A NOTE ON THE CATEGORY OF QUASI-PROXIMITY SPACES

Muammer Kula and Samed Özkan

Abstract. We characterize the separation properties $T_0$ and $T_1$ at a point $p$ in the category of quasi-proximity spaces. Moreover, the (strongly) closed and (strongly) open subobjects of an object, and each of the various notions of connected and compact objects are characterized in this topological category.

1. Introduction

Proximity structure was introduced by Efremovich in 1951 [15,16]. He characterized the proximity relation “$A$ is close to $B$” as a binary relation on subsets of a set $X$. Some researchers such as Leader, Lodato and Pervin have worked with weaker axioms than Efremovich’s proximity axioms. In this way, some generalized proximities were appeared. One of these is quasi-proximity relation. The concept of quasi-proximities introduced by Pervin [24] in 1963.

Proximity, quasi-proximity and uniformity are important concepts close to topology and they have rich topological properties. For this reason, in recent years, these notions constitute a significant research area in the field of topological spaces.


In [20], we characterized various topological notions such as separation, closedness, connectedness in the category of proximity spaces. In this paper, we obtain more general results such as $\overline{T}_0$ at $p$, $T_1$ at $p$, (strongly) closedness, connectedness, compactness for quasi-proximity spaces and furthermore, we investigate the relationship between the notion of closedness in usual sense and the notion of (strongly) closedness (in our sense) in quasi-proximity spaces.

2010 Mathematics Subject Classification: Primary 54B30; Secondary 54D10; 54A05; 54E05; 18B99.

Key words and phrases: topological category, quasi-proximity space, separation, closedness, connectedness, compactness.

Communicated by Miloš Kurilić.
2. Preliminaries

The following are some basic definitions and notations which we will use throughout the paper.

**Definition 2.1.** [23] A quasi-proximity or P-proximity space is a pair \((X, \delta)\), where \(X\) is a set and \(\delta\) is a binary relation on the powerset of \(X\) such that

1. \(A \delta B\) implies \(A, B \neq \emptyset\);
2. \((A \cup B)\delta C\) if \(A\delta C\) or \(B\delta C\);
3. \(C\delta (A \cup B)\) if \(C\delta A\) or \(C\delta B\);
4. \(A \cap B \neq \emptyset\) implies \(A\delta B\);
5. \(A \delta B\) implies there is an \(E \subseteq X\) such that \(A\delta E\) and \((X - E)\delta B\);

where \(A\delta B\) means it is not true that \(A\ldots\).

If \(\delta\) satisfies the symmetry condition \(A\delta B\) if \(B\delta A\), then it is called an (Efremovitch) proximity on \(X\). The \(Q5\) axiom is called strong axiom, and it plays an important role in the theory of proximity spaces.

A function \(f : (X, \delta) \rightarrow (Y, \delta')\) between two (quasi-)proximity spaces is called a (quasi-)proximity mapping iff \(f(A)\delta' f(B)\) whenever \(A\delta B\). It can easily be shown that \(f\) is a (quasi-)proximity mapping iff \(f^{-1}(C)\delta f^{-1}(D)\) whenever \(C\delta D\).

In a (quasi-)proximity space \((X, \delta)\), we write \(A \preceq B\) if and only if \(\bar{A}\delta(X - B)\). The relation \(\preceq\) is called \(p\)-neighborhood relation or the strong inclusion. When \(A \preceq B\), we say that \(B\) is a \(p\)-neighborhood of \(A\) or \(A\) is strongly contained in \(B\) [17].

We denote the category of proximity spaces and proximity mappings by \(\mathbf{Prox}\), and the category of quasi-proximity spaces and quasi-proximity mappings by \(\mathbf{QProx}\). Hunsaker and Sharma [18] showed that the functor \(\mathcal{U} : \mathbf{Prox} \rightarrow \mathbf{Set}\) is topological. Similarly, \(\mathbf{QProx}\) is a topological category over \(\mathbf{Set}\) [13] p. 31], and \(\mathbf{Prox}\) is a full subcategory of \(\mathbf{QProx}\) [27] p. 147].

**Definition 2.2.** Let \(\mathfrak{B}\) be a (quasi-)proximity-base on a set \(X\) and let a binary relation \(\delta\) on \(P(X)\) be defined as follows: \((A, B) \in \delta\) if, given any finite covers \(\{A_i : 1 \leq i \leq n\}\) and \(\{B_j : 1 \leq j \leq m\}\) of \(A\) and \(B\) respectively, then there exists a pair \((i, j)\) such that \((A_i, B_j) \in \mathfrak{B}\). \(\delta\) is a (quasi-)proximity on \(X\) finer than the relation \(\mathfrak{B}\) [18] [22] [26].

**Definition 2.3.** Let \(X\) be a non-empty set, for each \(i \in I\), \((X_i, \delta_i)\) be a (quasi-)proximity space and \(f_i : X_i \rightarrow (X_j, \delta_j)\) be a source in \(\mathbf{Set}\). Define a binary relation \(\mathfrak{B}\) on \(P(X)\) as follows: for \(A, B \in P(X)\), \(A\mathfrak{B}B\) if \(f_i(A)\delta_i f_i(B)\), for all \(i \in I\). \(\mathfrak{B}\) is a (quasi-)proximity-base on \(X\) [26] Theorem 3.8]. The initial (quasi-)proximity structure \(\delta\) on \(X\) generated by the (quasi-)proximity base \(\mathfrak{B}\) is given by for \(A, B \in P(X)\), \(A\delta B\) iff for any finite covers \(\{A_i : 1 \leq i \leq n\}\) and \(\{B_j : 1 \leq j \leq m\}\) of \(A\) and \(B\) respectively, then there exists a pair \((i, j)\) such that \((A_i, B_j) \in \mathfrak{B}\) [22] [26].

**Definition 2.4.** Let \((X, \delta)\) be a (quasi-)proximity space, \(Y\) a nonempty set and \(f\) be a function from a (quasi-)proximity space \((X, \delta)\) onto a set \(Y\). The strong inclusion \(\subseteq^*\) induced by the finest (quasi-)proximity \(\delta^*\) (the quotient (quasi-)proximity) on \(Y\) making \(f\) (quasi-)proximally continuous is given by: for every
A, B ⊂ Y, A ≪* B if and only if, for each binary rational s in [0, 1], there is some C_s ⊂ Y such that C_0 = A, C_1 = B and s < t implies f^{-1}(C_s) ≪ s f^{-1}(C_t) \[17\] or \[28\] p. 276], where ≪_δ represents the strong inclusion induced by the (quasi-) proximity δ on X. In addition, if f: (X, δ) \rightarrow (X, δ') be a one-to-one (quasi-) proximity quotient map, then Aδ^*B if and only if f^{-1}(A)δf^{-1}(B) \[17\] p. 591].

**Definition 2.5.** Let X be set and p ∈ X. Let X ∨_p X be the wedge at p \[2\], i.e., two disjoint copies of X identified at p, i.e., the pushout of p: 1 \rightarrow X along itself (where 1 is the terminal object in Set). An epi sink \{i_1, i_2: (X, δ) \rightarrow (X ∨_p X, δ')\} (quasi-proximity mappings), where i_1, i_2 are the canonical injections, in Prox(QProx) is a final lift if and only if the following statement holds. For each pair A, B in the different component of X ∨_p X, Aδ'B iff there exist sets C, D in X such that Cδ\{p\} and \{p\}δD with \(i_k^{-1}(A) = C\) and \(i_j^{-1}(B) = D\) for \(k, j = 1, 2\) and \(k \neq j\). If A and B are in the same component of wedge, then Aδ'B iff there exist sets C, D in X such that CδD and \(i_k^{-1}(A) = C\) and \(i_k^{-1}(B) = D\) for some \(k = 1, 2\). Specially, if \(i_k(C) = A\) and \(i_k(D) = B\), then \((i_k(C), i_k(D)) \in δ'\) iff \((i_k^{-1}(i_k(C)), i_k^{-1}(i_k(D))) = (C, D) \in δ\).

**Definition 2.6.** Let X be a non-empty set. The discrete (quasi-)proximity structure δ on X is given by for A, B ⊂ X, AδB iff A ∩ B ≠ \emptyset and the indiscrete (quasi-)proximity structure δ on X is given by for A, B ⊂ X, AδB iff A ≠ \emptyset and B ≠ \emptyset \[23\] p. 9].

### 3. T₀ and T₁ quasi-proximity spaces at a point

In this section, we give the characterizations of T₀ and T₁ quasi-proximity spaces at a point p.

Let B be set and p ∈ B. Let B ∨_p B be the wedge at p. A point x in B ∨_p B will be denoted by x_1(x_2) if x is in the first (resp. second) component of B ∨_p B. Note that \(p_1 = p_2\).

The principal p-axis map, \(A_p: B ∨_p B \rightarrow B^2\) is defined by \(A_p(x_1) = (x, p)\) and \(A_p(x_2) = (p, x)\). The skewed p-axis map, \(S_p: B ∨_p B \rightarrow B^2\) is defined by \(S_p(x_1) = (x, x)\) and \(S_p(x_2) = (p, x)\). The fold map at p, \(∇_p: B ∨_p B \rightarrow B\) is given by \(∇_p(x_i) = x\) for \(i = 1, 2\) \[2\].

**Definition 3.1.** (cf. \[2\]) Let \(U: 𝕀 \rightarrow Set\) be a topological functor, X an object in 𝕀, p be a point in 𝕀(X) = B.

1. X is T₀ at p iff the initial lift of the U-source \{A_p: B ∨_p B \rightarrow 𝕀D(B) = B\} and \(∇_p: B ∨_p B \rightarrow 𝕀D(B) = B\) is discrete, where D is the discrete functor which is a left adjoint to U.

2. X is T₀′ at p iff the initial lift of the U-source \{id: B ∨_p B \rightarrow 𝕀(X ∨_p X) = B \forall p B and \(∇_p: B ∨_p B \rightarrow 𝕀D(B) = B\) is discrete, where X ∨_p X is the wedge in 𝕀, i.e., the final lift of the U-sink \{i_1, i_2: 𝕀(U) = B ∨_p B\} where i_1, i_2 denote the canonical injections.

3. X is T₁ at p iff the initial lift of the U-source \{S_p: B ∨_p B \rightarrow 𝕀(X^2) = B^2\} and \(∇_p: B ∨_p B \rightarrow 𝕀D(B) = B\) is discrete.
THEOREM 3.1. Let \((X, \delta)\) be a quasi-proximity space and \(p \in X\). \((X, \delta)\) is \(\overline{T_0}\) at \(p\) iff for each \(x \neq p\), \((\{x\}, \{p\}) \notin \delta\) or \((\{p\}, \{x\}) \notin \delta\).

PROOF. Let \((X, \delta)\) is \(\overline{T_0}\) at \(p\). We shall show that the condition holds.

Suppose for some \(x \in X\), \((\{x\}, \{p\}) \in \delta\) and \((\{p\}, \{x\}) \in \delta\) with \(x \neq p\). Then, by Definitions 2.3 and 2.6 for \((U, V) \in \delta'\) (\(\delta'\) is a quasi-proximity structure on \(X \vee_p X\)) with \(U = \{x_1\}\) and \(V = \{x_2\}\),

\[
\begin{align*}
\pi_1 A_p(U) \delta \pi_1 A_p(V) &= \{x\} \delta \{p\}, \\
\pi_2 A_p(U) \delta \pi_2 A_p(V) &= \{p\} \delta \{x\},
\end{align*}
\]

where \(\pi_i : X^2 \to X\), \(i = 1, 2\), are the projection maps, and

\[\nabla_p(\{x_1\}) \delta \nabla_p(\{x_2\}) = \{x\} \delta \{x\}.
\]

But \(U \cap V = \emptyset\). This is a contradiction to the fact that \((X, \delta)\) is \(\overline{T_0}\) at \(p\). For \((V, U) \in \delta'\), similar contradiction can be handled. Hence if \((\{x\}, \{p\}) \in \delta\) and \((\{p\}, \{x\}) \notin \delta\), then \(x = p\).

Conversely, suppose that for each \(x \neq p\), \((\{x\}, \{p\}) \notin \delta\) or \((\{p\}, \{x\}) \notin \delta\). We need to show that \((X, \delta)\) is \(\overline{T_0}\) at \(p\).

Let \((U, V)\) be any set in \(\delta'\), where \(\delta'\) is the initial on \(X \vee_p X\) induced by the maps \(A_p\) and \(\nabla_p\), \(\pi_1 A_p(U) \delta \pi_1 A_p(V)\), \(\pi_2 A_p(U) \delta \pi_2 A_p(V)\) and \(\nabla_p U \delta \nabla_p V\).

Since \(\delta_d\) is the discrete quasi-proximity structure and \(\nabla_p U \delta \nabla_p V\), then \(\nabla_p U \cap \nabla_p V \neq \emptyset\). It follows that there exists \(x \in \nabla_p U \cap \nabla_p V\). Hence, there exist \(y \in U\) and \(z \in V\) such that \(\nabla_p y = x = \nabla_p z\).

If \(x = p\), then \(y = p_i = z\), \((i = 1, 2)\) and \(p_i \in U \cap V\).

If \(x \neq p\), then \(y = x_i\), \(z = x_j\), \((i, j = 1, 2)\). We need to show that \(U \cap V \neq \emptyset\).

If \(p \in \nabla_p U \cap \nabla_p V\), then \(p_i \in U \cap V\), \((i = 1, 2)\). Suppose that \(p \notin \nabla_p U \cap \nabla_p V\). We show that both \(U\) and \(V\) are in the first or in the second or in both component of \(X \vee_p X\).

If \(U\) subset of the first component of \(X \vee_p X\) and \(V\) subset of the second component of \(X \vee_p X\), then \(\{x_1\} \subseteq U\) and \(\{x_2\} \subseteq V\). It follows that

\[
\begin{align*}
\pi_1 A_p(\{x_1\}) \delta \pi_1 A_p(\{x_2\}) &= \{x\} \delta \{p\}, \\
\pi_2 A_p(\{x_1\}) \delta \pi_2 A_p(\{x_2\}) &= \{p\} \delta \{x\}.
\end{align*}
\]

Since \((\{x\}, \{p\}) \notin \delta\) or \((\{p\}, \{x\}) \notin \delta\) (by assumption), \((U, V) \notin \delta'\) by the conditions (Q2) and (Q3) of Definition 2.7.

The case \(U\) subset of the second component of \(X \vee_p X\) and \(V\) subset of the first component of \(X \vee_p X\) can be handled similarly. Hence \(U\) and \(V\) can not be in different component of \(X \vee_p X\).

If \(U\) (resp. \(V\)) subset of the first component of \(X \vee_p X\) and \(V\) (resp. \(U\)) subset of both component of \(X \vee_p X\), then \(U \supseteq \{x_1\}\) and \(V \supseteq \{x_1, x_2\}\). If \(U\) (resp. \(V\)) subset of both component of \(X \vee_p X\) and \(V\) (resp. \(U\)) subset of the second component of \(X \vee_p X\), then \(U \supseteq \{x_1, x_2\}\) and \(V \supseteq \{x_2\}\). Hence \(U \cap V \neq \emptyset\).
Similarly if $U$ and $V$ are in the first (resp. second) component of $X \lor_p X$, then $U \supseteq \{x_1\}$ and $V \supseteq \{x_1\} (U \supseteq \{x_2\}$ and $V \supseteq \{x_2\}$). If $U$ and $V$ are in both component of $X \lor_p X$, then $U \supseteq \{x_1, x_2\}$ and $V \supseteq \{x_1, x_2\}$. Hence $U \cap V \neq \emptyset$.

If $(\{x_1\}, \{x_1\}) \in \delta' (i = 1, 2)$, then

\[\pi_1A_p(\{x_1\}) \delta_1A_p(\{x_1\}) = \{x\} \delta\{x\}, \quad \pi_2A_p(\{x_1\}) \delta_2A_p(\{x_1\}) = \{p\} \delta\{p\},\]

\[\pi_1A_p(\{x_2\}) \delta_1A_p(\{x_2\}) = \{p\} \delta\{p\}, \quad \pi_2A_p(\{x_2\}) \delta_2A_p(\{x_2\}) = \{x\} \delta\{x\}.\]

We must have $(U, V) \supseteq ((\{x_1\}, \{x_1\}), (i = 1, 2)$, i.e., $U \cap V \neq \emptyset$ and consequently, by Definitions 2.3, 2.6 and 3.1, $(X, \delta)$ is $T_0$ at $p$.

**Theorem 3.2.** A quasi-proximity space is $T_0$ at $p$ for every $p \in X$.

**Proof.** It is similar to the proof in [20] Theorem 3.7. □

**Theorem 3.3.** Let $(X, \delta)$ be a quasi-proximity space and $p \in X$. $(X, \delta)$ is $T_1$ at $p$ if for each $x \neq p$, $((\{x\}, \{p\}) \notin \delta$ or $((\{p\}, \{x\}) \notin \delta$.

**Proof.** The proof is similar to the proof of Theorem 3.1 by using the map $S_p$ instead of the map $A_p$. □

**Remark 3.1.** (1) Note that for the category Top of topological spaces, $\bar{T}_0$ at $p$, $T_0'$ at $p$, or $T_1$ at $p$ reduce to for each $x \in X$, a topological space, with $x \neq p$, there exists a neighborhood of $x$ not containing $p$ or (resp. and) there exists a neighborhood of $p$ not containing $x$ [5].

(2)(a) If $\mathcal{U}: \mathcal{E} \rightarrow \text{Set}$ be a normalized topological functor, then each of $\bar{T}_0$ at $p$ and $T_1$ at $p$ implies $T_0'$ at $p$ [4] Corollary 2.11.

(b) In a topological category, $T_0$ at $p$ and $T_1$ at $p$ objects may be equivalent, see [11,19] and all objects may be $T_1$ at $p$, for example, it is shown, in [7], that all prebornological spaces are $T_1$ at $p$. Moreover, $T_1$ at $p$ objects could be only discrete objects, see [12].

**4. (Strongly) Closed and (Strongly) Open Subspaces**

In this section, the (strongly) closed and (strongly) open subobjects of an object are characterized in the category of quasi-proximity spaces, QProx.

Let $B$ be set and $p \in B$. The infinite wedge product $\lor_p B$ is formed by taking countably many disjoint copies of $B$ and identifying them at the point $p$. Let $B^\infty = B \times B \times \ldots$ be the countable cartesian product of $B$. Define $A^\infty_p: \lor_p B \rightarrow B^\infty$ by $A^\infty_p(\{x_i\}) = (p, p, \ldots, p, x, p, \ldots)$, where $x_i$ is in the $i$-th component of the infinite wedge and $x$ is in the $i$-th place in $(p, p, \ldots, p, x, p, \ldots)$ (infinite principal $p$-axis map), and $\nabla^\infty_p: \lor_p B \rightarrow B$ by $\nabla^\infty_p(x_i) = x$ for all $i \in I$ (infinite fold map), [3].

**Definition 4.1.** (cf. [3]) Let $\mathcal{U}: \mathcal{E} \rightarrow \text{Set}$ be a topological functor, $X$ an object in $\mathcal{E}$ with $\mathcal{U}(X) = B$. Let $F$ be a nonempty subset of $B$. We denote by $X/F$ the final lift of the epi $\mathcal{U}$-sink $q: \mathcal{U}(X) = B \rightarrow B/F = (B\setminus F) \cup \{\ast\}$, where $q$ is the epi map that is the identity on $B\setminus F$ and identifying $F$ with a point $\{\ast\}$.

Let $p$ be a point in $B$. 


(1) $p$ is closed iff the initial lift of the $U$-source $\{A^\infty_p : \forall^\infty B \to U(X^\infty) = B^\infty$ and $\nabla^\infty_p : \forall^\infty B \to U(D(B) = B$) is discrete.
(2) $F \subset X$ is closed iff $\{\ast\}$, the image of $F$, is closed in $X/F$ or $F = \emptyset$.
(3) $F \subset X$ is strongly closed iff $X/F$ is $T_1$ at $\{\ast\}$ or $F = \emptyset$.
(4) If $B = F = \emptyset$, then we define $F$ to be both closed and strongly closed.

**Theorem 4.1.** Let $(X, \delta)$ be a quasi-proximity space and $p \in X$. \{p\} is closed in $X$ iff for any $B \subset X$, if $\{p\} \delta B$ or $B \delta \{p\}$, then $p \in B$.

**Proof.** Let $\{p\}$ is closed in $X$. We show that for any $B \subset X$, if $\{p\} \delta B$ or $B \delta \{p\}$, then $p \in B$. Suppose that $\{p\} \delta B$ or $B \delta \{p\}$ while $p \notin B$, for some $B \subset X$. Then for some $x \neq p$ and $x \in B$, we get $\{(x), \{p\}\} \in \delta$ and $\{\{p\}, \{x\}\} \in \delta$ by the conditions (Q2) and (Q3) of Definition 2.1. Let $U = \{x_1\}$ and $V = \{x_2\}$. Note that $(U, V) \in \delta'$ since

\[ \pi_1 A^\infty_p U \delta \pi_1 A^\infty_p V = \{x\} \delta \{p\}, \quad \pi_2 A^\infty_p U \delta \pi_2 A^\infty_p V = \{p\} \delta \{x\}, \]

and for $i \geq 3$,

\[ \pi_i A^\infty_p U \delta \pi_i A^\infty_p V = \{p\} \delta \{p\}, \]

where $\pi_i : X^\infty \to X$ are the projection maps, and

\[ \nabla^\infty_p (\{x_1\}) \delta_\delta \nabla^\infty_p (\{x_2\}) = \{x\} \delta \{x\}, \]

where $\delta_\delta$ is the discrete quasi-proximity structure on $X$. But $U \cap V = \emptyset$. This is a contradiction to the fact that $\delta'$ is discrete. For $(V, U) \in \delta'$ can be handled similarly.

If the condition holds, then by using the similar argument in the proof in [20] Theorem 4.5, $\{p\}$ is closed in $X$. \hfill $\Box$

**Theorem 4.2.** Let $(X, \delta)$ be a quasi-proximity space. Then $\emptyset \neq F \subset X$ is (strongly) closed iff $x \in F$ whenever $\{x\} \delta F$ or $F \delta \{x\}$ for any $x \in X$.

**Proof.** It can be proven by using the same argument in the proof in [20] Theorems 4.6 and 4.7. \hfill $\Box$

**Definition 4.2.** Let $\mathcal{E}$ be a topological category over $\text{Set}$, $X$ an object in $\mathcal{E}$ and $F$ be a nonempty subset of $X$.

1. $F \subset X$ is open iff $F^c$, the complement of $F$, is closed in $X$.
2. $F \subset X$ is strongly open iff $F^c$, the complement of $F$, is strongly closed in $X$ [10].

**Theorem 4.3.** Let $(X, \delta)$ be a quasi-proximity space. $\emptyset \neq F \subset X$ is (strongly) open iff $x \in F^c$ whenever $\{x\} \delta F^c$ or $F^c \delta \{x\}$ for all $x \in X$.

**Proof.** It follows from Theorem 4.2 and Definition 4.2. \hfill $\Box$

**Definition 4.3.** [23] p. 106] Let $(X, \delta)$ be a quasi-proximity space and $A \subset X$. Define $\bar{A} = \{x \mid x \delta A \text{ or } A \delta x\}$ and if $\bar{A} = A$, then $A$ is said to be closed.

**Theorem 4.4.** Let $(X, \delta)$ be a quasi-proximity space and $A \subset X$. $A$ is closed (in the usual sense) iff $A$ is (strongly) closed (in our sense).


Proof. It follows from Theorem 4.2 and Definition 4.3. □

In Top, the notion of closedness coincides with the usual closedness, and F is strongly closed iff F is closed and for each x \notin F there exists a neighbourhood of F missing x. In QProx, by Theorem 4.4, the notion of (strong) closedness coincide with the usual notion of closedness. The notion of (strong) closedness induces closure operators in sense of Dikranjan and Giuli in convergence spaces, preordered spaces and semiuniform convergence spaces.

5. Connectedness and Compactness

In this section, the characterization of each of the notions of (strongly) connected and (strongly) compact objects in the category of quasi-proximity spaces are given.

Definition 5.1. Let E be a topological category over Set and X be an object in E.

1) X is connected iff the only subsets of X both strongly open and strongly closed are X and ∅.
2) X is strongly connected iff the only subsets of X both open and closed are X and ∅.
3) X is D-connected iff any morphism from X to any discrete object is constant (cf. [1, 10, 21, 25]).

Theorem 5.1. Let (X, δ) be a quasi-proximity space.

1) (X, δ) is (strongly) connected iff for any non-empty proper subset F of X, either the condition (a) or (b) holds.
   (a) x \notin F whenever \{x\}δF or Fδ\{x\} for some x ∈ X.
   (b) x \notin Fc whenever \{x\}δFc or Fcδ\{x\} for some x ∈ X.
2) (X, δ) is D-connected iff \{x\}δ\{y\} or \{y\}δ\{x\} for all x, y ∈ X with x \neq y.


Definition 5.2. Let E be a topological category over Set, X and Y be objects in E, and f : X → Y a morphism in E. Then,

1) f is said to be closed if the image of each closed subobject of X is a closed subobject of Y.
2) f is said to be strongly closed if the image of each strongly closed subobject of X is a strongly closed subobject of Y.
3) X is compact iff the projection π2: X × Y → Y is closed for each object Y in E.
4) X is strongly compact iff the projection π2: X × Y → Y is strongly closed for each object Y in E.

Theorem 5.2. (1) Let f: (X, δ) → (Y, δ') be a quasi-proximity mapping. If D ⊂ Y is (strongly) closed, so also is f−1(D).
(2) Let (Y, δ') be a quasi-proximity space. If N ⊂ Y is (strongly) closed and M ⊂ N is (strongly) closed, so also is M ⊂ Y.
Proof. (1) Suppose $D \subset Y$ is (strongly) closed and $x \in f^{-1}(D)$. By Theorem 4.2, $y \in D$ whenever $\{y\} \delta D$ or $D \delta \{y\}$ for all $y \in Y$. We need to show that $x \in f^{-1}(D)$ whenever $\{x\} \delta f^{-1}(D)$ or $f^{-1}(D) \delta \{x\}$ for all $x \in X$. Note that $f(x) \in f(f^{-1}(D)) \subset D$ and $\{f(x)\} \delta D$ or $D \delta \{f(x)\}$ since $f$ is a quasi-proximity mapping and $D \subset Y$ is (strongly) closed. Thus, $f^{-1}(D)$ is (strongly) closed.

(2) Suppose $N \subset Y$ and $M \subset N$ are (strongly) closed, $y \in Y$ and there exists $a \in M$ such that $g y a$ or $a \delta y$. By Theorem 4.2, we need to show that $y \in M$. Since $N \subset Y$ is (strongly) closed and $M \subset N$, by Theorem 4.2, $y \in N$. It follows that $y \in M$ since $M \subset N$ is (strongly) closed. □

Theorem 5.3. Every quasi-proximity space is (strongly) compact.

Proof. Let $(X, \delta)$ be a quasi-proximity space. By Definition 5.2(3) (5.2(4)), we need to show that for each proximity space $(Y, \delta')$, the projection $\pi_2 : (X, \delta) \times (Y, \delta') \to (Y, \delta')$ is (strongly) closed.

Suppose $M \subset X \times Y$ is (strongly) closed. To show that $\pi_2 M$ is (strongly) closed, we assume the contrary and apply Theorem 4.2. So, for some point $b \in Y$, $b \notin \pi_2 M$ whenever $\{b\} \delta' \pi_2 M$ or $\pi_2 M \delta' \{b\}$. Since $M \subset X \times Y$ is (strongly) closed, $(a, b) \in M$ whenever $(a, b) \delta'' M$ or $M \delta'' \{(a, b)\}$ for all $(a, b) \in X \times Y$, where $\delta''$ is the product quasi-proximity structure on $X \times Y$. Hence $\pi_2 \{(a, b)\} \delta' \pi_2 M = \{b\} \delta' \pi_2 M$ or $\pi_2 M \delta' \pi_2 \{(a, b)\} = \pi_2 M \delta' \{b\}$, by definition of the product quasi-proximity structure. Since $(a, b) \in M$, $\pi_2 (a, b) = b \in \pi_2 M$. This is a contradiction to the fact that $M$ is (strongly) closed. Hence, by Theorem 4.2, $\pi_2 M$ must be (strongly) closed and consequently, by Definition 5.2(3) (5.2(4)), $(X, \delta)$ is (strongly) compact. □

Theorem 5.4. Let $f : (X, \delta) \to (Y, \delta')$ be a quasi-proximity mapping. If $(X, \delta)$ is (strongly) compact, then the subspace $f(X)$ is (strongly) compact.

Proof. It follows from Theorem 5.3. □

Acknowledgement. We would like to thank the referee for his/her valuable and helpful suggestions that improved the paper.

References

1. V. A. Arhangel’skii, R. Wiegandt, Connectedness and disconnectedness in topology, Gen. Topology Appl. 5 (1975), 9–33.