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WHEN A TOTAL GRAPH ASSOCIATED WITH A COMMUTATIVE RING IS PERFECT?

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ABSTRACT. Let R be a commutative ring with identity, and let $Z(R)$ be the set of zero-divisors of R. The total graph of R is the graph $T(\Gamma(R))$ whose vertices are all elements of R, and two distinct vertices x and y are adjacent if and only if $x + y \in Z(R)$. We investigate the perfectness of the graphs $Z_0(\Gamma(R))$, $T_0(\Gamma(R))$ and $T(\Gamma(R))$, where $Z_0(\Gamma(R))$ and $T_0(\Gamma(R))$ are (induced) subgraphs of $T(\Gamma(R))$ on $Z(R)^* = Z(R) \setminus \{0\}$ and $R^* = R \setminus \{0\}$, respectively.

1. Introduction

The idea of a total graph associated with a commutative ring was first introduced by Anderson and Badawi [**3**]. They studied the connectedness, diameter and girth of total graphs. The Hamiltonian total graphs were investigated by Akbari et al. [**2**]. Moreover, Eulerian total graphs in [**17**], total graphs without zero element in [**4**], complement of total graphs in [**5**], planar total graphs in [**14**], total graphs of noncommutative rings in [**12**], characterization of balanced signed total graphs in [**16**], total graphs of semirings in [**13**] and coloring of some special total graphs in [**1**, **8**] were studied. Some other investigations into properties of total graphs of a commutative ring may be found in [**9**, **15**, **18**]. This paper is devoted to study perfect total graphs.

Throughout this paper, all rings are assumed to be commutative with identity. We denote by $Z(R)$, $\mathrm{Reg}(R)$, $\mathrm{Max}(R)$, $\mathrm{Nil}(R)$ and $U(R)$, the set of all zero-divisor elements of R, the set of regular elements of R, the set of all maximal ideals of R, the set of all nilpotent elements of R and the set of all invertible elements of R, respectively. A non-zero ideal I of R is called *essential*, if I has a non-zero intersection with any non-zero ideal of R. For a subset T of R such that $0 \in T$, let $T^* = T \setminus \{0\}$. For any undefined notation or terminology in poset theory, we refer the reader to [**6**, **7**].

Let G be a graph with the vertex set $V(G)$ and edge set $E(G)$. By $\bar{G}$, we mean the complement graph of G. We write $u - v$, to denote an edge with ends

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u, v. A graph $H = (V_0, E_0)$ is called a *subgraph of* G if $V_0 \subseteq V$ and $E_0 \subseteq E$. Moreover, H is called an *induced subgraph by* V_0, denoted by $G[V_0]$, if $V_0 \subseteq V$ and $E_0 = \{\{u, v\} \in E(G) \mid u, v \in V_0\}$. A complete graph of order n and a complete bipartite graph with part sizes m, n are denoted by K_n and $K_{m,n}$, respectively. Let G_1 and G_2 be two disjoint graphs. The *join* of G_1 and G_2, denoted by $G_1 \vee G_2$, is a graph with the vertex set $V(G_1 \vee G_2) = V(G_1) \cup V(G_2)$ and edge set $E(G_1 \vee G_2) = E(G_1) \cup E(G_2) \cup \{uv \mid u \in V(G_1), v \in V(G_2)\}$. For a graph G, $S \subseteq V(G)$ is called a *clique* if the subgraph induced on S is complete. The number of vertices in the largest clique of graph G is called the *clique number* of G and is often denoted by $\omega(G)$. For a graph G, let $\chi(G)$ denote the *chromatic number* of G, i.e., the minimal number of colors which can be assigned to the vertices of G in such a way that every two adjacent vertices have different colors. Clearly, for every graph G, $\omega(G) \leqslant \chi(G)$. A graph G is said to be *weakly perfect* if $\omega(G) = \chi(G)$. A *perfect graph* G is a graph in which every induced subgraph is weakly perfect. For any undefined notation or terminology in graph theory, we refer the reader to [**19**].

Let R be a commutative ring with identity, and let $Z(R)$ be the set of zero-divisors of R. *The total graph of* R is the (undirected) graph $T(\Gamma(R))$ with vertices all elements of R, and two distinct vertices x and y are adjacent if and only if $x + y \in Z(R)$. Let $\text{Reg}(\Gamma(R))$, $Z_0(\Gamma(R))$ and $T_0(\Gamma(R))$ be (induced) subgraphs of $T(\Gamma(R))$ with vertices $\text{Reg}(R)$, $Z(R)^* = Z(R) \setminus \{0\}$ and $R^* = R \setminus \{0\}$, respectively. In this paper, we investigate the perfectness of the graphs $Z_0(\Gamma(R))$, $T_0(\Gamma(R))$ and $T(\Gamma(R))$. Indeed, for a ring R which is not an integral domain and $\omega(Z_0(\Gamma(R))) < \infty$, we show that the following statements hold (Theorem 2.5).

(1) If $|\text{Max}(R)| \leqslant 2$, then $Z_0(\Gamma(R))$ is perfect.
(2) If $|\text{Max}(R)| \geqslant 3$, then $Z_0(\Gamma(R))$ is perfect if and only if $u + v \neq 1$, for every $u, v \in U(R)$.

Also, for such a ring R we show that, if $|\text{Max}(R)| \neq 2$, then the following statements are equivalent (Theorem 3.1).

(1) $Z_0(\Gamma(R))$ is perfect. (2) $T_0(\Gamma(R))$ is perfect. (3) $T(\Gamma(R))$ is perfect.

2. When $Z_0(\Gamma(R))$ is perfect?

Let R be a ring and $\omega(Z_0(\Gamma(R))) < \infty$. In this section, we give condition under which $Z_0(\Gamma(R))$ is perfect. First, we recall the following results from [**3**].

THEOREM 2.1. [**3**, Theorem 2.1]. *Let R be a ring such that $Z(R)$ is an ideal of R. Then $Z(\Gamma(R))$ is a complete (induced) subgraph of $T(\Gamma(R))$ and $Z(\Gamma(R))$ is disjoint from* $\text{Reg}(\Gamma(R))$.

THEOREM 2.2. [**3**, Theorem 2.2]. *Let R be a ring such that $Z(R)$ is an ideal of R, and let $|Z(R)| = \alpha$ and $|R/Z(R)| = \beta$.*

(1) *If $2 \in Z(R)$, then* $\text{Reg}(\Gamma(R))$ *is the union of $\beta - 1$ disjoint K'_αs.*
(2) *If $2 \notin Z(R)$, then* $\text{Reg}(\Gamma(R))$ *is the union of $(\beta - 1)/2$ disjoint $K'_{\alpha,\alpha}$s.*

Using Theorems 2.1 and 2.2, we state the following result.

THEOREM 2.3. *Let R be an Artinian local ring. Then the graphs $Z_0(\Gamma(R))$, $Z(\Gamma(R))$, $\mathrm{Reg}(\Gamma(R))$, $T_0(\Gamma(R))$ and $T(\Gamma(R))$ are perfect.*

PROOF. Let $|Z(R)| = \alpha$ and $|R/Z(R)| = \beta$. By Theorems 2.1 and 2.2,

$$T(\Gamma(R)) = \begin{cases} \underbrace{K^\alpha \cup K^\alpha \cup \cdots \cup K^\alpha}_{\beta \text{ copies}} & \text{if } 2 \in Z(R) \\ K^\alpha \cup \underbrace{K^{\alpha,\alpha} \cup K^{\alpha,\alpha} \cup \cdots \cup K^{\alpha,\alpha}}_{(\beta-1)/2 \text{ copies}} & \text{if } 2 \notin Z(R) \end{cases}$$

Clearly, $T(\Gamma(R))$ is perfect and thus all subgraphs $Z_0(\Gamma(R))$, $Z(\Gamma(R))$, $\mathrm{Reg}(\Gamma(R))$ and $T_0(\Gamma(R))$ are perfect. $\square$

The following lemmas will be used frequently in this paper. A different proof for Lemma 2.1 may be found in [**4**].

LEMMA 2.1. *Let R be a ring. If $a \in \mathrm{Nil}(R)$ and $x \in Z(R)$, then $a + x \in Z(R)$.*

PROOF. Assume that $a \in \mathrm{Nil}(R)$. We claim that $\mathrm{ann}_R(a)$ is an essential ideal of R. Suppose to the contrary, there exists an ideal I such that $I \cap \mathrm{ann}_R(a) = (0)$. If $x \in I^*$, then $0 \neq ax \in I$, and thus $aax = a^2x \neq 0$. By continuing this procedure, $a^n x \neq 0$, for every positive integer n, a contradiction and so the claim is proved. Therefore, $\mathrm{ann}_R(a) \cap \mathrm{ann}_R(x) \neq (0)$, for every $x \in Z(R)^*$ and thus $z(a + x) = 0$, for every $0 \neq z \in \mathrm{ann}_R(a) \cap \mathrm{ann}_R(x)$. This means that $a + x \in Z(R)$. $\square$

LEMMA 2.2. *Let R be a ring which is not an integral domain. Then the following statements are equivalent:*

(1) $\omega(Z_0(\Gamma(R))) < \infty$. (2) $\chi(Z_0(\Gamma(R))) < \infty$. (3) *$R$ is a finite ring.*

PROOF. (3) $\Rightarrow$ (1), (2) are clear. If we prove (1) $\Rightarrow$ (3), then (2) $\Rightarrow$ (3) is easily obtained. Thus (1) $\Rightarrow$ (3) is the only thing to prove.

(1) $\Rightarrow$ (3) Assume that $\omega(Z_0(\Gamma(R))) < \infty$. We show that $|R| < \infty$. If $\mathrm{Nil}(R) = Z(R)$, then by Lemma 2.1, $Z_0(\Gamma(R))$ is a complete graph. This means that $|Z(R)| < \infty$ and thus R is a finite ring (see [**11**]). If $\mathrm{Nil}(R) \neq Z(R)$, let $x \in Z(R) \setminus \mathrm{Nil}(R)$. The set $\{x^n \mid n \in \mathbb{N}\}$ is a clique of $Z_0(\Gamma(R))$, and thus $x^n = x^m$, for some distinct positive integers n, m. We can let $n < m$ and hence $x^n(1 - x^{m-n}) = 0$. This implies that $Rx^{m-n} + \mathrm{ann}_R(x^n) = R$. Therefore, for some $k \geqslant n$, $Rx^k + \mathrm{ann}_R(x^n) = R$. Since $(Rx^k)(\mathrm{ann}_R(x^n)) = (0)$, we have $R \cong R_1 \times R_2$, where R_1 and R_2 are two rings. If $a, b \in R_1^*$, then $(a, 0)$ is adjacent to $(b, 0)$. Therefore, the set $\{(a, 0) \mid a \in R_1^*\}$ is finite and thus R_1 is a finite ring. Similarly, R_2 is a finite ring, as desired. $\square$

To state our main result in this section, we need the following celebrate result.

THEOREM 2.4. [**10**, The Strong Perfect Graph Theorem]. *A graph G is perfect if and only if neither G nor $\bar{G}$ contain an induced odd cycle of length at least* 5.

THEOREM 2.5. *Let R be a ring which is not an integral domain and $\omega(Z_0(\Gamma(R))) < \infty$. Then the following statements hold.*

(1) *If $|\mathrm{Max}(R)| \leqslant 2$, then $Z_0(\Gamma(R))$ is perfect.*

(2) *If* $|\operatorname{Max}(R)| \geqslant 3$*, then* $Z_0(\Gamma(R))$ *is perfect if and only if* $u+v \neq 1$*, for every* $u, v \in U(R)$.

Proof. (1) If $|\operatorname{Max}(R)| = 1$, then $Z(R) = \operatorname{Nil}(R)$ and hence $Z_0(\Gamma(R))$ is a complete graph, by Lemma 2.1. Thus $Z_0(\Gamma(R))$ is perfect. So, one may suppose that $\operatorname{Max}(R) = \{\mathfrak{m}_1, \mathfrak{m}_2\}$. Since R is an Artinian ring, $Z(R) = \mathfrak{m}_1 \cup \mathfrak{m}_2$. We show that

$$Z_0(\Gamma(R)) = K_{|\mathfrak{m}_1^* \cap \mathfrak{m}_2^*|} \vee (K_{|\mathfrak{m}_1 \setminus \mathfrak{m}_2|} \cup K_{|\mathfrak{m}_2 \setminus \mathfrak{m}_1|}).$$

Since $\mathfrak{m}_1 \cap \mathfrak{m}_2 = \operatorname{Nil}(R)$, by Lemma 2.1, every $x \in \mathfrak{m}_1^* \cap \mathfrak{m}_2^*$ is adjacent to all other vertices in $Z_0(\Gamma(R))$. Now, let $x, y \in \mathfrak{m}_1 \setminus \mathfrak{m}_2$. Clearly, x is adjacent to y. This means that $Z_0(\Gamma(R))[\mathfrak{m}_1 \setminus \mathfrak{m}_2]$ is a complete subgraph of $Z_0(\Gamma(R))$. Similarly, $Z_0(\Gamma(R))[\mathfrak{m}_2 \setminus \mathfrak{m}_1]$ is a complete subgraph of $Z_0(\Gamma(R))$. If $x \in \mathfrak{m}_1 \setminus \mathfrak{m}_2$ and $y \in \mathfrak{m}_2 \setminus \mathfrak{m}_1$, then $x+y \in U(R)$ and thus x is not adjacent to y. Hence

$$Z_0(\Gamma(R)) = K_{|\mathfrak{m}_1^* \cap \mathfrak{m}_2^*|} \vee (K_{|\mathfrak{m}_1 \setminus \mathfrak{m}_2|} \cup K_{|\mathfrak{m}_2 \setminus \mathfrak{m}_1|}).$$

Clearly, in this case $Z_0(\Gamma(R))$ is perfect, too.

(2) Let $|\operatorname{Max}(R)| = n \geqslant 3$ and $Z_0(\Gamma(R))$ be perfect. By Lemma 2.2 and [**6**, Theorem 8.7], $R \cong R_1 \times \cdots \times R_n$, where R_i is an Artinian local ring, for every $1 \leqslant i \leqslant n$. If there exist $u, v \in U(R)$ such that $u+v=1$, then there are $u_i, v_i \in U(R_i)$ with $u_i + v_i = 1$, for every $1 \leqslant i \leqslant n$. Let $A = \{x, y, z, w, e\}$, where

$$\begin{aligned} x &= (u_1, 0, -u_3, u_4, 0, -u_6, \ldots), \quad & y &= (-v_1, 0, v_3, -v_4, 0, v_6, \ldots), \\ z &= (v_1, u_2, 0, v_4, u_5, 0, \ldots), & w &= (0, -u_2, 0, 0, -u_5, 0, \ldots), \\ e &= (0, v_2, u_3, 0, v_5, u_6, \ldots). \end{aligned}$$

It is not hard to check that $Z_0(\Gamma(R))[A]$ is an induced cycle of length 5 (see the following figure).

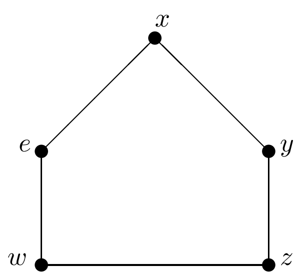

Figure 1. $Z_0(\Gamma(R))[A]$

This contradicts Theorem 2.4. Thus $u+v \neq 1$, for every $u, v \in U(R)$.

Conversely, assume that $u+v \neq 1$, for every $u, v \in U(R)$. Since $\omega(Z_0(\Gamma(R))) < \infty$, by Lemma 2.2 and [**6**, Theorem 8.7], $R \cong R_1 \times \cdots \times R_n$, where every R_i is an Artinian local ring. We show that $Z_0(\Gamma(R))$ is perfect. By Theorem 2.4, it is enough to show that $Z_0(\Gamma(R))$ and $\overline{Z_0(\Gamma(R))}$ contain no induced odd cycle of length at least 5. Indeed, we have the following claims:

Claim 2.1. $Z_0(\Gamma(R))$ contains no induced odd cycle of length at least 5. Assume to the contrary, $a_1 - a_2 - \cdots - a_n - a_1$ is an induced odd cycle of length at least 5 in $Z_0(\Gamma(R))$. We show that $u_1 + v_1 = w_1$, for some $u_1, v_1, w_1 \in U(R_1)$. To see this, let

$$a_1 = (x_{1_{a_1}}, x_{2_{a_1}}, \ldots, x_{n_{a_1}}), \quad a_2 = (x_{1_{a_2}}, x_{2_{a_2}}, \ldots, x_{n_{a_2}}), \ \ldots,$$
$$a_n = (x_{1_{a_n}}, x_{2_{a_n}}, \ldots, x_{n_{a_n}}), \quad A = \{x_{1_{a_1}}, x_{1_{a_2}}, \ldots, x_{1_{a_n}}\}.$$

Since $a_1 - a_2 - \cdots - a_n - a_1$ is an induced odd cycle of length at least 5, $|A \cap Z(R_1)| \leqslant 2$. This together with $R_1 = Z(R_1) \cup U(R_1)$ implies that the other elements in A are invertible in R_1. With no loss of generality, we may assume that

$$x_{1_{a_1}}, x_{1_{a_2}}, x_{1_{a_3}} \in U(R_1).$$

Since a_1 is not adjacent to a_3, $x_{1_{a_1}} + x_{1_{a_3}} = w_1$, for some $w_1 \in U(R_1)$. Hence there exist $u_1, v_1 \in U(R_1)$ such that $u_1 + v_1 = 1$. Similarly, one may show that there exist $u_i, v_i \in U(R_i)$ such that $u_i + v_i = 1$, for every $2 \leqslant i \leqslant n$. Thus $u + v = 1$, for some $u, v \in U(R)$, a contradiction. Therefore $Z_0(\Gamma(R))$ contains no induced odd cycle of length at least 5.

Claim 2.2. $\overline{Z_0(\Gamma(R))}$ contains no induced odd cycle of length at least 5. Assume to the contrary, $a_1 - a_2 - \cdots - a_n - a_1$ is an induced odd cycle of length at least 5 in $\overline{Z_0(\Gamma(R))}$. First, we show that $u_1 + v_1 = w_1$, for some $u_1, v_1, w_1 \in U(R_1)$. To see this, let

$$a_1 = (x_{1_{a_1}}, x_{2_{a_1}}, \ldots, x_{n_{a_1}}), \ a_2 = (x_{1_{a_2}}, x_{2_{a_2}}, \ldots, x_{n_{a_2}}), \ \ldots,$$
$$a_n = (x_{1_{a_n}}, x_{2_{a_n}}, \ldots, x_{n_{a_n}}).$$

Since a_1 is adjacent to a_2, at least one of the $x_{1_{a_1}}, x_{1_{a_2}}$ is invertible in R_1. With no loss of generality, we may assume that $x_{1_{a_1}} \in U(R_1)$. If $x_{1_{a_2}} \in U(R_1)$, then $x_{1_{a_1}} + x_{1_{a_2}} \in U(R_1)$ and thus $u_1 + v_1 = w_1$, for some $u_1, v_1, w_1 \in U(R_1)$. If $x_{1_{a_2}} \notin U(R_1)$, then $x_{1_{a_3}} \in U(R_1)$. If $x_{1_{a_4}} \in U(R_1)$, then $x_{1_{a_3}} + x_{1_{a_4}} \in U(R_1)$ and hence there exist $u_1, v_1, w_1 \in U(R_1)$ such that $u_1 + v_1 = w_1$. Otherwise, $x_{1_{a_5}} \in U(R_1)$. Since n is an odd integer, if we continue this procedure up to $t \leqslant n$, we have $x_{1_{a_1}} + x_{1_{a_t}} \in U(R_1)$ and so $u_1 + v_1 = w_1$, for some $u_1, v_1, w_1 \in U(R_1)$. This implies that there exist $u_1, v_1 \in U(R_1)$ such that $u_1 + v_1 = 1$. Similarly, one may show that there exist $u_i, v_i \in U(R_i)$ such that $u_i + v_i = 1$, for every $2 \leqslant i \leqslant n$. Hence $u + v = 1$, for some $u, v \in U(R)$, a contradiction. Therefore $\overline{Z_0(\Gamma(R))}$ contains no induced odd cycle of length at least 5.

Now, by Claims 2.1, 2.2 and Theorem 2.4, $Z_0(\Gamma(R))$ is a perfect graph. □

Corollary 2.1. *Let R be a ring which is not an integral domain and*

$$\omega(Z_0(\Gamma(R))) < \infty.$$

If $|\operatorname{Max}(R)| \geqslant 3$ and $2 \in U(R)$, then $Z_0(\Gamma(R))$ is not perfect.

3. When $T_0(\Gamma(R))$ and $T(\Gamma(R))$ are perfect?

In this section, we investigate the perfectness of $T_0(\Gamma(R))$ and $T(\Gamma(R))$. First consider the following simple example.

EXAMPLE 3.1. (1) Let $R = \mathbb{Z}_3 \times \mathbb{Z}_5$. Then $Z_0(\Gamma(R)) = K_2 \cup K_4$ and thus it is perfect.

(2) Let $R = \mathbb{Z}_3 \times \mathbb{Z}_5$. The following cycle shows that $T_0(\Gamma(R))$ has an induced cycle of length 5, i.e., it is not perfect and hence $T(\Gamma(R))$ is not perfect.

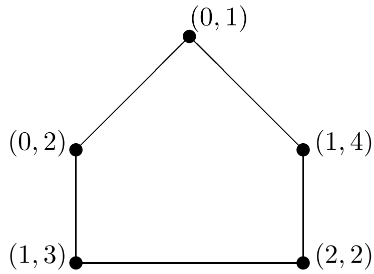

FIGURE 2. Induced cycle of length 5 in $T_0(\Gamma(\mathbb{Z}_3 \times \mathbb{Z}_5))$

(3) Let $R = \mathbb{Z}_3 \times \mathbb{Z}_3$. The following figure shows that $\overline{T_0(\Gamma(R))}$ is perfect. Thus by Theorem 2.4, $T_0(\Gamma(R))$ is perfect.

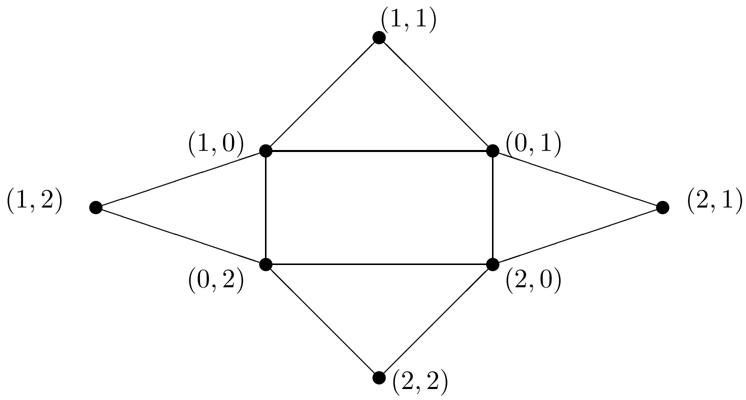

FIGURE 3. $\overline{T_0(\Gamma(\mathbb{Z}_3 \times \mathbb{Z}_3))}$

(4) Let $R = \mathbb{Z}_3 \times \mathbb{Z}_3$. The following cycle shows that $T(\Gamma(R))$ has an induced cycle of length 5 and thus it is not perfect.

Example 3.1 shows that if $|\operatorname{Max}(R)| = 2$, then we do not have definite results on perfectness of $T_0(\Gamma(R))$ and $T(\Gamma(R))$.

THEOREM 3.1. *Let R be a ring which is not an integral domain and $\omega(Z_0(\Gamma(R))) < \infty$. If $|\operatorname{Max}(R)| \neq 2$, then the following statements are equivalent:*

(1) $Z_0(\Gamma(R))$ *is perfect.* (2) $T_0(\Gamma(R))$ *is perfect.* (3) $T(\Gamma(R))$ *is perfect.*

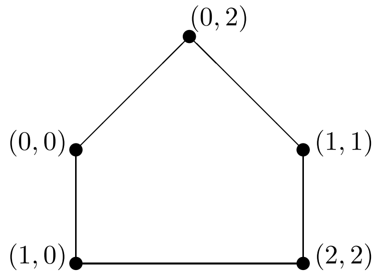

FIGURE 4. Induced cycle of length 5 in $T(\Gamma(\mathbb{Z}_3 \times \mathbb{Z}_3))$

PROOF. If $|\operatorname{Max}(R)| = 1$, then the proof is obtained from Theorem 2.3. Hence suppose that $|\operatorname{Max}(R)| \geqslant 3$.

(1) $\Rightarrow$ (2) The argument here is a refinement of the proof of Theorem 2.5. By this proof it is easily seen that neither $T_0(\Gamma(R))$ nor $\overline{T_0(\Gamma(R))}$ contains an induced odd cycle of length at least 5 and thus is perfect (replace $T_0(\Gamma(R))$ with $Z_0(\Gamma(R))$).

(2) $\Rightarrow$ (3) We show that $T(\Gamma(R))$ and $\overline{T(\Gamma(R))}$ contain no induced odd cycle of length at least 5. Indeed, we have the following claims:

CLAIM 3.1. $T(\Gamma(R))$ contains no induced odd cycle of length at least 5. Assume to the contrary, $a_1 - a_2 - \cdots - a_n - a_1$ is an induced cycle of length at least 5 in $T(\Gamma(R))$. Since $T_0(\Gamma(R))$ is perfect, $\{a_1, a_2, \ldots, a_n\} \nsubseteq V(T_0(\Gamma(R)))$. Hence there exists $1 \leqslant i \leqslant n$ such that $a_i = 0$ and so $a_{i-1}, a_{i+1} \in Z(R)$. Hence $\{a_1, a_2, \ldots, a_n\} \setminus \{a_{i-1}, a_i, a_{i+1}\} \subseteq U(R)$. Also, since a_{i-1} is not adjacent to a_{i+1}, $a_{i-1} + a_{i+1} \in U(R)$. This implies that if $x_i \in Z(R_i)$, for some $1 \leqslant i \leqslant n$, then $y_i \in U(R_i)$ and if there exists $1 \leqslant j \leqslant n$ such that $y_j \in Z(R_j)$, then $x_j \in U(R_j)$, where $a_{i+1} = (x_1, x_2, \ldots, x_n)$ and $a_{i-1} = (y_1, y_2, \ldots, y_n)$. Let $x' = (z_1, z_2, \ldots, z_n)$, where $z_i \in U(R_i)$ such that $z_i = x_i$ or $z_i = y_i$. Let $a_{i+2} = (u_1, u_2, \ldots, u_n)$, $a_{i+3} = (v_1, v_2, \ldots, v_n)$ and put $y' = (w_1, w_2, \ldots, w_n)$ such that $w_i = u_i$ if $z_i = y_i$ in x' and $w_j = v_j$ if $z_j = x_j$.

Now, we can easily see that $y', x' \in U(R)$ such that $y' + x' \in U(R)$. Thus by Theorem 2.3, $Z_0(\Gamma(R))$ is not perfect, a contradiction.

CLAIM 3.2. $\overline{T(\Gamma(R))}$ contains no induced odd cycle of length at least 5. This proof is similar to the proof of Claim 1.

By Claims 3.1, 3.2 and Theorem 2.4, $T(\Gamma(R))$ is a perfect graph.

(3) $\Rightarrow$ (1) is clear. □

We close this paper, with the following question.

Let R be a ring, $\omega(Z_0(\Gamma(R))) < \infty$ and $|\operatorname{Max}(R)| = 2$. What condition $T(\Gamma(R))$ is perfect under?

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