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IMPROVING A CONSTANT IN HIGH-DIMENSIONAL DISCREPANCY ESTIMATES

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Abstract. For all $s \geqslant 1$ and $N \geqslant 1$ there exist sequences $(z_1, \dots, z_N)$ in $[0,1]^s$ such that the star-discrepancy of these points can be bounded by

$$D_N^*(z_1, \dots, z_N) \leqslant c\frac{\sqrt{s}}{\sqrt{N}}.$$

In practice it is desirable to obtain low values of c. The best known value for the constant is $c = 10$ as has been calculated by Aistleitner. In this paper we improve the bound to $c = 9$.

1. Introduction

When quasi-Monte Carlo methods are applied in practice to answer financial mathematical questions, the occurring problems frequently involve to explicitly or implicitly calculate integrals. Often the arithmetic mean of some function evaluations $f(z_1), \dots, f(z_N)$ is taken as an approximation of the integral under consideration. A theoretic justification for this approach is the Koksma–Hlawka inequality which states that the difference between the arithmetic mean and the integral of a function f over the s-dimensional unit cube is bounded by the product of the total variation of f in the sense of Hardy–Krause and the so-called star-discrepancy $D_N^*(z_1, \dots, z_N)$.

As the problems occurring in practice are in addition typically high-dimensional ($s \gg 0$) and function evaluation is expensive ($N \ll \infty$), see e.g. [**2, 8**], classical (finite) low-discrepancy sequences which satisfy the inequality

$$D_N^*(z_1, \dots, z_N) \leqslant c\frac{(\log N)^{s-1}}{N}$$

are of limited use because the numerator grows exponentially with the dimension s (and $s \gg 0$). This observation is known as the *curse of dimensionality*, compare e.g. [**9**, Chapter 1], [**10**]. Instead of using low-discrepancy sequences, it is hence desirable

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to construct sequences which have a small star-discrepancy if N is relatively small in comparison to s. This is the main task of this paper.

In [**7**] it was shown that for every $s \geqslant 1$ and $N \geqslant 1$, there exists a finite sequence $(z_1, \dots, z_N)$ of elements of the s-dimensional unit cube such that the star-discrepancy of this sequence satisfies

$$D_N^*(z_1, \dots, z_N) \leqslant c\frac{\sqrt{s}}{\sqrt{N}}$$

for some constant c independent of s and N. These finite sequences have a lower discrepancy than classical low-discrepancy sequences if s is relatively big in comparison to N. However, no concrete value for c was calculated in this paper. In [**1**], a new proof of the result was given including the explicit upper bound $c = 10$. In this paper, we improve the upper constant to $c = 9$. More precisely, we show:

Theorem 1.1. *For any $s \geqslant 1$ and $N \geqslant 1$, there exists a sequence $(z_1, \dots, z_N)$ of elements of the s-dimensional unit cube such that*

$$D_N^*(z_1, \dots, z_N) < 9\frac{\sqrt{s}}{\sqrt{N}}. \tag{1.1}$$

An improvement of the constant c is of important practical use: for bounding the discrepancy of a sequence $(z_1, \dots, z_N)$ deterministically by 1 we need a sequence $(z_1, \dots, z_N)$ of length $N > c^2 s$. So N depends on c^2 which means it is a matter of interest to find the best possible theoretical value of c. Our work is a contribution to this aim.

Our proof closely follows the one presented in [**1**]. As was already mentioned therein, an improvement of Gnewuch's upper bound for the smallest cardinality of a δ-cover, Theorem 2.1, should result in a better value for c. Indeed, in Proposition 2.1 we are able to improve Gnewuch's result, which is another main result of our paper. Afterwards, we only need to slightly amend Aistleitner's proof for the new upper bound and find a lower value for c. A main ingredient of the proof is the use of Monte Carlo samples. Instead of using such *completely random sequences*, it is another idea to use *less random* samples, e.g. Latin hypercube sampling. Such alternative approaches have been conducted in unpublished work, [**4, 6**] and also lead to improved values of c in comparison to [**1**]. Finally, it should be mentioned, that the rate of convergence $\sqrt{s/N}$ is in some sense best possible: it was shown in [**3**] by Doerr that for a random set of independent, uniformly distributed points $(z_1, \dots, z_N)$ the inequality

$$E(D_N^*(z_1, \dots, z_N)) \geqslant \tilde{c}\frac{\sqrt{s}}{\sqrt{N}}$$

for the expected value holds.

2. Proof of the main results

Before we come to the proof of Theorem 1.1 we collect some of the necessary background.

Discrepancy. Let $Z = (z_n)_{n\geqslant 0}$ be a sequence in $[0,1)^s$. Then the *star-discrepancy* of the first N points of the sequence is defined by

$$D_N^*(Z) := \sup_{B\subset[0,1)^d} \Big|\frac{A_N(B)}{N} - \lambda_s(B)\Big|,$$

where the supremum is taken over all intervals $B = [0,a_1)\times[0,a_2)\times\cdots\times[0,a_s)\subset[0,1)^s$ and $A_N(B) := |\{n \mid 0\leqslant n < N,\ z_n\in B\}|$ and λ_s denotes the s-dimensional Lebesgue-measure. If $D_N^*(Z)$ satisfies

$$D_N(Z) = O(N^{-1}(\log N)^{s-1})$$

then Z is called a *low-discrepancy sequence*. For more details we refer the reader to [**9**].

δ-bracketing Covers. In this paper we will use the notation of δ-bracketing covers as in [**5**]: let $\mathcal{F}\subset L^1([0,1]^s)$ be a subset of the real valued Lebesgue integrable functions. For $0<\delta\leqslant 1$ and $f,g\in\mathcal{F}$ with

$$\int_{[0,1]^s}(g(x)-f(x))\mathrm{d}x \leqslant \delta.$$

we call the set

$$[f,g]_{\mathcal{F}} := \{h\in\mathcal{F} \mid f\leqslant h\leqslant g\}$$

a *δ-bracket of $\mathcal{F}$*. A finite subset $\Gamma\subset\mathcal{F}$ is called a *δ-cover of $\mathcal{F}$*, if for every $h\in\mathcal{F}$, there exists $f,g\in\Gamma$ with $h\in[f,g]_{\mathcal{F}}$. A *δ-bracketing cover of $\mathcal{F}$* is a set of δ-brackets whose union is $\mathcal{F}$. The number $\mathcal{N}(\mathcal{F},d)$ denotes the smallest cardinality of a δ-cover of $\mathcal{F}$, i.e.

$$\mathcal{N}(\mathcal{F},d) := \min\{|\Gamma| \mid \Gamma \text{ is a } \delta\text{-cover}\}.$$

Similarly, $N_{[\ ]}(\mathcal{F},\delta)$ denotes the smallest cardinality of a δ-bracketing cover. In the following we will restrict to the specific subset of $\mathcal{F}$ which consists of all indicator function of the form $\mathbb{1}_{[0,x)}$ for some $x<1$ and use the notation $\mathcal{N}(s,d)$ and $N_{[\ ]}(s,\delta)$ in this case.

Gnewuch's inequality. In [**5**], Gnewuch proved the following inequality for $N_{[\ ]}(s,\delta)$.

THEOREM 2.1. [**5**, Theorem 1.15] *Let $s\in\mathbb{N}$ and $0<\delta\leqslant 1$. Then*

$$N_{[\ ]}(s,\delta) \leqslant 2^{s-1}\Big(\frac{s^s}{s!}\Big)(\delta^{-1}+1)^s. \tag{2.1}$$

We will focus here on an intermediate result of Gnewuch which he derived during the proof of Theorem 2.1 and state it as a lemma. Afterwards we will show that it can be used to strengthen inequality (2.1).

LEMMA 2.1. *Let $s\in\mathbb{N}$ and $0<\delta\leqslant 1$. Then*

$$N_{[\ ]}(s,\delta) \leqslant \sum_{k=0}^{s-2}\binom{s}{k+1}2^{s-k-2}\frac{s^s}{(s-k)!}\Big(\delta^{-1}+\frac{1}{2}\Big)^{s-k} + \delta^{-1} + 1.$$

Indeed, we prove here the following stronger version of Gnewuch's inequality. Note that

$$2^{s-2}\Big(\frac{s^s}{s!}\Big)(\delta^{-1}+1)^s + \frac{1}{2}(\delta^{-1}+1) < 2^{s-1}\Big(\frac{s^s}{s!}\Big)(\delta^{-1}+1)^s$$

for all $s \geqslant 2$.

Proposition 2.1 (Upper bound for covering numbers). *Let* $s \in \mathbb{N}$ *and* $0 < \delta \leqslant 1$. *Then*

$$N_{[\,]}(s,\delta) \leqslant 2^{s-2}\Big(\frac{s^s}{s!}\Big)(\delta^{-1}+1)^s + \frac{1}{2}(\delta^{-1}+1).$$

Proof. We prove our claim by induction on s. Let $n := \lceil \delta^{-1} \rceil$. For $s = 1$ we have $N_{[\,]}(s,\delta) \leqslant n \leqslant \delta^{-1}+1$. So let $s \geqslant 2$. With Lemma 2.1 we have

$$\begin{aligned} N_{[\,]}(s,\delta) &\leqslant \sum_{k=0}^{s-2}\binom{s}{k+1}2^{s-k-2}\frac{s^s}{(s-k)!}\Big(\delta^{-1}+\frac{1}{2}\Big)^{s-k} + \delta^{-1}+1 \\ &\leqslant \sum_{k=0}^{s-2}\binom{s}{k}2^{s-k-2}\frac{s^s}{s!}\Big(\delta^{-1}+\frac{1}{2}\Big)^{s-k} + \delta^{-1}+1 \end{aligned}$$

For the right hand side we get

$$\begin{aligned} &\sum_{k=0}^{s-2}\binom{s}{k}2^{s-k-2}\frac{s^s}{s!}\Big(\delta^{-1}+\frac{1}{2}\Big)^{s-k} + \delta^{-1}+1 \\ &\quad = \sum_{k=0}^{s}\binom{s}{k}2^{s-k-2}\frac{s^s}{s!}\Big(\delta^{-1}+\frac{1}{2}\Big)^{s-k} - \sum_{k=s-1}^{s}\binom{s}{k}2^{s-k-2}\frac{s^s}{s!}\Big(\delta^{-1}+\frac{1}{2}\Big)^{s-k} \\ &\qquad + \delta^{-1}+1 \end{aligned}$$

Finally

$$-\sum_{k=s-1}^{s}\binom{s}{k}2^{s-k-2}\frac{s^s}{s!}\Big(\delta^{-1}+\frac{1}{2}\Big)^{s-k} + \delta^{-1}+1 \leqslant \frac{1}{2}(\delta^{-1}+1)$$

and

$$\sum_{k=0}^{s}\binom{s}{k}2^{s-k-2}\frac{s^s}{s!}\Big(\delta^{-1}+\frac{1}{2}\Big)^{s-k} = 2^{s-2}\frac{s^s}{s!}(\delta^{-1}+1)^s$$

imply

$$N_{[\,]}(s,\delta) \leqslant 2^{s-2}\Big(\frac{s^s}{s!}\Big)(\delta^{-1}+1)^s + \frac{1}{2}(\delta^{-1}+1). \qquad \square$$

Proof of Theorem 1.1. We closely follow the proof of [**1**, Theorem 1], and amend the arguments therein in order to take into account our improved version of Gwenuch's inequality. For $s = 1$, the points of distance $1/N$ satisfy the inequality and for $s = 2$, the Hammersley sequence with base 2 does the job. Therefore let $s \geqslant 3$. Without loss of generality we may assume that $N > 81s$ because the claim trivially follows otherwise. For a clear presentation, we subdivide our proof into 5 steps. Since steps 1, 2 and 5 are essentially the same as in [**1**], we will not go into details here but still present them for the sake of completeness and for introducing notation. On the other hand steps 3 and 4 include some additional aspects in comparison to [**1**].

STEP 1: Define subsets A_k and bound their cardinality by Proposition 2.1. Let

$$K := \lceil (\log_2(N) - \log_2(s))/2 \rceil.$$

Then $K \geqslant 3$ and

$$2^{-K} \in \left[\frac{\sqrt{s}}{2\sqrt{N}}, \frac{\sqrt{s}}{\sqrt{N}} \right].$$

By Proposition 2.1, there exists a 2^{-k}-cover of $[0,1]^s$ for $1 \leqslant k \leqslant K-1$, denoted by Γ_k, such that Stirling's formula yields

$$|\Gamma_k| \leqslant 2^{s-1} \Big(\frac{s^s}{s!} \Big) (2^k+1)^s + (2^k+1) \leqslant \frac{1}{\sqrt{2\pi s}} 2^{s-1} \exp(s) (2^k+1)^s + (2^k+1),$$

because $\mathcal{N}(s,d) \leqslant 2N_{[\,]}(s,d)$. Analogously, there exists a 2^{-K}-bracketing cover Δ_K with

$$|\Delta_K| \leqslant \frac{1}{\sqrt{2\pi s}} 2^{s-2} \exp(s) (2^K+1)^s + (2^K+1).$$

Moreover we set

$$\Gamma_K := \{ v \in [0,1]^s \mid (v,w) \in \Delta_K \text{ for some } w \}.$$

Fix $x \in [0,1]^s$ arbitrarily. We want to canonically define two sequences v_k, w_k with $v_k, w_k \in \Gamma_k \cup \{0\}$ such that $0 \leqslant v_1 \leqslant v_2 \leqslant \ldots \leqslant v_{K-1} \leqslant v_K \leqslant x \leqslant w_K$ holds. First we choose $v_K, w_K = (v_K(x), w_K(x))$ with $v_K \leqslant x \leqslant w_K$ and $\lambda_s[v_K, w_K] \leqslant 2^{-K}$. For every k, $2 \leqslant k \leqslant K$ and $\gamma \in \Gamma_k$, there exist $v_{k-1} = v_{k-1}(\gamma)$ and $w_{k-1} = w_{k-1}(\gamma)$ with $v_{k-1}, w_{k-1} \in \Gamma_{k-1} \cup \{0\}$, $v_k \leqslant \gamma \leqslant w_k$ and $\lambda_s[v_k, w_k] \leqslant 2^{-k+1}$. Recursively we set $p_K(x) =: v_K(x)$ and $p_k := v_k(v_{k+1}(x))$ for $1 \leqslant k \leqslant K-1$. Moreover we define $p_0 = 0$. Finally, for $x, y \in [0,1]$ let

$$\overline{[x,y]} := \begin{cases} [0,y] \setminus [0,x] & \text{if } x \neq 0 \\ [0,y] & \text{if } x = 0,\ y \neq 0 \\ \emptyset & \text{if } x = y = 0 \end{cases}.$$

For $0 \leqslant k \leqslant K-1$, the sets $[p_k(x), p_{k+1}(x)]$ are bounded by

$$\lambda_s \overline{[p_k(x), p_{k+1}(x)]} \leqslant 2^{-k}$$

and $[p_K(x), w_K(x)]$ by

$$\lambda_s \overline{[p_K(x), w_K(x)]} \leqslant 2^{-K}.$$

We define A_k as the set of all sets of the form $[p_k(x), p_{k+1}(x)]$ for $0 \leqslant k \leqslant K-1$ and A_K as the set of all sets of the form $[p_K(x), w_K(x)]$. It was proven in [**1**] that

$\lambda_s(A_k) \leqslant 2^{-k}$ for all $0 \leqslant k \leqslant K$. Moreover, every $p_{k+1} \in \Gamma_{k+1}$ is contained in some A_k and hence $|A_k| \leqslant |\Gamma_{k+1}|$. Therefore we have

$$|A_k| \leqslant \frac{1}{\sqrt{2\pi s}} 2^{s-1} \exp(s)(2^{k+1}+1)^s + (2^{k+1}+1),$$
$$|A_K| \leqslant \frac{1}{\sqrt{2\pi s}} 2^{s-2} \exp(s)(2^{K}+1)^s + \frac{1}{2}(2^{K}+1).$$

Step 2: Calculate the lower bound for expected value of indicator functions.

Let $X_1, \dots, X_n$ be a sequence of i.i.d random variables defined on some probability space $(\Omega, \mathcal{A}, \mathbb{P})$ having uniform distribution on $[0,1]^s$. For $I \in A_k$ set $Z_i := \mathbb{1}_I(X_i) - \lambda_s(I)$. In [**1**], it is shown by using Bernstein's and Hoeffding's inequality that for arbitrary $c > 0$

$$\mathbb{P}\Big(\Big|\sum_{i=1}^{N} Z_i\Big| > c\sqrt{sN}\Big) \leqslant \begin{cases} 2\exp\big(-\frac{c^2 s}{2^{-k+1}(1-2^{-k})+4c^2 2^{-K}/3}\big) & \text{for } 2 \leqslant k \leqslant K \\ 2\exp(-2c^2 s) & \text{for } k = 0,1 \end{cases}$$

Step 3: Show that

$$\mathbb{P}\Big(\bigcup_{k=0}^{K} \bigcup_{I \in A_k} \Big(\Big|\sum_{i=1}^{N} \mathbb{1}_I(X_i) - N\lambda_s(I)\Big| > c_k\sqrt{sN}\Big)\Big) < 1,$$

where the coefficients c_k will be chosen in the following.

Let

$$B_k := \bigcup_{I \in A_k} \Big(\Big|\sum_{i=1}^{N} \mathbb{1}_I(X_i) - N\lambda_s(I)\Big| > c_k\sqrt{sN}\Big).$$

For $k = 0$ we get

$$\begin{aligned}\mathbb{P}(B_0) &\leqslant 2\exp(-2c_0^2 s)|A_0| \\ &\leqslant 2\exp(-2c_0^2 s)\Big(\frac{1}{\sqrt{2\pi s}} 2^{s-1}\exp(s)3^s + 3\Big) \\ &\overset{s \geqslant 3}{\leqslant} \exp(-2c_0^2 s)\exp(s)6^s\Big(\frac{1}{\sqrt{6\pi}} + 6\exp(-3)6^{-3}\Big) \\ &\leqslant \frac{1}{4}\exp(-2c_0 s^2)\exp(s)6^s.\end{aligned}$$

Thus $\mathbb{P}(B_0) \leqslant 1/4$ holds after choosing $c_0 = \sqrt{(\log(6)+1)/2} \leqslant 1.19$. Analogously, we choose $c_1 = \sqrt{(\log(10)+1)/2} \leqslant 1.29$ and get $\mathbb{P}(B_1) \leqslant 1/4$. Now let $2 \leqslant k \leqslant K-1$. Then

$$\begin{aligned}2^k\mathbb{P}(B_k) \leqslant &\underbrace{\Big(\frac{2^{s-1}}{\sqrt{2\pi s}}\exp(s)(2^{k+1}+1)^s + (2^{k+1}+1)\Big)2^{k+1}}_{=:G} \\ &\cdot \exp\Big(-\frac{c_k^2 s}{2^{-k+1}(1-2^{-k}) + \frac{4c_k}{3}2^{-K}}\Big)\end{aligned}$$

At first we bound G by

$$\begin{aligned} G &\overset{s\geqslant 3}{\leqslant} \Big(\frac{1}{2\sqrt{6\pi}} + 2^{-3}(2^{k+1}+1)^{-2}\exp(-3)\Big)2^{s+1+k}(2^{k+1}+1)^s\exp(s) \\ &\leqslant \tfrac{1}{8}2^{s+1+k}(2^{k+1}+1)^s\exp(s) \\ &\leqslant \tfrac{1}{8}2^{k+1}\exp(s(\log(2)+1+\log(2^{k+1}+1))) \\ &\overset{s\geqslant 3}{\leqslant} \exp\left(s\underbrace{\left(\tfrac{4}{3}(1+k)\log(2)+1+\log(2^{-k-1}+1)\right)}_{:=a_k}\right). \end{aligned}$$

Finally we define c_k as the positive solution of the equation

$$c_k = \sqrt{a_k}\cdot\sqrt{2^{-k+1}(1-2^k)+\tfrac{4}{3}c_k 2^{-K}},$$

which yields $|c_k| \leqslant 1.58$ and

$$2^k\mathbb{P}(B_k) \leqslant \exp(s\cdot a_k)\exp\Big(-\frac{c_k^2 s}{2^{-k+1}(1-2^{-k})+\frac{4c_k}{3}2^{-K}}\Big) \leqslant 1$$

and thus $\mathbb{P}(B_k) \leqslant 2^{-k}$. For $k = K$ the set A_K contains at most

$$|A_K| \leqslant |\Delta_K| \leqslant \frac{1}{\sqrt{2\pi s}}2^{s-2}\exp(s)(2^K+1)^s + (2^K+1)$$

elements. Similarly to the last case we obtain

$$\begin{aligned} 2^K\mathbb{P}(B_K) \leqslant &\exp\left(s\left(\underbrace{\frac{4}{3}K\log(2)+1+\log(1+2^{-K})}_{:=a_K}\right)\right) \\ &\cdot\exp\Big(-\frac{c_K^2 s}{2^{-K+1}(1-2^{-K})+\frac{4c_K}{3}2^{-K}}\Big). \end{aligned}$$

Defining c_K via the equation

$$c_K = \sqrt{a_K}\cdot\sqrt{2^{-K+1}(1-2^{-K})+\tfrac{4}{3}c_K 2^{-K}}$$

we arrive at $|c_K| \leqslant 1.33$ and $\mathbb{P}(B_K) \leqslant 2^{-K}$. This completes step 3.

STEP 4: Show that

$$\sum_{k=0}^{K} c_k < 8. \tag{2.2}$$

The fact that the choice of the c_k depends on K, will be reflected by the notation $c_{k,K}$ in this step. For $K \leqslant 31$, the desired inequality can be checked by computer calculation. In this range, the maximal value is achieved for $K = 31$ and $\sum_{k=0}^{31} c_{k,31} \leqslant 7.99789995$. Hence let $K \geqslant 32$. Since the $c_{k,K}$ are monotonically decreasing for increasing K, we have

$$\sum_{k=0}^{31} c_{k,K} \leqslant 7.99789995$$

and $c_{K,K} \leqslant c_{32,32} \leqslant 5 \cdot 10^{-9}$ for $K \geqslant 32$. Solving the equation that defines $c_{k,K}$, we find

$$c_{k,K} \leqslant 0.2480726 \cdot k2^{-k/2}$$

for $k, K \geqslant 32$, $k \leqslant K-1$ and thus end up with the desired bound.

Step 5: Derive inequality (1.1) According to step 3 we may choose a realization $X_1(\omega), \ldots, X_N(\omega)$ with $\omega \notin \bigcup_{k=0}^{K} B_k$ and set $z_n := X_n(\omega)$ for $1 \leqslant n \leqslant N$. In [**1**], it is proven that

$$N\lambda_s([0,x]) - \left(\sum_{k=0}^{K-1} c_k + 1\right)\sqrt{sN} \leqslant \sum_{n=1}^{N} \mathbb{1}_{[0,x)}(z_n) \leqslant N\lambda_s([0,x]) + \left(\sum_{k=0}^{K-1} c_k + 1\right)\sqrt{sN}$$

holds for arbitrary $x \in [0,1]^s$. Thus (1.1) follows from (2.2). □

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