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ON THE JACK LEMMA AND ITS GENERALIZATION

Mamoru Nunokawa, Nak Eun Cho, and Janusz Sokół

Abstract. We present a new geometric approach to some problems in differential subordination theory and discuss the new results closely related to the Jack lemma.

1. Introduction

Let $\mathcal{H}$ denote the class of analytic functions in the unit disc $\mathbb{D} = \{z: |z| < 1\}$ on the complex plane $\mathbb{C}$. Let $f, g \in \mathcal{H}$, $f(|z| < |z_0|) \subset g(|z| < |\zeta_0|)$ and $f(z_0) = g(\zeta_0)$, $|z_0| < |\zeta_0| = r \leqslant 1$. In this paper we are looking for the relation between $f(z_0)$ and $z_0 f'(z_0)$ for the case when $g(z)$ maps the unit disc $\mathbb{D}$ onto the right half-plane $\mathrm{Re}\{w\} > 0$. In order to prove our main result we recall the following lemma, well known as the Jack lemma. This lemma describes the case when $g(z) = z$.

Lemma 1.1. [**1**] *Let $w(z)$ be a nonconstant and analytic function in the unit disc $\mathbb{D}$ with $w(0) = 0$. If $|w(z)|$ attains its maximum value on the disc $|z| \leqslant r$ at the point z_0, $|z_0| = r$, then $z_0 w'(z_0) = s w(z_0)$ and $s \geqslant 1$.*

The Jack lemma has several applications and generalizations in the theory of differential subordinations, see for instance [**2**], [**3**] and [**4**]. In this paper we generalize the following Nunokawa lemma, [**5**], see also [**6**] for its angle version.

Lemma 1.2. *Let p be analytic function in $|z| < 1$, with $p(0) = 1$. If there exists a point z_0, $|z_0| < 1$, such that $\mathrm{Re}\{p(z)\} > 0$ for $|z| < |z_0|$ and $p(z_0) = \pm ia$ for some $a > 0$, then we have*

$$\frac{z_0 p'(z_0)}{p(z_0)} = \frac{2ik \arg\{p(z_0)\}}{\pi},$$

for some $k \geqslant (a + a^{-1})/2 \geqslant 1$.

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2. Main results

THEOREM 2.1. *Let p be an analytic function in $|z|<1$. If there exists a point z_0, $|z_0|<1$, such that* $\mathrm{Re}\{p(z)\}>0$ *for* $|z|<|z_0|$ *and* $p(z_0)=\pm ia$ *for some* $a>0$, *then we have* $z_0p'(z_0)=ikp(z_0)$, *where*

$$k \geqslant \frac{|p(0)|^2 - 2a\,\mathrm{Im}\{p(0)\} + a^2}{2a\,\mathrm{Re}\{p(0)\}} > 0 \quad \textit{when} \quad \arg\{p(z_0)\} = \frac{\pi}{2} \tag{2.1}$$

$$k \leqslant -\frac{|p(0)|^2 + 2a\,\mathrm{Im}\{p(0)\} + a^2}{2a\,\mathrm{Re}\{p(0)\}} < 0 \quad \textit{when} \quad \arg\{p(z_0)\} = -\frac{\pi}{2} \tag{2.2}$$

PROOF. Let

$$q(z) = \frac{p(0) - p(z)}{\overline{p(0)} + p(z)}, \quad z \in \mathbb{D}. \tag{2.3}$$

Then $q(z)$ is an analytic function in the unit disc $\mathbb{D}$ with $q(0)=0$. If there exists a point z_0, $|z_0|=r<1$, such that $\mathrm{Re}\{p(z)\}>0$ for $|z|<|z_0|$ and $p(z_0)=\pm ia$ for some $a>0$, then $|q(z)|$ attains its maximum value $|q(z_0)|=1$ on the disc $|z|\leqslant r$ at the point z_0. By the Jack lemma 1.1 we have

$$z_0q'(z_0) = sq(z_0), \quad s \geqslant 1. \tag{2.4}$$

Moreover, by (2.3), we have

$$z_0q'(z_0) = -\frac{2z_0p'(z_0)\,\mathrm{Re}\{p(0)\}}{(\overline{p(0)} + p(z_0))^2}, \quad sq(z_0) = s\frac{p(0) - p(z_0)}{\overline{p(0)} + p(z_0)}. \tag{2.5}$$

Substituting (2.5) in (2.4), we obtain

$$\begin{aligned} z_0p'(z_0) &= -s\frac{(p(0) - p(z_0))}{\overline{p(0)} + p(z_0)}\frac{(\overline{p(0)} + p(z_0))^2}{2\,\mathrm{Re}\{p(0)\}} \\ &= -s\frac{|p(0)|^2 + 2ip(z_0)\,\mathrm{Im}\{p(0)\} - p^2(z_0)}{2\,\mathrm{Re}\{p(0)\}} \\ &= ip(z_0)s\frac{|p(0)|^2 + 2ip(z_0)\,\mathrm{Im}\{p(0)\} - p^2(z_0)}{-2ip(z_0)\,\mathrm{Re}\{p(0)\}} = ip(z_0)k, \end{aligned}$$

where from

$$k = s\frac{|p(0)|^2 + 2ip(z_0)\,\mathrm{Im}\{p(0)\} - p^2(z_0)}{-2ip(z_0)\,\mathrm{Re}\{p(0)\}}. \tag{2.6}$$

For the case $p(z_0)=ia$, we have

$$\begin{aligned} k &= s\frac{|p(0)|^2 + 2ip(z_0)\,\mathrm{Im}\{p(0)\} - p^2(z_0)}{-2ip(z_0)\,\mathrm{Re}\{p(0)\}} \\ &= s\frac{|p(0)|^2 - 2a\,\mathrm{Im}\{p(0)\} + a^2}{2a\,\mathrm{Re}\{p(0)\}} \geqslant \frac{|p(0)|^2 - 2a\,\mathrm{Im}\{p(0)\} + a^2}{2a\,\mathrm{Re}\{p(0)\}}, \end{aligned}$$

because $s \geqslant 1$, and

$$\frac{|p(0)|^2 - 2a\,\mathrm{Im}\{p(0)\} + a^2}{2a\,\mathrm{Re}\{p(0)\}} = \frac{(a - |p(0)|)^2 + 2a(|p(0)| - \mathrm{Im}\{p(0)\})}{2a\,\mathrm{Re}\{p(0)\}} > 0.$$

Therefore, we obtain (2.1). For the case $p(z_0) = -a$, we have from (2.6)

$$\begin{aligned} k &= s\frac{|p(0)|^2 + 2ip(z_0)\operatorname{Im}\{p(0)\} - p^2(z_0)}{-2ip(z_0)\operatorname{Re}\{p(0)\}} \\ &= s\frac{|p(0)|^2 + 2a\operatorname{Im}\{p(0)\} + a^2}{-2a\operatorname{Re}\{p(0)\}} \\ &\leqslant -\frac{|p(0)|^2 + 2a\operatorname{Im}\{p(0)\} + a^2}{2a\operatorname{Re}\{p(0)\}}, \end{aligned}$$

because $s \geqslant 1$, and

$$\frac{|p(0)|^2 + 2a\operatorname{Im}\{p(0)\} + a^2}{2a\operatorname{Re}\{p(0)\}} = \frac{(a - |p(0)|)^2 + 2a(|p(0)| + \operatorname{Im}\{p(0)\})}{2a\operatorname{Re}\{p(0)\}} > 0.$$

Therefore, we obtain (2.2). □

Corollary 2.2. *Let p be an analytic function in $|z| < 1$ with $p(0) = b \in \mathbb{R}$. If there exists a point z_0, $|z_0| < 1$, such that $\operatorname{Re}\{p(z)\} > 0$ for $|z| < |z_0|$ and $p(z_0) = \pm ia$ for some $a > 0$, then we have $z_0p'(z_0) = ikp(z_0)$, where*

$$\begin{aligned} k &\geqslant \frac{|b|^2 + a^2}{2ab} > 0 \quad \textit{when} \quad \arg\{p(z_0)\} = \frac{\pi}{2} \\ k &\leqslant -\frac{|b|^2 + a^2}{2ab} < 0 \quad \textit{when} \quad \arg\{p(z_0)\} = -\frac{\pi}{2}. \end{aligned}$$

For $b = 1$ Corollary 2.2 becomes Nunokawa's lemma 1.2.

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University of Gunma, Hoshikuki-cho 798-8
Chuou-Ward, Chiba, Japan
mamoru_nuno@doctor.nifty.jp

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Department of Applied Mathematics, College of Natural Sciences
Pukyong National University
Busan, Korea
necho@pknu.ac.kr

College of Natural Sciences, University of Rzeszów
Rzeszów, Poland
jsokol@ur.edu.pl