_1=0}^{m-j_2-\dots-j_{n-1}}, \\ N_{j_1, j_2, \dots, j_{n-1}}^m & = \frac{m!}{j_1! j_2! \cdots (m-j_1-\dots-j_{n-1})!}. \end{aligned}$$

PROOF. Using (3.1), we obtain

$$\prod_{k=1}^n e^{xt} \cos\left(\frac{yt}{2^k}\right) = \frac{e^{xnt} \sin(yt)}{2^n \sin\left(\frac{yt}{2^n}\right)}.$$

Combing the above equation with (1.10), (1.11) and (3.2), we get the following functional equation:

$$G_C\left(t, x, \frac{y}{2}\right) G_C\left(t, x, \frac{y}{2^2}\right) \cdots G_C\left(t, x, \frac{y}{2^n}\right) = \frac{1}{2^n} \csc\left(\frac{yt}{2^n}\right) G_S(t, nx, y).$$

Therefore

$$\begin{aligned} & \sum_{m=0}^{\infty} \frac{t^m}{m!} \sum_{j_1+j_2+\dots+j_{n-1}=m} N_{j_1, j_2, \dots, j_{n-1}}^m C_{j_{n-1}} \left(x, \frac{y}{2^n}\right) \\ & \quad \times C_{j_{n-2}} \left(x, \frac{y}{2^{n-1}}\right) \cdots C_{m-j_1-j_2-\dots-j_{n-1}} \left(x, \frac{y}{2}\right) \\ & = \sum_{m=0}^{\infty} \sum_{j=0}^{[(m+1)/2]} (-1)^j \binom{m+1}{2j} \frac{B_{2j} \left(\frac{1}{2}\right) y^{2j-1} 2^{2j-2nj}}{m+1} S_{m+1-2j}(xn, y) \frac{t^m}{m!}. \end{aligned}$$

Comparing the coefficients of $\frac{t^m}{m!}$ on both sides of the aforementioned equation, we arrive at the desired result. \square

Substituting $n = 1$ into Theorem 3.1, we get the following corollary:

COROLLARY 3.1. *Let $m \in \mathbb{N}_0$. Then we have*

$$C_m\left(x, \frac{y}{2}\right) = \sum_{j=0}^{[(m+1)/2]} (-1)^j \binom{m+1}{2j} \frac{y^{2j-1}}{m+1} B_{2j}\left(\frac{1}{2}\right) S_{m+1-2j}(x, y).$$

REMARK 3.1. Substituting $n = 2$ into Theorem 3.1, we have

$$\begin{aligned} \sum_{j_1=0}^m \binom{m}{j_1} C_{j_1}\left(x, \frac{y}{2^2}\right) C_{m-j_1}\left(x, \frac{y}{2}\right) \\ = \sum_{j=0}^{[(m+1)/2]} (-1)^j \binom{m+1}{2j} \frac{y^{2j-1} 2^{-2j}}{m+1} B_{2j}\left(\frac{1}{2}\right) S_{m+1-2j}(2x, y). \end{aligned}$$

In addition, if we set $n = 3$ in Theorem 3.1, we have

$$\begin{aligned} \sum_{j_2=0}^m \sum_{j_1=0}^{m-j_2} \binom{m}{j_2} \binom{m-j_2}{j_1} C_{j_2}\left(x, \frac{y}{2^3}\right) C_{j_1}\left(x, \frac{y}{2^2}\right) C_{m-j_1-j_2}\left(x, \frac{y}{2}\right) \\ = \sum_{j=0}^{[(m+1)/2]} (-1)^j \binom{m+1}{2j} \frac{y^{2j-1} 2^{-4j}}{m+1} B_{2j}\left(\frac{1}{2}\right) S_{m+1-2j}(3x, y). \end{aligned}$$

THEOREM 3.2. *Let $n \in \mathbb{N}$ and $m \in \mathbb{N}_0$. Then we have*

$$\begin{aligned} \sum_{j_1+\dots+j_{2n-2}=m} N_{j_1, j_2, \dots, j_{2n-2}}^m S_{j_{2n-2}}(x, y) S_{j_{2n-3}}(x, y) \cdots S_{m-j_1-\dots-j_{2n-2}}(x, y) \\ = \frac{(-1)^{n-1}}{2^{2n-2}} \sum_{j=0}^{n-1} (-1)^j \binom{2n-1}{j} S_m((2n-1)x, (2n-1-2j)y). \end{aligned}$$

PROOF. Using (3.3), we obtain

$$\prod_{k=1}^{2n-1} e^{xt} \sin(yt) = \frac{(-1)^{n-1}}{2^{2n-2}} \sum_{j=0}^{n-1} (-1)^j \binom{2n-1}{j} e^{(2n-1)xt} \sin((2n-1-2j)yt).$$

Combining the above equation with (1.11), we have the following functional equation:

$$\underbrace{G_S(t, x, y) G_S(t, x, y) \cdots G_S(t, x, y)}_{(2n-1) \text{ times}} = \frac{(-1)^{n-1}}{2^{2n-2}} \sum_{j=0}^{n-1} (-1)^j \binom{2n-1}{j} G_S(t, (2n-1)x, (2n-1-2j)y).$$

Therefore

$$\begin{aligned} & \sum_{m=0}^{\infty} \frac{t^m}{m!} \sum_{j_1+j_2+\dots+j_{2n-2}=m} N_{j_1, j_2, \dots, j_{2n-2}}^m S_{j_{2n-2}}(x, y) \\ & \quad \times S_{j_{2n-3}}(x, y) \cdots S_{m-j_1-j_2-\dots-j_{2n-2}}(x, y) \\ & = \sum_{m=0}^{\infty} \frac{(-1)^{n-1}}{2^{2n-2}} \sum_{j=0}^{n-1} (-1)^j \binom{2n-1}{j} S_m((2n-1)x, (2n-1-2j)y) \frac{t^m}{m!}. \end{aligned}$$

Comparing the coefficients of $\frac{t^m}{m!}$ on both sides of the aforementioned equation, we arrive at the desired result. \square

THEOREM 3.3. *Let $n \in \mathbb{N}$ and $m \in \mathbb{N}_0$. Then we have*

$$\begin{aligned} & \sum_{j_1+\dots+j_{2n-1}=m} N_{j_1, j_2, \dots, j_{2n-1}}^m S_{j_{2n-1}}(x, y) S_{j_{2n-2}}(x, y) \cdots S_{m-j_1-\dots-j_{2n-1}}(x, y) \\ & = \frac{(2nx)^m}{2^{2n}} \binom{2n}{n} + \frac{(-1)^n}{2^{2n-1}} \sum_{j=0}^{n-1} (-1)^j \binom{2n}{j} C_m(2nx, (2n-2j)y). \end{aligned}$$

PROOF. By using equation (3.4), we get

$$\prod_{k=1}^{2n} e^{xt} \sin(yt) = \frac{e^{2nxt}}{2^{2n}} \binom{2n}{n} + \frac{(-1)^n}{2^{2n-1}} \sum_{j=0}^{n-1} (-1)^j \binom{2n}{j} e^{2nxt} \cos((2n-2j)yt).$$

Combining the above equation with (1.10) and (1.11), we obtain the following functional equation:

$$\begin{aligned} & \underbrace{G_S(t, x, y) G_S(t, x, y) \dots G_S(t, x, y)}_{2n \text{ times}} \\ & = \frac{e^{2nxt}}{2^{2n}} \binom{2n}{n} + \frac{(-1)^n}{2^{2n-1}} \sum_{j=0}^{n-1} (-1)^j \binom{2n}{j} G_C(t, 2nx, (2n-2j)y). \end{aligned}$$

Applying the Cauchy product rule in the left-hand side of the previous functional equation, after some elementary calculations, we get

$$\begin{aligned} & \sum_{m=0}^{\infty} \frac{t^m}{m!} \sum_{j_1+j_2+\dots+j_{2n-1}=m} N_{j_1, j_2, \dots, j_{2n-1}}^m S_{j_{2n-1}}(x, y) \\ & \quad \times S_{j_{2n-2}}(x, y) \cdots S_{m-j_1-j_2-\dots-j_{2n-1}}(x, y) \\ & = \binom{2n}{n} \sum_{m=0}^{\infty} \frac{(2nx)^m}{2^{2n}} \frac{t^m}{m!} + \sum_{m=0}^{\infty} \frac{(-1)^n}{2^{2n-1}} \sum_{j=0}^{n-1} (-1)^j \binom{2n}{j} C_m(2nx, (2n-2j)y) \frac{t^m}{m!}. \end{aligned}$$

Comparing the coefficients of $\frac{t^m}{m!}$ on both sides of the aforementioned equation, we arrive at the desired result. \square

THEOREM 3.4. *Let $n \in \mathbb{N}_0$. Then we have*

$$(3.5) \quad x^n = \sum_{k=0}^{[n/2]} (-1)^k \binom{n}{2k} y^{2k} E_{2k}^* C_{n-2k}(x, y).$$

PROOF. Using (1.6) and (1.10), we have the following functional equation:

$$e^{ixt} = G_{E2}(yt)G_C(it, x, y).$$

From the above equation, we get

$$\sum_{n=0}^{\infty} (ix)^n \frac{t^n}{n!} = \sum_{n=0}^{\infty} y^n E_n^* \frac{t^n}{n!} \sum_{n=0}^{\infty} i^n C_n(x, y) \frac{t^n}{n!}.$$

Therefore

$$\sum_{n=0}^{\infty} (ix)^n \frac{t^n}{n!} = \sum_{n=0}^{\infty} \sum_{k=0}^n \binom{n}{k} i^{n-k} y^k E_k^* C_{n-k}(x, y) \frac{t^n}{n!}.$$

Comparing the coefficients of $\frac{t^n}{n!}$ on both sides of the above equation, we get

$$(3.6) \quad x^n = \sum_{k=0}^n \binom{n}{k} i^{-k} y^k E_k^* C_{n-k}(x, y).$$

Since $E_{2m+1}^* = 0$, ($m \in \mathbb{N}_0$) equation (3.6) reduces to the following relation:

$$x^n = \sum_{k=0}^{[n/2]} (-1)^k \binom{n}{2k} y^{2k} E_{2k}^* C_{n-2k}(x, y).$$

Thus, we get the assertion of the theorem. □

THEOREM 3.5. *Let $n \in \mathbb{N}_0$. Then we have*

$$(3.7) \quad \sum_{k=0}^{[n/2]} (-1)^k \binom{n}{2k} 4^k y^{2k-1} B_{2k} \left(\frac{1}{2}\right) S_{n-2k}(x, y) = B_n(x+1) - B_n(x).$$

PROOF. Using (1.11) and (3.2), we have the following functional equation:

$$e^{xt} = \csc(yt)G_S(t, x, y).$$

With the help of the above functional equation, we obtain

$$y \sum_{n=0}^{\infty} n x^{n-1} \frac{t^n}{n!} = \sum_{n=0}^{\infty} \sum_{k=0}^{[n/2]} (-1)^k \binom{n}{2k} (2y)^{2k} B_{2k} \left(\frac{1}{2}\right) S_{n-2k}(x, y) \frac{t^n}{n!}.$$

Combining the above equation with (2.3), we get

$$y \sum_{n=0}^{\infty} (B_n(x+1) - B_n(x)) \frac{t^n}{n!} = \sum_{n=0}^{\infty} \sum_{k=0}^{[n/2]} (-1)^k \binom{n}{2k} (2y)^{2k} B_{2k} \left(\frac{1}{2}\right) S_{n-2k}(x, y) \frac{t^n}{n!}.$$

Comparing the coefficients of $\frac{t^n}{n!}$ on both sides of the aforementioned equation, we arrive at the desired result. □

4. Identities and relations involving Fubini numbers and Tangent numbers

It is well known that the Fubini numbers and the Stirling numbers have many applications in combinatoric analysis, in number theory, and also in other areas (cf. [6, 12, 43]; and the references cited therein). By using generating functions with their functional equations, we derive some identities involving the Fubini numbers, the tangent numbers and the Stirling numbers of the second kind.

THEOREM 4.1. *Let $n \in \mathbb{N}$. Then we have*

$$(4.1) \quad w_g(n) = \sum_{k=0}^{[(n-1)/2]} \sum_{j=0}^{n-2k-1} \frac{(-1)^k}{2^{2k+1}} \binom{n}{2k+1} \binom{n-2k-1}{j} T_{2k+1} w_g(j) \\ + \sum_{k=0}^{[(n-1)/2]} \frac{(-1)^k}{2^{2k+1}} \binom{n}{2k+1} T_{2k+1} w_g(n-2k-1).$$

PROOF. Using (1.8) and (1.14), we obtain $-1 = (F(t/2)(e^t + 1) - 1)G_w(t)$ wherefrom we get

$$-1 = \sum_{n=0}^{\infty} (-1)^n \frac{T_{2n+1}}{2^{2n+1}} \frac{t^{2n+1}}{(2n+1)!} \sum_{n=0}^{\infty} \frac{t^n}{n!} \sum_{n=0}^{\infty} w_g(n) \frac{t^n}{n!} \\ + \sum_{n=0}^{\infty} (-1)^n \frac{T_{2n+1}}{2^{2n+1}} \frac{t^{2n+1}}{(2n+1)!} \sum_{n=0}^{\infty} w_g(n) \frac{t^n}{n!} - \sum_{n=0}^{\infty} w_g(n) \frac{t^n}{n!}.$$

Therefore

$$\sum_{n=1}^{\infty} w_g(n) \frac{t^n}{n!} = \sum_{n=1}^{\infty} \sum_{k=0}^{[(n-1)/2]} \sum_{j=0}^{n-2k-1} \frac{(-1)^k}{2^{2k+1}} \binom{n}{2k+1} \binom{n-2k-1}{j} T_{2k+1} w_g(j) \frac{t^n}{n!} \\ + \sum_{n=1}^{\infty} \sum_{k=0}^{[(n-1)/2]} \frac{(-1)^k}{2^{2k+1}} \binom{n}{2k+1} T_{2k+1} w_g(n-2k-1) \frac{t^n}{n!}.$$

Comparing the coefficients of $\frac{t^n}{n!}$ on both sides of the aforementioned equation, we arrive at the desired result. \square

Combining (4.1) with (1.17), we arrive at the following corollary:

COROLLARY 4.1. *Let $n \in \mathbb{N}$. Then we have*

$$w_g(n) = \sum_{k=0}^{[(n-1)/2]} (-1)^k \binom{n}{2k+1} \frac{1}{2^{2k+1}} \lim_{x \rightarrow 0} \frac{d^{2k+1}}{dx^{2k+1}} \{\tan(x)\} \\ \times \left(\sum_{j=0}^{n-2k-1} \binom{n-2k-1}{j} w_g(j) + w_g(n-2k-1) \right).$$

By combining (4.1) with (1.9), we also derive a relation between the tangent numbers and the Stirling numbers of the second kind by the following corollary:

COROLLARY 4.2. *Let $n \in \mathbb{N}$. Then we have*

$$\begin{aligned} \sum_{v=0}^n v! S_2(n, v) &= \sum_{k=0}^{[(n-1)/2]} \sum_{j=0}^{n-2k-1} \sum_{v=0}^j \frac{(-1)^k v!}{2^{2k+1}} \binom{n}{2k+1} \binom{n-2k-1}{j} S_2(j, v) T_{2k+1} \\ &+ \sum_{k=0}^{[(n-1)/2]} \sum_{v=0}^{n-2k-1} \frac{(-1)^k v!}{2^{2k+1}} \binom{n}{2k+1} S_2(n-2k-1, v) T_{2k+1}. \end{aligned}$$

**5. Formulas and combinatorial sums
derive from Riemann integral and p -adic integrals**

In this section, new formulas and combinatorial sums are given by applying both Riemann integrals and p -adic integrals to the formulas and relations presented in the previous sections. Firstly, some combinatorial sums, which are found with the help of the Riemann integral, are given. Secondly, some formulas and combinatorial sums, which are found with the help of p -adic integrals, are given.

5.1. Formulas and combinatorial sums derive from Riemann integral.

Here, by the aid of applications of the Riemann integral to equation (3.5) and equation (3.7), we derive formulas and combinatorial sums.

THEOREM 5.1. *Let $n \in \mathbb{N}_0$. Then we have*

$$\frac{1}{n+1} = \sum_{k=0}^{[n/2]} \sum_{j=0}^{[(n-2k)/2]} (-1)^{j+k} \binom{n-2k}{2j} \binom{n}{2k} \frac{y^{2k+2j}}{n-2k-2j+1} E_{2k}^*.$$

PROOF. Integrating both sides of (3.5) from 0 to 1 with respect to x , we have

$$\frac{1}{n+1} = \sum_{k=0}^{[n/2]} (-1)^k \binom{n}{2k} y^{2k} E_{2k}^* \int_0^1 C_{n-2k}(x, y) dx.$$

Combining the above resulting equation with (1.12), after some elementary calculations, we arrive at the desired result. □

THEOREM 5.2. *Let $n \in \mathbb{N}$. Then we have*

$$1 = \sum_{k=0}^{[n/2]} \sum_{j=0}^{[(n-2k-1)/2]} (-1)^{j+k} \binom{n-2k}{2j+1} \binom{n}{2k} \frac{2^{2k} y^{2k+2j}}{n-2k-2j} B_{2k} \left(\frac{1}{2} \right).$$

PROOF. Integrating both sides of (3.7) from 0 to 1 with respect to x , we have

$$\frac{1}{n} = \frac{1}{n} \sum_{k=0}^{[n/2]} (-1)^k \binom{n}{2k} 2^{2k} y^{2k-1} B_{2k} \left(\frac{1}{2} \right) \int_0^1 S_{n-2k}(x, y) dx.$$

Combining the above resulting equation with (1.13), after some elementary calculations, we arrive at the desired result. □

5.2. Formulas and combinatorial sums derive from p -adic integrals.

There are many applications of p -adic integrals involving the Volkenborn integral and the fermionic integral. The most important application of these is to construct generating functions of special number and polynomial classes containing Bernoulli and Euler numbers and polynomials. Another application of p -adic integrals is that there are formulas that directly calculate Bernoulli and Euler numbers and polynomials. Here, with the help of these formulas, very interesting and useful identities, relations and combinatoric sums, including the Bernoulli numbers and polynomials, the Euler numbers and polynomials, are given.

Now it is time to briefly give introduction for p -adic integrals, involving the Volkenborn integral and the fermionic integral.

Let \mathbb{Z}_p denote the set of p -adic integers. Let \mathbb{K} be a field with a complete valuation. Let $f \in C^1(\mathbb{Z}_p \rightarrow \mathbb{K})$, set of continuous derivative functions.

The Volkenborn integral (bosonic p -adic integral) of function f on \mathbb{Z}_p is given by

$$(5.1) \quad \int_{\mathbb{Z}_p} f(x) d\mu_1(x) = \lim_{N \rightarrow \infty} \frac{1}{p^N} \sum_{x=0}^{p^N-1} f(x),$$

where $\mu_1(x) = \mu_1(x + p^N \mathbb{Z}_p) = 1/p^N$ (cf. [21–23, 36, 40] and the references therein).

Using (5.1), the Bernoulli numbers B_n is also given as follows:

$$(5.2) \quad \int_{\mathbb{Z}_p} x^n d\mu_1(x) = B_n$$

(cf. [21–23, 36, 40] and the references therein).

The fermionic p -adic integral of function f on \mathbb{Z}_p is given by

$$(5.3) \quad \int_{\mathbb{Z}_p} f(x) d\mu_{-1}(x) = \lim_{N \rightarrow \infty} \sum_{x=0}^{p^N-1} (-1)^x f(x),$$

where $\mu_{-1}(x) = \mu_{-1}(x + p^N \mathbb{Z}_p) = (-1)^x$ (cf. [25, 26, 40] and the references therein).

Using (5.3), the Euler numbers E_n is also given as follows:

$$(5.4) \quad \int_{\mathbb{Z}_p} x^n d\mu_{-1}(x) = E_n$$

(cf. [25, 26, 40] and the references therein).

Here, applying the p -adic integrals formulas (5.2) and (5.4) to (3.5) and (3.7), we derive formulas including the Bernoulli numbers and the Euler numbers, respectively.

THEOREM 5.3. *Let $n \in \mathbb{N}_0$. Then we have*

$$B_n = \sum_{k=0}^{[n/2]} \sum_{j=0}^{[(n-2k)/2]} (-1)^{j+k} \binom{n-2k}{2j} \binom{n}{2k} B_{2k+2j} B_{n-2k-2j} E_{2k}^*.$$

PROOF. By applying the Volkenborn integral with respect to x and y to (3.5), we have

$$\int_{\mathbb{Z}_p} \int_{\mathbb{Z}_p} x^n d\mu_1(x) d\mu_1(y) = \sum_{k=0}^{[n/2]} (-1)^k \binom{n}{2k} E_{2k}^* \int_{\mathbb{Z}_p} \int_{\mathbb{Z}_p} y^{2k} C_{n-2k}(x, y) d\mu_1(x) d\mu_1(y).$$

Combining the above resulting equation with (1.12) and (5.2), we arrive at the desired result. \square

THEOREM 5.4. *Let $n \in \mathbb{N}_0$. Then we have*

$$E_n = \sum_{k=0}^{[n/2]} \sum_{j=0}^{[(n-2k)/2]} (-1)^{j+k} \binom{n-2k}{2j} \binom{n}{2k} E_{2k+2j} E_{n-2k-2j} E_{2k}^*.$$

PROOF. By applying the fermionic p -adic integral with respect to x and y to (3.5), we have

$$\begin{aligned} \int_{\mathbb{Z}_p} \int_{\mathbb{Z}_p} x^n d\mu_{-1}(x) d\mu_{-1}(y) \\ = \sum_{k=0}^{[n/2]} (-1)^k \binom{n}{2k} E_{2k}^* \int_{\mathbb{Z}_p} \int_{\mathbb{Z}_p} y^{2k} C_{n-2k}(x, y) d\mu_{-1}(x) d\mu_{-1}(y). \end{aligned}$$

Combining the above resulting equation with (1.12) and (5.4), we arrive at the desired result. \square

THEOREM 5.5. *Let $n \in \mathbb{N}$. Then we have*

$$B_{n-1} = \frac{1}{n} \sum_{k=0}^{[n/2]} \sum_{j=0}^{[(n-2k-1)/2]} (-1)^{j+k} \binom{n}{2k} \binom{n-2k}{2j+1} 2^{2k} B_{2k} \left(\frac{1}{2}\right) B_{2k+2j} B_{n-2k-2j-1}.$$

PROOF. By applying the Volkenborn integral with respect to x and y to (3.7), we have

$$\begin{aligned} \int_{\mathbb{Z}_p} \int_{\mathbb{Z}_p} x^{n-1} d\mu_1(x) d\mu_1(y) \\ = \frac{1}{n} \sum_{k=0}^{[n/2]} (-1)^k \binom{n}{2k} 2^{2k} B_{2k} \left(\frac{1}{2}\right) \int_{\mathbb{Z}_p} \int_{\mathbb{Z}_p} y^{2k-1} S_{n-2k}(x, y) d\mu_1(x) d\mu_1(y). \end{aligned}$$

Combining the above resulting equation with (1.13) and (5.2), we arrive at the desired result. \square

THEOREM 5.6. *Let $n \in \mathbb{N}$. Then we have*

$$E_{n-1} = \frac{1}{n} \sum_{k=0}^{[n/2]} \sum_{j=0}^{[\frac{n-2k-1}{2}]} (-1)^{j+k} \binom{n}{2k} \binom{n-2k}{2j+1} 2^{2k} B_{2k} \left(\frac{1}{2}\right) E_{2k+2j} E_{n-2k-2j-1}.$$

PROOF. By applying the fermionic p -adic integral with respect to x and y to (3.7), we have

$$\begin{aligned} & \int_{\mathbb{Z}_p} \int_{\mathbb{Z}_p} x^{n-1} d\mu_{-1}(x) d\mu_{-1}(y) \\ &= \frac{1}{n} \sum_{k=0}^{[n/2]} (-1)^k \binom{n}{2k} 2^{2k} B_{2k} \left(\frac{1}{2}\right) \int_{\mathbb{Z}_p} \int_{\mathbb{Z}_p} y^{2k-1} S_{n-2k}(x, y) d\mu_{-1}(x) d\mu_{-1}(y). \end{aligned}$$

Combining the above resulting equation with (1.13) and (5.4), we arrive at the desired result. \square

Acknowledgment. The present paper was supported by Scientific Research Project Administration of Akdeniz University with Project Number: FDK-2020-5276.

References

1. M. Abramowitz, I. A. Stegun (Eds), *Handbook of Mathematical Functions with Formulas, Graphs, and Mathematical Tables*, U.S. Dept. of Commerce, National Bureau of Standards, Washington, D.C., 1972.
2. E. T. Bell, *Umbral symmetric functions and algebraic analogues of the Bernoullian and Eulerian numbers and functions*, Math. Z. **19** (1924), 35–49.
3. P. L. Butzer, K. Schmidt, E. L. Stark, L. Vogt, *Central factorial numbers; their main properties and some applications*, Numer. Funct. Anal. Optim. **10**(5–6) (1989), 419–488, <https://doi.org/10.1080/01630568908816313>.
4. L. Carlitz, *Some formulas for the Bernoulli and Euler polynomials*, Math. Nachr. **25** (1963), 223–231.
5. ———, *Recurrences for the Bernoulli and Euler numbers. II*, Math. Nachr. **29** (1965), 151–160.
6. L. Comtet, *Advanced Combinatorics*, Reidel, Dordrecht, Boston, 1974.
7. J. H. Conway, R. K. Guy, *The Book of Numbers*, Springer-Verlag, New York, 1996.
8. R. Dere, Y. Simsek, H. M. Srivastava, *A unified presentation of three families of generalized Apostol type polynomials based upon the theory of the umbral calculus and the umbral algebra*, J. Number Theory **133** (2013), 3245–3263.
9. A. W. F. Edwards, *A quick route to sums of powers*, Amer. Math. Monthly **93** (1986), 451–455.
10. J. Faulhaber, *Academia Algebra, Darinnen die miraculosische Inventiones zu den höchsten Cossen weiters continuirt und profitiert werden*, call number QA154.8 F3 1631a f MATH at Stanford University Libraries, Johann Ulrich Schönigk, Augspurg [sic], 1631.
11. I. M. Gessel, X. G. Viennot, *Determinants, paths, and plane partitions*, Preprint, 1989.
12. I. J. Good, *The number of ordering of n candidates when ties are permitted*, Fibonacci Quart. **13** (1975), 11–18.
13. H. W. Gould, *Combinatorial Identities: Table III: Binomial Identities Derived from Trigonometric and Exponential Series*, May 3, 2010, <https://math.wvu.edu/~hgould/Vol.6.PDF>
14. D. Jackson, *The Theory of Approximation*, Am. Math. Soc. Coll. Publ. **11**, New York, 1958.

15. C. G. J. Jacobi, *De usu legitimo formulae summatoriae Maclaurinianae*, J. Reine Angew. Math. **12** (1834), 263–272.
16. A. Jeffrey, H-H. Dai, *Handbook of Mathematical Formulas and Integrals*, Academic Press, Amsterdam, 2008.
17. N. Kilar, Y. Simsek, *A new family of Fubini numbers and polynomials associated with Apostol-Bernoulli numbers and polynomials*, J. Korean Math. Soc. **54**(5) (2017), 1605–1621.
18. ———, *Identities and relations for Fubini type numbers and polynomials via generating functions and p -adic integral approach*, Publ. Inst. Math., Nouv. Sér. **106**(120) (2019), 113–123.
19. ———, *Relations on Bernoulli and Euler polynomials related to trigonometric functions*, Adv. Stud. Contemp. Math. **29**(2) (2019), 191–198.
20. ———, *Two parametric kinds of Eulerian-type polynomials associated with Euler’s formula*, Symmetry **11** (2019), 1–19.
21. D. S. Kim, T. Kim, *Some p -adic integrals on \mathbb{Z}_p associated with trigonometric functions*, Russ. J. Math. Phys. **25**(3) (2018), 300–308.
22. T. Kim, *On a q -analogue of the p -adic log gamma functions and related integrals*, J. Number Theory **76** (1999), 320–329.
23. ———, *q -Volkenborn integration*, Russ. J. Math. Phys. **19** (2002), 288–299.
24. ———, *Sums of powers of consecutive q -integers*, Adv. Stud. Contemp. Math. (Kyungshang) **9**(1) (2004), 15–18.
25. ———, *A note on q -Volkenborn integration*, Proc. Jangjeon Math. Soc. **8**(1) (2005), 13–17.
26. ———, *On the analogs of Euler numbers and polynomials associated with p -adic q -integral on \mathbb{Z}_p at $q = -1$* , J. Math. Anal. Appl. **331**(2) (2007), 779–792.
27. T. Kim, D. S. Kim, J. Kwon, *Analogues of Faulhaber’s formula for poly-Bernoulli and type 2 poly-Bernoulli polynomials*, Montes Taurus J. Pure Appl. Math. Article ID: MTJPAM-D-20-00033, **3**(1) (2021) 1–6
28. T. Kim, S.-H. Rim, Y. Simsek, *A note on the alternating sums of powers of consecutive q -integers*, Adv. Stud. Contemp. Math. (Kyungshang) **13**(2) (2006), 159–164.
29. I. Kucukoglu, Y. Simsek, *Identities and relations on the q -Apostol type Frobenius-Euler numbers and polynomials*, J. Korean Math. Soc. **56**(1) (2019), 265–284.
30. D. E. Knuth, *Johann Faulhaber and sums of powers*, Math. Comput. **61**(203) (1993), 277–294.
31. M. Masjed-Jamei, W. Koepf, *Symbolic computation of some power-trigonometric series*, J. Symb. Comput. **80**(2) (2017), 273–284.
32. M. Masjed-Jamei, M. R. Beyki, W. Koepf, *An extension of the Euler-Maclaurin quadrature formula using a parametric type of Bernoulli polynomials*, Bull. Sci. Math. **156** (2019), 1–26.
33. A. D. Polyinin, A. V. Manzhirov, *Handbook of Mathematics for Engineers and Scientists*, Chapman and Hall/CRC, Boca Raton, 2007.
34. F. Qi, *Derivatives of tangent functions and tangent numbers*, Appl. Math. Comput. **268** (2015), 844–858.
35. J. Riordan, *Combinatorial Identities*, Wiley, New York, 1968.
36. W. H. Schikhof, *Ultrametric Calculus: An Introduction to p -adic Analysis*, Camb. Stud. Adv. Math. 4, Cambridge University Press, Cambridge, 1984.
37. Y. Simsek, *q -Analogue of twisted l -series and q -twisted Euler numbers*, J. Number Theory **110**(2) (2005), 267–278.
38. ———, *Special functions related to Dedekind-type DC-sums and their applications*, Russ. J. Math. Phys. **17**(4) (2010), 495–508.
39. ———, *Complete sum of products of (h, q) -extension of Euler polynomials and numbers*, J. Difference Equ. Appl. **16**(11) (2010), 1331–1348.
40. ———, *Explicit formulas for p -adic integrals: Approach to p -adic distributions and some families of special numbers and polynomials*, Montes Taurus J. Pure Appl. Math. **1**(1) (2019), 1–76.
41. ———, *Applications of constructed new families of generating-type functions interpolating new and known classes of polynomials and numbers*, Preprint.

42. Y. Simsek, D. Kim, T. Kim, S.-H. Rim, *A note on the sums of powers of consecutive q -integers*, J. Appl. Funct. Differ. Equ. **1**(1) (2006), 81–88.
43. H. M. Srivastava, J. Choi, *Zeta and q -Zeta Functions and Associated Series and Integrals*, Elsevier, Amsterdam, 2012.
44. L. Tits, *Sur la sommation des puissances numériques*, Mathesis **37** (1923), 353–355.

Department of Mathematics
Faculty of Science University of Akdeniz
Antalya, Turkey
neslihankilar@gmail.com
ysimsek@akdeniz.edu.tr

(Received 15 07 2020)