PUBLICATIONS DE L'INSTITUT MATHÉMATIQUE
Nouvelle série, tome 107(121) (2020), 75–83 DOI: https://doi.org/10.2298/PIM2021075K

A NOTE ON THE CATEGORY OF QUASI-PROXIMITY SPACES

Muammer Kula and Samed Özkan

Abstract. We characterize the separation properties T_0 and T_1 at a point p in the category of quasi-proximity spaces. Moreover, the (strongly) closed and (strongly) open subobjects of an object, and each of the various notions of connected and compact objects are characterized in this topological category.

1. Introduction

Proximity structure was introduced by Efremovich in 1951 [**15, 16**]. He characterized the proximity relation "A is close to B" as a binary relation on subsets of a set X. Some researchers such as Leader, Lodato and Pervin have worked with weaker axioms than Efremovich's proximity axioms. In this way, some generalized proximities were appeared. One of these is quasi-proximity relation. The concept of quasi-proximities introduced by Pervin [**24**] in 1963.

Proximity, quasi-proximity and uniformity are important concepts close to topology and they have rich topological properties. For this reason, in recent years, these notions constitute a significant research area in the field of topological spaces.

Baran [**2**] gave various generalizations of the usual separation properties of topology for an arbitrary topological category over sets. One of the uses of local separation properties is to define the notions of closedness and strong closedness [**3**] in set based topological categories. These notions form an appropriate closure operator in the sense of Dikranjan and Giuli [**14**] in convergence spaces [**8**], preordered spaces [**9**] and semiuniform convergence spaces [**11**].

In [**20**], we characterized various topological notions such as separation, closedness, connectedness in the category of proximity spaces. In this paper, we obtain more general results such as $\bar{T}_0$ at p, T_1 at p, (strongly) closedness, connectedness, compactness for quasi-proximity spaces and furthermore, we investigate the relationship between the notion of closedness in usual sense and the notion of (strongly) closedness (in our sense) in quasi-proximity spaces.

2010 *Mathematics Subject Classification:* Primary 54B30; Secondary 54D10; 54A05; 54E05; 18B99.

Key words and phrases: topological category, quasi-proximity space, separation, closedness, connectedness, compactness.

Communicated by Miloš Kurilić.

2. Preliminaries

The following are some basic definitions and notations which we will use throughout the paper.

Definition 2.1. [**23**] A quasi-proximity or P-proximity space is a pair (X, δ), where X is a set and δ is a binary relation on the powerset of X such that

(Q1) $A\delta B$ implies $A, B \neq \emptyset$;
(Q2) $(A \cup B)\delta C$ iff $A\delta C$ or $B\delta C$;
(Q3) $C\delta(A \cup B)$ iff $C\delta A$ or $C\delta B$;
(Q4) $A \cap B \neq \emptyset$ implies $A\delta B$;
(Q5) $A\bar{\delta}B$ implies there is an $E \subseteq X$ such that $A\bar{\delta}E$ and $(X - E)\bar{\delta}B$;

where $A\bar{\delta}B$ means it is not true that $A\delta B$.

If δ satisfies the symmetry condition $A\delta B$ iff $B\delta A$, then it is called a (Efremovich) proximity on X. The $(Q5)$ axiom is called *strong axiom*, and it plays an important role in the theory of proximity spaces.

A function $f :: (X, \delta) \to (Y, \delta')$ between two (quasi-)proximity spaces is called a *(quasi-)proximity mapping* iff $f(A)\delta' f(B)$ whenever $A\delta B$. It can easily be shown that f is a (quasi-)proximity mapping iff $f^{-1}(C)\bar{\delta}f^{-1}(D)$ whenever $C\bar{\delta}'D$.

In a (quasi-)proximity space (X, δ), we write $A \ll B$ if and only if $A\bar{\delta}(X - B)$. The relation $\ll$ is called p-neighborhood relation or the strong inclusion. When $A \ll B$, we say that B is a p-neighborhood of A or A is strongly contained in B [**17**, **23**].

We denote the category of proximity spaces and proximity mappings by **Prox**, and the category of quasi-proximity spaces and quasi-proximity mappings by **QProx**. Hunsaker and Sharma [**18**] showed that the functor $\mathcal{U}$: **Prox** $\to$ **Set** is topological. Similarly, **QProx** is a topological category over **Set** [**13**, p. 31], and **Prox** is a full subcategory of **QProx** [**27**, p. 147].

Definition 2.2. Let $\mathfrak{B}$ be a (quasi-)proximity-base on a set X and let a binary relation δ on $P(X)$ be defined as follows: $(A, B) \in \delta$ if, given any finite covers $\{A_i : 1 \leqslant i \leqslant n\}$ and $\{B_j : 1 \leqslant j \leqslant m\}$ of A and B respectively, then there exists a pair (i, j) such that $(A_i, B_j) \in \mathfrak{B}$. δ is a (quasi-)proximity on X finer than the relation $\mathfrak{B}$ [**18**, **22**, **26**].

Definition 2.3. Let X be a non-empty set, for each $i \in I$, (X_i, δ_i) be a (quasi-) proximity space and $f_i \colon X \to (X_i, \delta_i)$ be a source in **Set**. Define a binary relation $\mathfrak{B}$ on $P(X)$ as follows: for $A, B \in P(X)$, $A\mathfrak{B}B$ iff $f_i(A)\delta_i f_i(B)$, for all $i \in I$. $\mathfrak{B}$ is a (quasi-)proximity-base on X [**26**, Theorem 3.8]. The initial (quasi-)proximity structure δ on X generated by the (quasi-)proximity base $\mathfrak{B}$ is given by for $A, B \in P(X)$, $A\delta B$ iff for any finite covers $\{A_i : 1 \leqslant i \leqslant n\}$ and $\{B_j : 1 \leqslant j \leqslant m\}$ of A and B respectively, then there exists a pair (i, j) such that $(A_i, B_j) \in \mathfrak{B}$ [**22**, **26**].

Definition 2.4. Let (X, δ) be a (quasi-)proximity space, Y a nonempty set and f be a function from a (quasi-)proximity space (X, δ) onto a set Y. The strong inclusion $\ll^*$ induced by the finest (quasi-)proximity δ^* (the quotient (quasi-) proximity) on Y making f (quasi-)proximally continuous is given by: for every

$A, B \subset Y$, $A \ll^* B$ if and only if, for each binary rational s in $[0,1]$, there is some $C_s \subset Y$ such that $C_0 = A$, $C_1 = B$ and $s < t$ implies $f^{-1}(C_s) \ll_\delta f^{-1}(C_t)$ [**17**] or [**28**, p. 276], where $\ll_\delta$ represents the strong inclusion induced by the (quasi-) proximity δ on X. In addition, if $f\colon (X,\delta) \to (X,\delta^*)$ be a one-to-one (quasi-) proximity quotient map, then $A\delta^* B$ if and only if $f^{-1}(A)\delta f^{-1}(B)$ [**17**, p. 591].

Definition 2.5. Let X be set and $p \in X$. Let $X \vee_p X$ be the wedge at p [**2**], i.e., two disjoint copies of X identified at p, i.e., the pushout of $p\colon 1 \to X$ along itself (where 1 is the terminal object in **Set**). An epi sink $\{i_1, i_2\colon (X,\delta) \to (X \vee_p X, \delta')\}$ (quasi-proximity mappings), where i_1, i_2 are the canonical injections, in **Prox**(**QProx**) is a final lift if and only if the following statement holds. For each pair A, B in the different component of $X \vee_p X$, $A\delta' B$ iff there exist sets C, D in X such that $C\delta\{p\}$ and $\{p\}\delta D$ with $i_k^{-1}(A) = C$ and $i_j^{-1}(B) = D$ for $k, j = 1, 2$ and $k \neq j$. If A and B are in the same component of wedge, then $A\delta' B$ iff there exist sets C, D in X such that $C\delta D$ and $i_k^{-1}(A) = C$ and $i_k^{-1}(B) = D$ for some $k = 1, 2$. Specially, if $i_k(C) = A$ and $i_k(D) = B$, then $(i_k(C), i_k(D)) \in \delta'$ iff $(i_k^{-1}(i_k(C)), i_k^{-1}(i_k(D))) = (C, D) \in \delta$.

Definition 2.6. Let X be a non-empty set. The discrete (quasi-)proximity structure δ on X is given by for $A, B \subset X$, $A\delta B$ iff $A \cap B \neq \emptyset$, and the indiscrete (quasi-)proximity structure δ on X is given by for $A, B \subset X$, $A\delta B$ iff $A \neq \emptyset$ and $B \neq \emptyset$ [**23**, p. 9].

3. T_0 and T_1 quasi-proximity spaces at a point

In this section, we give the characterizations of T_0 and T_1 quasi-proximity spaces at a point p.

Let B be set and $p \in B$. Let $B \vee_p B$ be the wedge at p. A point x in $B \vee_p B$ will be denoted by $x_1(x_2)$ if x is in the first (resp. second) component of $B \vee_p B$. Note that $p_1 = p_2$.

The principal p-axis map, $A_p\colon B \vee_p B \to B^2$ is defined by $A_p(x_1) = (x, p)$ and $A_p(x_2) = (p, x)$. The skewed p-axis map, $S_p\colon B \vee_p B \to B^2$ is defined by $S_p(x_1) = (x, x)$ and $S_p(x_2) = (p, x)$. The fold map at p, $\nabla_p\colon B \vee_p B \to B$ is given by $\nabla_p(x_i) = x$ for $i = 1, 2$ [**2**].

Definition 3.1. (cf. [**2**]) Let $\mathcal{U}\colon \mathcal{E} \to$ **Set** be a topological functor, X an object in $\mathcal{E}$, p be a point in $\mathcal{U}(X) = B$.

(1) X is $\bar{T}_0$ at p iff the initial lift of the $\mathcal{U}$-source $\{A_p\colon B \vee_p B \to \mathcal{U}(X^2) = B^2$ and $\nabla_p\colon B \vee_p B \to \mathcal{U}\mathcal{D}(B) = B\}$ is discrete, where $\mathcal{D}$ is the discrete functor which is a left adjoint to $\mathcal{U}$.
(2) X is T_0' at p iff the initial lift of the $\mathcal{U}$-source $\{id\colon B \vee_p B \to \mathcal{U}(X \vee_p X) = B \vee_p B$ and $\nabla_p\colon B \vee_p B \to \mathcal{U}\mathcal{D}(B) = B\}$ is discrete, where $X \vee_p X$ is the wedge in $\mathcal{E}$, i.e., the final lift of the $\mathcal{U}$-sink $\{i_1, i_2\colon \mathcal{U}(X) = B \to B \vee_p B\}$ where i_1, i_2 denote the canonical injections.
(3) X is T_1 at p iff the initial lift of the $\mathcal{U}$-source $\{S_p\colon B \vee_p B \to \mathcal{U}(X^2) = B^2$ and $\nabla_p\colon B \vee_p B \to \mathcal{U}\mathcal{D}(B) = B\}$ is discrete.

Theorem 3.1. *Let (X,δ) be a quasi-proximity space and $p \in X$. (X,δ) is $\bar{T}_0$ at p iff for each $x \neq p$, $(\{x\},\{p\}) \notin \delta$ or $(\{p\},\{x\}) \notin \delta$.*

Proof. Let (X,δ) is $\bar{T}_0$ at p. We shall show that the condition holds.

Suppose for some $x \in X$, $(\{x\},\{p\}) \in \delta$ and $(\{p\},\{x\}) \in \delta$ with $x \neq p$. Then, by Definitions 2.3 and 2.6, for $(U,V) \in \delta'$ (δ' is a quasi-proximity structure on $X \vee_p X$) with $U = \{x_1\}$ and $V = \{x_2\}$,

$$\pi_1 A_p(U)\delta\pi_1 A_p(V) = \{x\}\delta\{p\},$$
$$\pi_2 A_p(U)\delta\pi_2 A_p(V) = \{p\}\delta\{x\},$$

where $\pi_i \colon X^2 \to X$, $i = 1,2$, are the projection maps, and

$$\nabla_p(\{x_1\})\delta_d\nabla_p(\{x_2\}) = \{x\}\delta_d\{x\}.$$

But $U \cap V = \emptyset$. This is a contradiction to the fact that (X,δ) is $\bar{T}_0$ at p. For $(V,U) \in \delta'$, similar contradiction can be handled. Hence if $(\{x\},\{p\}) \in \delta$ and $(\{p\},\{x\}) \in \delta$, then $x = p$.

Conversely, suppose that for each $x \neq p$, $(\{x\},\{p\}) \notin \delta$ or $(\{p\},\{x\}) \notin \delta$. We need to show that (X,δ) is $\bar{T}_0$ at p.

Let (U,V) be any set in δ', where δ' is the initial on $X \vee_p X$ induced by the maps A_p and ∇_p, $\pi_1 A_p(U)\delta\pi_1 A_p(V)$, $\pi_2 A_p(U)\delta\pi_2 A_p(V)$ and $\nabla_p(U)\delta_d\nabla_p(V)$.

Since δ_d is the discrete quasi-proximity structure and $\nabla_p U\delta_d\nabla_p V$, then $\nabla_p U \cap \nabla_p V \neq \emptyset$. It follows that there exists $x \in \nabla_p U \cap \nabla_p V$. Hence, there exist $y \in U$ and $z \in V$ such that $\nabla_p y = x = \nabla_p z$.

If $x = p$, then $y = p_i = z$, $(i = 1,2)$ and $p_i \in U \cap V$.

If $x \neq p$, then $y = x_i$, $z = x_j$ $(i,j = 1,2)$. We need to show that $U \cap V \neq \emptyset$.

If $p \in \nabla_p U \cap \nabla_p V$, then $p_i \in U \cap V$ $(i = 1,2)$. Suppose that $p \notin \nabla_p U \cap \nabla_p V$. We show that both U and V are in the first or in the second or in both component of $X \vee_p X$.

If U subset of the first component of $X \vee_p X$ and V subset of the second component of $X \vee_p X$, then $\{x_1\} \subseteq U$ and $\{x_2\} \subseteq V$. It follows that

$$\pi_1 A_p(\{x_1\})\delta\pi_1 A_p(\{x_2\}) = \{x\}\delta\{p\},$$
$$\pi_2 A_p(\{x_1\})\delta\pi_2 A_p(\{x_2\}) = \{p\}\delta\{x\}.$$

Since $(\{x\},\{p\}) \notin \delta$ or $(\{p\},\{x\}) \notin \delta$ (by assumption), $(U,V) \notin \delta'$ by the conditions $(Q2)$ and $(Q3)$ of Definition 2.1.

The case U subset of the second component of $X \vee_p X$ and V subset of the first component of $X \vee_p X$ can be handled similarly. Hence U and V can not be in different component of $X \vee_p X$.

If U (resp. V) subset of the first component of $X \vee_p X$ and V (resp. U) subset of both component of $X \vee_p X$, then $U \supseteq \{x_1\}$ and $V \supseteq \{x_1, x_2\}$ (resp. $V \supseteq \{x_1\}$ and $U \supseteq \{x_1, x_2\}$). If U (resp. V) subset of both component of $X \vee_p X$ and V (resp. U) subset of the second component of $X \vee_p X$, then $U \supseteq \{x_1, x_2\}$ and $V \supseteq \{x_2\}$ (resp. $V \supseteq \{x_1, x_2\}$ and $U \supseteq \{x_2\}$). Hence $U \cap V \neq \emptyset$.

Similarly if U and V are in the first (resp. second) component of $X \vee_p X$, then $U \supseteq \{x_1\}$ and $V \supseteq \{x_1\}$ ($U \supseteq \{x_2\}$ and $V \supseteq \{x_2\}$). If U and V are in both component of $X \vee_p X$, then $U \supseteq \{x_1, x_2\}$ and $V \supseteq \{x_1, x_2\}$. Hence $U \cap V \neq \emptyset$.

If $(\{x_i\}, \{x_i\}) \in \delta'$ $(i = 1, 2)$, then

$$\pi_1 A_p(\{x_1\}) \delta \pi_1 A_p(\{x_1\}) = \{x\} \delta \{x\}, \qquad \pi_2 A_p(\{x_1\}) \delta \pi_2 A_p(\{x_1\}) = \{p\} \delta \{p\},$$
$$\pi_1 A_p(\{x_2\}) \delta \pi_1 A_p(\{x_2\}) = \{p\} \delta \{p\}, \qquad \pi_2 A_p(\{x_2\}) \delta \pi_2 A_p(\{x_2\}) = \{x\} \delta \{x\}.$$

We must have $(U, V) \supseteq (\{x_i\}, \{x_i\})$, $(i = 1, 2)$, i.e., $U \cap V \neq \emptyset$ and consequently, by Definitions 2.3, 2.6 and 3.1, (X, δ) is $\overline{T}_0$ at p. □

Theorem 3.2. *A quasi-proximity space is T_0' at p for every $p \in X$.*

Proof. It is similar to the proof in [**20**, Theorem 3.7]. □

Theorem 3.3. *Let (X, δ) be a quasi-proximity space and $p \in X$. (X, δ) is T_1 at p iff for each $x \neq p$, $(\{x\}, \{p\}) \notin \delta$ or $(\{p\}, \{x\}) \notin \delta$.*

Proof. The proof is similar to the proof of Theorem 3.1 by using the map S_p instead of the map A_p. □

Remark 3.1. (1) Note that for the category **Top** of topological spaces, $\overline{T}_0$ at p, T_0' at p, or T_1 at p reduce to for each $x \in X$, a topological space, with $x \neq p$, there exists a neighborhood of x not containing p or (resp. and) there exists a neighborhood of p not containing x [**5**].

(2)(a) If $\mathcal{U} \colon \mathcal{E} \to$ **Set** be a normalized topological functor, then each of $\overline{T}_0$ at p and T_1 at p implies T_0' at p [**4**, Corollary 2.11].

(b) In a topological category, $\overline{T}_0$ at p and T_1 at p objects may be equivalent, see [**11**,**19**] and all objects may be T_1 at p, for example, it is shown, in [**7**], that all prebornological spaces are T_1 at p. Moreover, T_1 at p objects could be only discrete objects, see [**12**].

4. (Strongly) Closed and (Strongly) Open Subspaces

In this section, the (strongly) closed and (strongly) open subobjects of an object are characterized in the category of quasi-proximity spaces, **QProx**.

Let B be set and $p \in B$. The infinite wedge product $\vee_p^\infty B$ is formed by taking countably many disjoint copies of B and identifying them at the point p. Let $B^\infty = B \times B \times \dots$ be the countable cartesian product of B. Define $A_p^\infty \colon \vee_p^\infty B \to B^\infty$ by $A_p^\infty(x_i) = (p, p, \dots, p, x, p,' \dots)$, where x_i is in the i-th component of the infinite wedge and x is in the i-th place in $(p, p, \dots, p, x, p, \dots)$ (infinite principal p-axis map), and $\nabla_p^\infty \colon \vee_p^\infty B \to B$ by $\nabla_p^\infty(x_i) = x$ for all $i \in I$ (infinite fold map), [**3**].

Definition 4.1. (cf. [**3**]) Let $\mathcal{U} \colon \mathcal{E} \to$ **Set** be a topological functor, X an object in $\mathcal{E}$ with $\mathcal{U}(X) = B$. Let F be a nonempty subset of B. We denote by X/F the final lift of the epi $\mathcal{U}$-sink $q : \mathcal{U}(X) = B \to B/F = (B \backslash F) \cup \{*\}$, where q is the epi map that is the identity on $B \backslash F$ and identifying F with a point $\{*\}$.

Let p be a point in B.

(1) p is closed iff the initial lift of the $\mathcal{U}$-source $\{A_p^\infty \colon \vee_p^\infty B \to \mathcal{U}(X^\infty) = B^\infty$ and $\nabla_p^\infty \colon \vee_p^\infty B \to \mathcal{U}\mathcal{D}(B) = B\}$ is discrete.
(2) $F \subset X$ is closed iff $\{*\}$, the image of F, is closed in X/F or $F = \emptyset$.
(3) $F \subset X$ is strongly closed iff X/F is T_1 at $\{*\}$ or $F = \emptyset$.
(4) If $B = F = \emptyset$, then we define F to be both closed and strongly closed.

THEOREM 4.1. *Let (X, δ) be a quasi-proximity space and $p \in X$. $\{p\}$ is closed in X iff for any $B \subset X$, if $\{p\}\delta B$ or $B\delta\{p\}$, then $p \in B$.*

PROOF. Let $\{p\}$ is closed in X. We show that for any $B \subset X$, if $\{p\}\delta B$ or $B\delta\{p\}$, then $p \in B$. Suppose that $\{p\}\delta B$ or $B\delta\{p\}$ while $p \notin B$, for some $B \subset X$. Then for some $x \neq p$ and $x \in B$, we get $(\{x\}, \{p\}) \in \delta$ and $(\{p\}, \{x\}) \in \delta$ by the conditions $(Q2)$ and $(Q3)$ of Definition 2.1. Let $U = \{x_1\}$ and $V = \{x_2\}$. Note that $(U, V) \in \delta'$ since

$$\pi_1 A_p^\infty U \delta \pi_1 A_p^\infty V = \{x\}\delta\{p\}, \quad \pi_2 A_p^\infty U \delta \pi_2 A_p^\infty V = \{p\}\delta\{x\},$$

and for $i \geqslant 3$,

$$\pi_i A_p^\infty U \delta \pi_i A_p^\infty V = \{p\}\delta\{p\},$$

where $\pi_i \colon X^\infty \to X$ are the projection maps, and

$$\nabla_p^\infty(\{x_1\})\delta_d \nabla_p^\infty(\{x_2\}) = \{x\}\delta_d\{x\},$$

where δ_d is the discrete quasi-proximity structure on X. But $U \cap V = \emptyset$. This is a contradiction to the fact that δ' is discrete. For $(V, U) \in \delta'$ can be handled similarly.

If the condition holds, then by using the similar argument in the proof in [**20**, Theorem 4.5], $\{p\}$ is closed in X. □

THEOREM 4.2. *Let (X, δ) be a quasi-proximity space. Then $\emptyset \neq F \subset X$ is (strongly) closed iff $x \in F$ whenever $\{x\}\delta F$ or $F\delta\{x\}$ for any $x \in X$.*

PROOF. It can be proven by using the same argument in the proof in [**20**, Theorems 4.6 and 4.7]. □

DEFINITION 4.2. Let $\mathcal{E}$ be a topological category over **Set**, X an object in $\mathcal{E}$ and F be a nonempty subset of X.

(1) $F \subset X$ is open iff F^c, the complement of F, is closed in X.
(2) $F \subset X$ is strongly open iff F^c, the complement of F, is strongly closed in X [**10**].

THEOREM 4.3. *Let (X, δ) be a quasi-proximity space. $\emptyset \neq F \subset X$ is (strongly) open iff $x \in F^c$ whenever $\{x\}\delta F^c$ or $F^c\delta\{x\}$ for all $x \in X$.*

PROOF. It follows from Theorem 4.2 and Definition 4.2. □

DEFINITION 4.3. [**23**, p. 106] Let (X, δ) be a quasi-proximity space and $A \subset X$. Define $\bar{A} = \{x \mid x\delta A \text{ or } A\delta x\}$ and if $\bar{A} = A$, then A is said to be closed.

THEOREM 4.4. *Let (X, δ) be a quasi-proximity space and $A \subset X$. A is closed (in the usual sense) iff A is (strongly) closed (in our sense).*

PROOF. It follows from Theorem 4.2 and Definition 4.3. □

In **Top**, the notion of closedness coincides with the usual closedness [**3**], and F is strongly closed iff F is closed and for each $x \notin F$ there exists a neighbourhood of F missing x, [**3**]. In **QProx**, by Theorem 4.4, the notion of (strong) closedness coincide with the usual notion of closedness. The notion of (strong) closedness induces closure operators in sense of Dikranjan and Giuli [**14**] in convergence spaces [**8**], preordered spaces [**9**] and semiuniform convergence spaces [**11**].

5. Connectedness and Compactness

In this section, the characterization of each of the notions of (strongly) connected and (strongly) compact objects in the category of quasi-proximity spaces are given.

DEFINITION 5.1. Let $\mathcal{E}$ be a topological category over **Set** and X be an object in $\mathcal{E}$.

(1) X is connected iff the only subsets of X both strongly open and strongly closed are X and $\emptyset$ [**10**].
(2) X is strongly connected iff the only subsets of X both open and closed are X and $\emptyset$ [**10**].
(3) X is D-connected iff any morphism from X to any discrete object is constant (cf. [**1, 10, 21, 25**]).

THEOREM 5.1. *Let (X, δ) be a quasi-proximity space.*

(1) *(X, δ) is (strongly) connected iff for any non-empty proper subset F of X, either the condition (a) or (b) holds.*
   (a) *$x \notin F$ whenever $\{x\}\delta F$ or $F\delta\{x\}$ for some $x \in X$.*
   (b) *$x \notin F^c$ whenever $\{x\}\delta F^c$ or $F^c\delta\{x\}$ for some $x \in X$.*
(2) *(X, δ) is D-connected iff $\{x\}\delta\{y\}$ or $\{y\}\delta\{x\}$ for all $x, y \in X$ with $x \neq y$.*

PROOF. It is similar to the proof in [**20**, Theorems 4.13 and 4.14]. □

DEFINITION 5.2. [**6**] Let $\mathcal{E}$ be a topological category over **Set**, X and Y be objects in $\mathcal{E}$, and $f : X \to Y$ a morphism in $\mathcal{E}$. Then,

(1) f is said to be closed iff the image of each closed subobject of X is a closed subobject of Y.
(2) f is said to be strongly closed iff the image of each strongly closed subobject of X is a strongly closed subobject of Y.
(3) X is compact iff the projection $\pi_2 : X \times Y \to Y$ is closed for each object Y in $\mathcal{E}$.
(4) X is strongly compact iff the projection $\pi_2 : X \times Y \to Y$ is strongly closed for each object Y in $\mathcal{E}$.

THEOREM 5.2. (1) *Let $f : (X, \delta) \to (Y, \delta')$ be a quasi-proximity mapping. If $D \subset Y$ is (strongly) closed, so also is $f^{-1}(D)$.*

(2) *Let (Y, δ') be a quasi-proximity space. If $N \subset Y$ is (strongly) closed and $M \subset N$ is (strongly) closed, so also is $M \subset Y$.*

Proof. (1) Suppose $D \subset Y$ is (strongly) closed and $x \in f^{-1}(D)$. By Theorem 4.2, $y \in D$ whenever $\{y\}\delta' D$ or $D\delta'\{y\}$ for all $y \in Y$. We need to show that, $x \in f^{-1}(D)$ whenever $\{x\}\delta f^{-1}(D)$ or $f^{-1}(D)\delta\{x\}$ for all $x \in X$. Note that $f(x) \in f(f^{-1}(D)) \subset D$ and $\{f(x)\}\delta' D$ or $D\delta'\{f(x)\}$ since f is a quasi-proximity mapping and $D \subset Y$ is (strongly) closed. Thus, $f^{-1}(D)$ is (strongly) closed.

(2) Suppose $N \subset Y$ and $M \subset N$ are (strongly) closed, $y \in Y$ and there exists $a \in M$ such that $y\delta' a$ or $a\delta' y$. By Theorem 4.2, we need to show that $y \in M$. Since $N \subset Y$ is (strongly) closed and $M \subset N$, by Theorem 4.2, $y \in N$. It follows that $y \in M$ since $M \subset N$ is (strongly) closed. □

Theorem 5.3. *Every quasi-proximity space is (strongly) compact.*

Proof. Let (X, δ) be a quasi-proximity space. By Definition 5.2(3) (5.2(4)), we need to show that for each proximity space (Y, δ'), the projection $\pi_2 \colon (X, \delta) \times (Y, \delta') \to (Y, \delta')$ is (strongly) closed.

Suppose $M \subset X \times Y$ is (strongly) closed. To show that $\pi_2 M$ is (strongly) closed, we assume the contrary and apply Theorem 4.2. So, for some point $b \in Y$, $b \notin \pi_2 M$ whenever $\{b\}\delta'\pi_2 M$ or $\pi_2 M\delta'\{b\}$. Since $M \subset X \times Y$ is (strongly) closed, $(a, b) \in M$ whenever $\{(a, b)\}\delta'' M$ or $M\delta''\{(a, b)\}$ for all $(a, b) \in X \times Y$, where δ'' is the product quasi-proximity structure on $X \times Y$. Hence $\pi_2\{(a, b)\}\delta'\pi_2 M = \{b\}\delta'\pi_2 M$ or $\pi_2 M\delta'\pi_2\{(a, b)\} = \pi_2 M\delta'\{b\}$, by definition of the product quasi-proximity structure. Since $(a, b) \in M$, $\pi_2(a, b) = b \in \pi_2 M$. This is a contradiction to the fact that M is (strongly) closed. Hence, by Theorem 4.2, $\pi_2 M$ must be (strongly) closed and consequently, by Definition 5.2(3) (5.2(4)), (X, δ) is (strongly) compact. □

Theorem 5.4. *Let $f \colon (X, \delta) \to (Y, \delta')$ be a quasi-proximity mapping. If (X, δ) is (strongly) compact, then the subspace $f(X)$ is (strongly) compact.*

Proof. It follows from Theorem 5.3. □

Acknowledgement. We would like to thank the referee for his/her valuable and helpful suggestions that improved the paper.

References

1. V. A. Arhangel'skii, R. Wiegandt, *Connectedness and disconnectedness in topology*, Gen. Topology Appl. **5** (1975), 9–33.
2. M. Baran, *Separation properties*, Indian J. Pure Appl. Math. **23** (1992), 333–341.
3. ______, *The notion of closedness in topological categories*, Commentat. Math. Univ. Carol. **34** (1993), 383–395.
4. ______, *Generalized local separation properties*, Indian J. Pure Appl. Math. **25** (1994), 615–620.
5. ______, *Separation properties in topological categories*, Math. Balk. **10** (1996), 39–48.
6. ______, *A notion of compactness in topological categories*, Publ. Math. **50**(3–4) (1997), 221–234.
7. ______, *Completely regular objects and normal objects in topological categories*, Acta Math. Hung. **80**(3) (1998), 211–224.
8. ______, *Closure operators in convergence spaces*, Acta Math. Hung. **87** (2000), 33–45.

9. M. Baran, J. Al-Safar, *Quotient-reflective and bireflective subcategories of the category of preordered sets*, Topology Appl. **158** (2011), 2076–2084.
10. M. Baran, M. Kula, *A note on connectedness*, Publ. Math. **68**(3–4) (2006), 489–501.
11. M. Baran, S. Kula, T. M. Baran, M. Qasim, *Closure operators in semiuniform convergence spaces*, Filomat **27**(4) (2013), 537–546.
12. T. M. Baran, M. Kula, T_1 *Extended pseudo-quasi-semi metric spaces*, Math. Sci. Appl. E-Notes **5** (2017), 40–45.
13. G. C. L. Brümmer, *Topological categories*, Topology Appl. **18** (1984), 27–41.
14. D. Dikranjan, E. Giuli, *Closure operators I*, Topology Appl. **27** (1987), 129–143.
15. V. A. Efremovich, *Infinitesimal spaces*, Dokl. Akad. Nauk SSSR **76** (1951), 341–343. (in Russian)
16. ———, *The geometry of proximity I*, Mat. Sb. **31** (1952), 189–200. (in Russian)
17. L. M. Friedler, *Quotients of proximity spaces*, Proc. Am. Math. Soc. **37**(2) (1973), 589–594.
18. W. N. Hunsaker, P. L. Sharma, *Proximity spaces and topological functors*, Proc. Am. Math. Soc. **45**(3) (1974), 419–425.
19. M. Kula, *A note on Cauchy spaces*, Acta Math. Hung. **133**(1–2) (2011), 14–32.
20. M. Kula, T. Maraşlı, S. Özkan, *A note on closedness and connectedness in the category of proximity spaces*, Filomat **28**(7) (2014), 1483–1492.
21. H. Lord, *Connectedness and disconnectedness*, Ann. N. Y. Acad. Sci. **767** (1995), 115–139.
22. J. M. Metzger, *Quasi-Uniform and Quasi-Proximity Spaces*, PhD Thesis, The University of Connecticut, 1970.
23. S. A. Naimpally, B. D. Warrack, *Proximity Spaces*, Cambridge University Press, Cambridge, 1970.
24. W. J. Pervin, *Quasi-proximities for topological spaces*, Math. Ann. **150** (1963), 325–326.
25. G. Preuss, *Theory of Topological Structures: An Approach to Topological Categories*, D. Reidel Publ. Co., Dordrecht, 1988.
26. P. L. Sharma, *Proximity bases and subbases*, Pac. J. Math. **37**(2) (1971), 515–526.
27. A. Tozzi, O. Wyler, *On categories of supertopological spaces*, Acta Univ. Carol., Math. Phys. **28**(2) (1987), 137–149.
28. S. Willard, *General Topology*, Addison-Wesley, Reading, Mass., 1970.

Department of Mathematics
Erciyes University
Kayseri
Turkey
kulam@erciyes.edu.tr

(Received 07 04 2016)
(Revised 05 02 2020)

Department of Mathematics
Nevşehir Hacı Bektaş Veli University
Nevşehir
Turkey
ozkans@nevsehir.edu.tr