ON THE JACK LEMMA
AND ITS GENERALIZATION

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Abstract. We present a new geometric approach to some problems in differential subordination theory and discuss the new results closely related to the Jack lemma.

1. Introduction

Let $H$ denote the class of analytic functions in the unit disc $D = \{z : |z| < 1\}$ on the complex plane $\mathbb{C}$. Let $f, g \in H$, $f(|z| < |z_0|) \subset g(|z| < |\zeta_0|)$ and $f(z_0) = g(\zeta_0)$. In this paper we are looking for the relation between $f(z_0)$ and $z_0 f'(z_0)$ for the case when $g(z)$ maps the unit disc $D$ onto the right half-plane $\Re\{w\} > 0$. In order to prove our main result we recall the following lemma, well known as the Jack lemma. This lemma describes the case when $g(z) = z$.

Lemma 1.1. [1] Let $w(z)$ be a nonconstant and analytic function in the unit disc $D$ with $w(0) = 0$. If $|w(z)|$ attains its maximum value on the disc $|z| \leq r$ at the point $z_0$, $|z_0| = r$, then $z_0 w'(z_0) = sw(z_0)$ and $s \geq 1$.

The Jack lemma has several applications and generalizations in the theory of differential subordinations, see for instance [2], [3] and [4]. In this paper we generalize the following Nunokawa lemma, [5], see also [6] for its angle version.

Lemma 1.2. Let $p$ be analytic function in $|z| < 1$, with $p(0) = 1$. If there exists a point $z_0$, $|z_0| < 1$, such that $\Re\{p(z)\} > 0$ for $|z| < |z_0|$ and $p(z_0) = \pm ia$ for some $a > 0$, then we have

$$
\frac{z_0 p'(z_0)}{p(z_0)} = \frac{2ik \arg\{p(z_0)\}}{\pi},
$$

for some $k \geq (a + a^{-1})/2 \geq 1$.

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2. Main results

THEOREM 2.1. Let \( p \) be an analytic function in \( |z| < 1 \). If there exists a point \( z_0, |z_0| < 1 \), such that \( \text{Re}\{p(z)\} > 0 \) for \( |z| < |z_0| \) and \( p(z_0) = \pm ia \) for some \( a > 0 \), then we have \( z_0 p'(z_0) = ikp(z_0) \), where

\[
\begin{align*}
(2.1) \quad k & \geq \frac{|p(0)|^2 - 2a \text{Im}\{p(0)\} + a^2}{2a \text{Re}\{p(0)\}} > 0 \quad \text{when} \quad \text{arg}\{p(z_0)\} = \frac{\pi}{2} \\
(2.2) \quad k & \leq -\frac{|p(0)|^2 + 2a \text{Im}\{p(0)\} + a^2}{2a \text{Re}\{p(0)\}} < 0 \quad \text{when} \quad \text{arg}\{p(z_0)\} = -\frac{\pi}{2}
\end{align*}
\]

PROOF. Let

\[
q(z) = \frac{p(0) - p(z)}{p(0) + p(z)}, \quad z \in \mathbb{D}.
\]

Then \( q(z) \) is an analytic function in the unit disc \( \mathbb{D} \) with \( q(0) = 0 \). If there exists a point \( z_0, |z_0| = r < 1 \), such that \( \text{Re}\{p(z)\} > 0 \) for \( |z| < |z_0| \) and \( p(z_0) = \pm ia \) for some \( a > 0 \), then \( |q(z)| \) attains its maximum value \( |q(z_0)| = 1 \) on the disc \( |z| \leq r \) at the point \( z_0 \). By the Jack lemma \[1\] we have

\[
(2.4) \quad z_0 q'(z_0) = sq(z_0), \quad s \geq 1.
\]

Moreover, by (2.3), we have

\[
(2.5) \quad z_0 q'(z_0) = -\frac{2z_0 p'(z_0) \text{Re}\{p(0)\}}{(p(0) + p(z_0))^2}, \quad sq(z_0) = s \frac{p(0) - p(z_0)}{p(0) + p(z_0)}.
\]

Substituting (2.5) in (2.4), we obtain

\[
z_0 p'(z_0) = -s \frac{(p(0) - p(z_0)) (p(0) + p(z_0))^2}{2 \text{Re}\{p(0)\}}
\]

\[
= -s \frac{|p(0)|^2 + 2ip(z_0) \text{Im}\{p(0)\} - p^2(z_0)}{2 \text{Re}\{p(0)\}}
\]

\[
= ip(z_0)s \frac{|p(0)|^2 + 2ip(z_0) \text{Im}\{p(0)\} - p^2(z_0)}{-2ip(z_0) \text{Re}\{p(0)\}} = ip(z_0)k,
\]

where from

\[
k = s \frac{|p(0)|^2 + 2ip(z_0) \text{Im}\{p(0)\} - p^2(z_0)}{-2ip(z_0) \text{Re}\{p(0)\}}.
\]

For the case \( p(z_0) = ia \), we have

\[
k = s \frac{|p(0)|^2 + 2ip(z_0) \text{Im}\{p(0)\} - p^2(z_0)}{-2ip(z_0) \text{Re}\{p(0)\}}
\]

\[
= s \frac{|p(0)|^2 - 2a \text{Im}\{p(0)\} + a^2}{2a \text{Re}\{p(0)\}} \geq \frac{|p(0)|^2 - 2a \text{Im}\{p(0)\} + a^2}{2a \text{Re}\{p(0)\}},
\]

because \( s \geq 1 \), and

\[
\frac{|p(0)|^2 - 2a \text{Im}\{p(0)\} + a^2}{2a \text{Re}\{p(0)\}} > 0.
\]
Therefore, we obtain (2.1). For the case \( p(z_0) = -a \), we have from (2.6)

\[
k = s \left| p(0) \right|^2 + 2a \left| \text{Im}\{p(0)\} \right| + a^2
\]

\[
\leq \frac{\left| p(0) \right|^2 + 2a \left| \text{Im}\{p(0)\} \right| + a^2}{2a \text{Re}\{p(0)\}}
\]

because \( s \geq 1 \), and

\[
\frac{\left| p(0) \right|^2 + 2a \left| \text{Im}\{p(0)\} \right| + a^2}{2a \text{Re}\{p(0)\}} = \frac{(a - \left| p(0) \right|)^2 + 2a(\left| p(0) \right| + \text{Im}\{p(0)\})}{2a \text{Re}\{p(0)\}} > 0.
\]

Therefore, we obtain (2.2). \( \square \)

**Corollary 2.2.** Let \( p \) be an analytic function in \( |z| < 1 \) with \( p(0) = b \in \mathbb{R} \). If there exists a point \( z_0, \left| z_0 \right| < 1 \), such that \( \text{Re}\{p(z)\} > 0 \) for \( |z| < |z_0| \) and \( p(z_0) = \pm ia \) for some \( a > 0 \), then we have \( z_0 \frac{p'(z_0)}{p(z_0)} = ikp(z_0) \), where

\[
k \geq \frac{|b|^2 + a^2}{2ab} > 0 \quad \text{when} \quad \arg\{p(z_0)\} = \frac{\pi}{2}
\]

\[
k \leq -\frac{|b|^2 + a^2}{2ab} < 0 \quad \text{when} \quad \arg\{p(z_0)\} = -\frac{\pi}{2}.
\]

For \( b = 1 \) Corollary 2.2 becomes Nunokawa’s lemma 1.2.

**References**