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HYPERFINITE LOGICS AND NON-STANDARD EXTENSIONS OF BOOLEAN ALGEBRAS

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Abstract. Infinitary propositional logics, i.e., propositional logics with infinite conjunction and disjunction, have some deficiencies, e.g., these logics fail to be compact or complete, in general. Such kind of infinitary propositional logics are introduced, called hyperfinite logics, which are defined in a non-standard framework of non-standard analysis and have hyperfinite conjunctions and disjunctions. They have more nice properties than infinitary logics have, in general. Furthermore, non-standard extensions of Boolean algebras are investigated. These algebras can be regarded as algebraizations of hyperfinite logics, they have several unusual properties. These Boolean algebras are closed under the hyperfinite sums and products, they are representable by hyperfinitely closed Boolean set algebras and they are omega-compact. It is proved that standard Boolean algebras are representable by Boolean set algebras with a hyperfinite unit.

1. Introduction

Infinitary logics (i.e., propositional logics with infinitary conjunctions and disjunctions) have been investigated intensively, there is an extended literature for these logics [**2**, **5**, **9**].

These logics have important applications in mathematics. For example, infinitary propositional logics appear in the theory of Boolean algebras, or in the foundations of probability theory. Their expressive power is strong, but completeness, or compactness holds only in exceptional cases. An important exception is the infinitary propositional logic L_{ω_1} with *countable* conjunction and disjunction, this logic is complete.

Here we introduce a so-called hyperfinite logic L_H, i.e. a propositional logic having *hyperfinite* conjunctions and disjunctions. Hyperfinite logic is defined in a non-standard framework of non-standard analysis. Hyperfinite logic is complete, as it is proven. Roughly speaking, as is known, "hyperfinite" means an "infinite" whose behaviour is "like the finite". The part of the logic L_H, where the ranks

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of the formulas are finite, is L_{HF} (hyperfinite logic with finite rank). L_{HF} can be considered as a *dual* of L_{ω_1}, in a sense, because *hyperfinite* conjunctions and disjunctions are not denumerable operations. Hyperfinite logic has the "countable saturation" property.

The topics hyperfinite logic and non-standard extensions (*-transforms) of Boolean algebras are closely related. The latter algebras can be regarded as algebraization of hyperfinite logic. Non-standard extensions of Boolean algebras are interesting also for themselves, they have several unusual properties. They are closed under hyperfinite unions and intersections (they are "hyperfinitely closed"), but, cannot include the denumerable unions and intersections. They are ω-compact and they are representable by hyperfinitely closed Boolean *set algebras* (as is known, Boolean λ-algebras are not representable by Boolean λ-set algebras, in general, where λ is any fixed infinite cardinality). A consequence is a variant of the Stone representation theorem: Boolean algebras are representable as Boolean set algebras with a *hyperfinite* unit. Applications of the topic is expected in probability theory and in the theory of Boolean algebras.

Some concepts of hyperfinite logic is listed in 2, 3 concerns the semantics of hyperfinite logic, the completeness is proved in 4. The connection of hyperfinite logic and the *-transforms of Boolean algebras is analysed in 5.

2. On the concept of hyperfinite logic

We work in a general *framework*, in a suitable *superstructure* and *enlargement,* defined in non-standard analysis. The knowledge of the basic concepts as *- transform, hyperfiniteness, hypernatural, internality, etc. are assumed as prerequisites (see [**1**, **8**, **10**, **12**, **13**]).

A classical propositional logic L is assumed. As regards its *language* $\mathcal{L}$, the only unusual feature is that instead of the binary operation symbols conjunction $\wedge$ and disjunction $\vee$, the symbols of infinitely-many n-ary operations, the n-ary conjunctions $\bigwedge_n$ and the n-ary disjunctions $\bigvee_n$ are assumed in $\mathcal{L}$, where n runs over the natural numbers being $\geqslant 2$. $\mathcal{L}$ contains also the unary operation symbol negation $\neg$. Furthermore, the language includes a set $\{B_j : j \in N\}$ of propositional symbols (N is the set of natural numbers).

The concept of formula, i.e. that of well-formed formula (wff), is the usual:

(i) The propositional symbols are formulas.
(ii) If $\alpha_1, \alpha_2, \dots \alpha_n$ is a finite sequence of formulas, where n is a fixed natural number, then $\bigwedge_n \alpha_i$, $\bigvee_n \alpha_i$ are formulas, and $\neg\alpha_i$ is a formula.

Formulas are obtained by applying finitely-many times the rules in (i) and (ii). Let W denote the set of formulas in L.The common term for a propositional symbol and its negated is the *literal*.

The semantics of propositional logic is the usual:

An interpretation function s is a mapping from the set W of the formulas into the Boolean algebra $\mathcal{B}$ of two elements 0 and 1, having the property (p) below:

(p) if $\alpha_1, \alpha_2, \dots \alpha_n$ are formulas, then

$$s\big(\bigwedge_n \alpha_i\big) = \min_{1\leqslant i\leqslant n}\{s(\alpha_i)\},\ \ s\big(\bigvee_n \alpha_i\big) = \max_{1\leqslant i\leqslant n}\{s(\alpha_i)\},\ \ s(\neg\alpha_1) = -s(\alpha_1).$$

Let S denote a fixed non-empty set (possible interpretations) of the interpretation functions. The *truth set* $[\alpha]$ of a formula α is the set $\{s \in S : s(\alpha) = 1\}$.

Roughly speaking, hyperfinite logic will be the *-transform of the classical propositional logic introduced above. Next, this *-transform will be described and reformulated.

Let us consider the collection of the formulas in $\mathcal{L}$, the operations of the language $\mathcal{L}$ and the set S of the possible interpretation functions. Let us form a superstructure built on this collection and let us form the usual enlargement. Let * denote the usual embedding function (see [**8**, **13**]). As is known, the cardinality of a hyperfinite set is not denumerable.

By a *Q-sequence* $\langle a_i : i \in Q \rangle$ we mean an *internal* sequence, where Q is a hypernatural, i.e. $\langle a_i : i \in Q \rangle$ is an internal function.

Next, the *hyperfinite logic* L_H and the *hyperfinite logic with finite ranks* L_{HF} are defined:

The set of the propositional symbols is the set $^*\{B_j : j \in N\}$ denoted by A. In addition to the negation $\neg$, the operation symbols $\bigwedge_Q$ and $\bigvee_Q$ are assumed in the language, where Q runs over the hypernaturals.

The concept of *formula* α (i.e. a well-formed formula α) in L_H:

(i) The members of A are formulas.
(ii) If $\langle \alpha_i \rangle_{i \in Q}$ is a Q-sequence of formulas, where Q is a fixed hypernatural, then $\bigwedge_Q \alpha_i$, $\bigvee_Q \alpha_i$ are formulas, and $\neg\alpha_i$ is a formula.

Formulas of L_H are obtained by applying hyperfinitely-times, say M-times, the rules (i) and (ii) (where M is a hypernatural), furthermore, any sequence of the subformulas resulting α, are assumed to be an M-sequence (internal sequence).

The *rank of a formula* is the M occuring in the definition of the formula. If M is finite, then the formula is said to be a formula of finite rank.

Let L_{HF} denote the part of L_H such that the ranks of the formulas are *finite*. L_{HF} is called *hyperfinite logic with finite ranks.*

An *interpretation function* t for the logic L_H is an internal function, mapping from the set of the formulas in L_H, into the Boolean algebra $\mathcal{B}$ of two elements 0 and 1, having the property (p') below:

(p') if $\alpha_1, \alpha_2, \dots \alpha_Q$ is a Q-sequence of formulas in L_H, then

$$t\big(\textstyle\bigwedge_Q \alpha_i\big) = \min_{1 \leqslant i \leqslant Q}\{t(\alpha_i)\}, \;\; t\big(\textstyle\bigvee_Q \alpha_i\big) = \max_{1 \leqslant i \leqslant Q}\{t(\alpha_i)\}, \;\; t(\neg\alpha_1) = -t(\alpha_1).$$

The restriction of the interpretation function t to A is called *elementary interpretation.*

The concept of interpretation function origins from the facts that the Boolean algebra $\mathcal{B}$ of two elements remains unchanged at the *-transform and the set S of interpretation functions of L goes into a set $^*S$ of internal functions mapping the formulas of L_H into $\mathcal{B}$. The property (p) is inherited at the *-transform from the finite set of formulas in L to the hyperfinite set of formulas, i.e. to the Q-sequences of formulas in L_H, this is why (p') is true. Furthermore, $^*\min = \min$ and $^*\max = \max$.

The concept of interpretation function in L_H implies the definitions of the main concepts of semantics. Thus, it implies the concept of satisfiability, or the logical equivalency of two formulas: α and β are *logical equivalent* if their interpretations t give the same value for every $t \in {}^*S$.

Similarly to classical logic, $\Sigma \models \beta$ is equivalent to the unsatisfiability of the formula set $\Sigma \cup \{\neg\beta\}$.

3. Some semantical properties of hyperfinite logic

The definition by *-transform implies that hyperfinite logics have many analogous properties with classical propositional logic. Many algebraic properties are transfered by *: associativity, commutativity, distributivity of the conjunctions and disjunctions.

The formulation of the normal form theorem for the logic L_H is the following:

THEOREM 3.1. *Every formula of L_H is logical equivalent to a hyperfinite disjunction of certain hyperfinite conjunctions of literals, i.e. equivalent to a formula of the form*

$$\bigvee_{i \leqslant Q} \left(\bigwedge_{j \leqslant K_i} L_{ij} \right) \tag{3.1}$$

where Q, K_i are hypernaturals, L_{ij} are literals and the members of the disjunctions and conjunctions in (3.1) *constitute Q-, and K_i-sequences (i.e. internal sequences).*

PROOF. Known theorem of classical propositional logic is: for every formula α in W, there is a formula β being a finite disjunction of certain finite conjunctions of literals, logical equivalent to α. Considering the *-transform, we get that for every formula α in $^*W$ there is a formula β being a hyperfinite disjunction of hyperfinite conjunctions of literals, logical equivalent to α. The finite sets of the members of the finite conjunctions and disjunctions go into internal hyperfinite sequences at the *-transform. $\square$

Similarly, the De Morgan laws remain true in L_H. That is, the following is true: *The pair of formulas $\neg \bigvee_Q \gamma_i$ and $\bigwedge_Q \neg\gamma$, furthermore $\neg \bigwedge_Q \gamma_i$ and $\bigvee_Q \neg\gamma_i$ are logical equivalent* in L_H *for every Q-sequence $\langle\gamma_i\rangle_{i\in Q}$.*

With the logic L_{ω_1} the logic L_{HF} can be associated rather than L_H because the ranks of the formulas are finite in both logics. There are many similar properties of L_{ω_1} and L_{HF}. Next, however such a property of L_{HF} is presented which makes a difference between these two logics.

An important semantical concept in infinitary logics is the semantical consistency (see [**9**]). A set Λ of formulas is *semantical consistent* if every *finite* subset of Λ is satisfiable.

Recall that for the logic L_{ω_1} the equivalency of satisfiability and semantical consistency for countable formula sets *fails* to be true (see [**9**]).

Let Λ be a countable set of formulas of the logics L_{HF}, or L_H.

THEOREM 3.2. *For any countable set Λ of formulas, Λ is satisfiable if and only if Λ is semantical consistent.*

PROOF. The satisfiability of Λ implies the semantical consistency of Λ, this is trivial. We check the other direction. Using the definition of formula in L_H, we prove that the truth sets $\{t \in {}^*S : t(\alpha) = 1\}$ of *formulas* α are internal sets.

Let us consider the truth set of a propositional symbol C, i.e. the set $\{t \in {}^*S : t(C) = 1,\ C \in A\}$, where A denotes the set ${}^*\{B_j : j \in N\}$. The internal definition principle (see [**13**]) implies that this set is internal.

Assume that the truth sets of the formulas α and β are internal sets. By definition, these are subsets of the internal set ${}^*S$. As is known, the complement, the union and the intersections of two internal sets are also internal. And, if the truth sets of the formulas form a Q-sequence $\langle \alpha_i : i \in Q \rangle$ (i.e. an internal sequence), then the truth sets of the formulas $\bigvee_{i \leqslant Q} \alpha_i$ and $\bigwedge_{i \leqslant Q} \alpha_i$ are also internal sets, because these are internal unions and intersections of internal sets (see [**13**]).

Then, if α is in L_{HF}, i.e. α can be composed in finitely-many steps, then ordinary formula induction applies.

If α is in L_H, then *internal formula induction* applies [**8**, Ch. 11.3] rather than ordinary formula induction. Let us consider the *internal sequence* $\langle q \rangle$ of the subformulas resulting α by definition. If α is atomic, we are ready. Assume that the truth set of the Kth member of $\langle q \rangle$ is internal, where K is a hypernatural. To prove that the truth set of the $(K+1)$th member of $\langle q \rangle$ is also internal, the same argument applies as above at the induction step.

To complete the proof, we use that every enlargement has the "countable saturation" property, i.e. the property that every countable collection of internal sets having the finite intersection property has a non-empty intersection [**8**]. $\square$

The theorem can be generalized from countable cardinality to cardinalities less than the fixed cardinal number κ, using so-called *κ-saturated* enlargements (see [**13**, Ch. 4]).

The developement of the first order variant of hyperfinite logic exceeds the frame of this paper, but all the technics needed for that appear here. For example, first order normal forms (prenex, Skolem) can be introduced and the completeness can be reduced to the propositional case, as in classical logic.

4. On the completeness of the hyperfinite logic

To prove the completeness of the hyperfinite logic L_H analytic tableaux is applied. The concept of analytic tableaux (for short, tableaux) for classical propositional logic is considered to be known (see [**3**]). Roughly speaking, producing an analytic tableaux for propositional logic is a kind of implementation for producing a disjunctive normal form.

Classical propositional tableaux has finitely-many nodes and the tableaux is defined by induction. The concept of analytic tableaux for the logic L_H will be the *-transform of that of classical propositional logic. Tableaux for L_H will have hyperfinitely-many nodes and the definition happens by internal induction (an argument for this is given later).

The description of the *-transformed tableaux is:

First step ($K = 1$). Let the root include (be labelled by) a formula α of L_H. If α is an atomic formula, or a negated atomic formula, the procedure is finished. Assume that α is a compound formula. If α is a disjunction, then hyperfinitely-many nodes follow the root, each includes a disjunction member of α, respectively, instead of α. If α is a conjunction, then one node follows the root, including the conjunction members of α, instead of α. If α is a negation, then, if it is a double negation, then the double negation is omitted from α in the next node, else, one of the De Morgan rules is applied and a is replaced by its equivalent in the next node.

Assume that the tableaux is defined for the first K numbers, where K is a hypernatural, i.e. assume that the Kth node $\mathcal{N}$ has been defined.

The $(K + 1)$th step. If $\mathcal{N}$ includes a contradictory pair of formulas, then the branch of $\mathcal{N}$ (or simply $\mathcal{N}$) is said to be closed and another node being not closed is selected. If $\mathcal{N}$ includes only literals (but there is no contradictory pair), then the procedure is finished. Assume that $\mathcal{N}$ includes a compound formula. Then the procedure described at the first step should be repeated.

The procedure is finished if all the branches are closed (in this case the tableaux is called closed), or there is a node including only literals without contradictory pair. By the internal induction the procedure is defined for ${}^*N$-many steps (where N is a natural number). The procedure is obviously finished in M steps, where M is a hypernatural, but the steps after M are considered to result empty nodes.

Applying internal induction in the definition is justifiable because of the following reasons. Following the steps of the procedure, we get a sequence $\langle r \rangle$ of subformulas of α (being decomposed in a step) with the properties: every member of the sequence is a subformula of the previous one, or a subformula of some preceding one in the sequence, or it is an equivalent of a subformula by some De Morgan rule. Let us consider the internal subformula tree τ of α and eliminate the negations using the De Morgan rules, i.e. transform the negations into the literals using the De Morgan rules, let π denote the subformula tree obtained. The conjunction-disjunction structures of τ and π are the same (conjunctions and disjunctions are changed, at most) and π inherits the internality of τ. With the sequence $\langle r \rangle$ a decomposition sequence of the nodes can be associated in π, therefore the internality of π implies the internality of $\langle r \rangle$.

The following theorem states the completeness of analytic tableaux for the logic L_H.

THEOREM 4.1. *A formula α in L_H is unsatisfiable if and only if α has a closed analytic tableaux. If α is in L_{HF}, then the branches of its analytic tableaux are finite and there is a natural number upper bound for the lenghts of the branches.*

PROOF. By *-transform, the completeness follows from the respective theorem for classical propositional logic.

If α is in L_{HF}, then α has a finite rank. Considering a fixed branch and the successors nodes on it, respectively, the maximal rank of the formulas on a node, strictly decreases node-by-node, or, at applying the De Morgan rule, this maximal rank strictly decreases after two nodes. Therefore, following these maximal ranks on a branch it must became 1, or 2 after finitely-many steps, i.e. until a lief (last node) appears. This means that the branch is finite.

But, the leafs of the tableaux can be arranged into some Q-sequence, where Q is a hypernatural (see the normal form theorem). This is valid also for the lengths (natural numbers) of the branches corresponding to the leafs. However, by the "underflow principle" there is a natural number being upper bound of the foregoing lengths (this upper bound can be considered as the depth of the tableaux). □

While the procedure of forming a classical analytic tableaux is finite, that of forming an analytic tableaux for a formula in hyperfinite logic is hyperfinite. So the *"lengths" of the "deductions" are hyperfinite* (i.e. it is a hyperfinite, internal sequence). By (ii), if α is in L_{HF}, then the "depth" of the tableaux is finite, but the "width" is maybe hyperfinite.

As usual, the equivalency of $\Sigma \models \beta$ and the unsatisfiableness of the formula set $\Sigma \cup \{\neg\beta\}$ yields a kind of concept of deducibility of β from Σ.

5. Hyperfinitely closed Boolean algebras

As is known, in classical proposition logic the truth sets $[\alpha]'$s form a Boolean subalgebra $\mathcal{M}$ of the power Boolean set algebra $\mathcal{P}(S)$, where S is the set of the possible interpretations. Of course, $\mathcal{M}$ is closed under the finite unions and intersections.

A Boolean algebra $\mathcal{D}$ is hyperfinitely closed if $\langle d_i\rangle_{i\in Q}$ is any Q-sequence of the elements d_i in D, then $\bigcup_{i\in Q} d_i \in D$ and $\bigcap_{i\in Q} d_i \in C$. $\mathcal{D}$ is ω-compact if every countable subset of D, having the finite product property, has a non-empty infimum.

THEOREM 5.1. (i) *The truth sets of the formulas in L_H form a Boolean set algebra $\mathcal{C}$ closed under the hyperfinite unions and intersections ($\mathcal{C}$ is hyperfinitely closed), i.e. if $\langle A_i\rangle_{i\in Q}$ is any Q-sequence of the sets A_i in $\mathcal{C}$, then $\bigcup_{i\in Q} A_i \in \mathcal{C}$ and $\bigcap_{i\in Q} A_i \in \mathcal{C}$, where $\bigcup$ and $\bigcap$ mean ordinary union and intersection.*

(ii) *$\mathcal{C}$ is ω-compact and it cannot inlude the denumerable unions, or intersections of its members.*

PROOF. (i) $\mathcal{M} \subset P(S)$, where S denotes the set of the possible interpretations in the classical logic and $\mathcal{M}$ denotes the Boolean set algebra of the truth sets in the classical logic. Forming the *-transform, we get that $^*\mathcal{M} \subset {}^*\mathcal{P}(S)$. Here $^*\mathcal{P}(S)$ is the collection of the internal sets in $^*\mathcal{S}$. Furthermore, $^*\mathcal{M}$ includes the Boolean set algebra of the truth sets for L_H. Therefore the truth sets are internal sets, more exactly: $^*\{s \in S : s(\alpha) = 1, \alpha \in W\} = \{s \in {}^*S : s(\alpha) = 1, \alpha \in {}^*W\}$ and, as is known, they are closed under hyperfinite unions and intersections included in (i).

Notice that (i) follows also from the proof of Theorem 3.2.

(ii) The members of $\mathcal{C}$ are internal sets, thus $\mathcal{C}$ is ω-compact. By the underflow principle, if $D_i \in \mathcal{C}$ $(i \in N)$, then $\bigcup_{i\in N} D_i = \bigcup_{i\leqslant n} D_i$ for some natural number n [**8**, Ch. 11.10]. The case of the intersection is analogous. □

The hyperfinitely closedness is a rather unusual property of Boolean set algebras. Hyperfinitely closed Boolean set algebras can be considered as some algebraization of hyperfinite logic L_H, since a kind of converse of the above theorem is true, as well: with every hyperfinitely closed Boolean set algebra some hyperfinite logic can be associated.

Next we show how to get the algebraizations of hyperfinite logic purely in algebraic way.

We set out from ordinary Boolean algebras and from their enlargements.

THEOREM 5.2. *Let $\mathcal{A}$ be a Boolean algebra and ${}^*\mathcal{A}$ the enlargement of $\mathcal{A}$. Then*

(i) *${}^*\mathcal{A}$ is hyperfinitely closed*

(ii) *${}^*\mathcal{A}$ has such a Boolean set algebra representation $\widetilde{\mathcal{A}}$ that $\widetilde{\mathcal{A}}$ is hyperfinitely closed as set algebra, the hyperfinite infima and suprema in ${}^*\mathcal{A}$ are preserved at the canonical isomorphism and the members of $\widetilde{\mathcal{A}}$ are internal sets*

(iii) *$\widetilde{\mathcal{A}}$ is ω-compact.*

PROOF. (i) $\mathcal{A}$ is finitely closed under the Boolean operations, thus, by *-transfer, ${}^*\mathcal{A}$ is hyperfinitely closed.

(ii) We set out from a set algebra representation $\mathcal{A}'$ of $\mathcal{A}$ and form the *-transform $\widetilde{\mathcal{A}}$ of $\mathcal{A}'$. We show that the hyperfinite infima and suprema in $\widetilde{\mathcal{A}}$ are exactly the set operations intersections and unions. As regards the union, for example, this follows from the definition of the union. Let us consider the union axiom for finite unions in $\mathcal{A}'$:

$$\forall x \in \mathcal{P}_F(\mathcal{A}')\, \exists y \in A'\, \forall u(u \in y \leftrightarrow \exists v(u \in v \wedge v \in x))$$

The *-transform of this formula is:

$$\forall x \in {}^*\mathcal{P}_F(\mathcal{A}')\, \exists y \in \tilde{A}\, \forall u(u \in y \leftrightarrow \exists v(u \in v \wedge v \in x)).$$

It says that for a hyperfinite set x ($x \in {}^*\mathcal{P}_F(\mathcal{A}')$) the union of the members of x exists in $\tilde{A}$.

The members of $\tilde{\mathcal{A}}$ are internal, by definition. Composing the inverse of the transformation * defined on $\mathcal{A}$, the ordinary canonical isomorphism and the embedding * of $\mathcal{A}'$, we get the canonical isomorphism from ${}^*\mathcal{A}$ onto $\tilde{\mathcal{A}}$. It obviously preserves the hyperfinite operations.

(iii) By (ii), the members of $\tilde{\mathcal{A}}$ are internal. The proposition follows from the known saturation property of the internal sets (see [**8**, 15.5]). $\square$

Notice that while by (ii) the algebra $\tilde{\mathcal{A}}$ includes the hyperfinite intersections and unions (these are uncountable operations if they are infinite ones), $\tilde{\mathcal{A}}$ really *does not include* countable intersections and unions only finite ones (because the saturation property of internal sets).

The following version of Theorem 5.2 is true:

Let $\mathcal{A}$ be a Boolean algebra. Then there exists an atomic and hyperfinite extension $\mathcal{A}^+$ of $\mathcal{A}$ such that $\mathcal{A} \subset \mathcal{A}^+ \subset {}^*\mathcal{A}$ holds, furthermore, replacing ${}^*\mathcal{A}$ by ${}^+\mathcal{A}$ in Theorem 5.2, *the properties* (i), (ii) *and* (iii) *are valid.*

The existence of $\mathcal{A}^+$ is proved in [**8**, 19.4]. The proofs of (i), (ii) and (iii) are similar to those of Theorem 5.2.

The following proposition is a version of Stone's theorem:

COROLLARY 5.1. *Every Boolean algebra is isomorphic to a Boolean set algebra with a hyperfinite unit.*

It follows from the previous version of Theorem 5.2. Because every atomic Boolean algebra can be considered as a Boolean *set algebra*, where the unit is the set of the atoms.

As regards the application of hyperfinite logic, hyperfinite sets play an important role in non-standard measure-, or probability theory, among others. For example, there is a hyperfinitely closed Boolean set algebra which includes the set of the *bounded* realizations for stochastic processes. This set is not included in the known σ-algebras. In general, certain crucial sets can be treated in terms of hyperfinitely closed Boolean algebras [**4**]. There are considerable connections also between hyperfinitely closed Boolean algebras and cylindric algebras, too (see, [**6**, **7**, **11**]), these could be a subject of a forthcoming paper.

References

1. P. V. Andreev, E. I. Gordon, *A theory of hyperfinite sets*, Ann. Pure Appl. Logic **143** (2006), 3–19.
2. J. Bairwise, *Infinitary logic and admissible sets*, J. Symb. Log. **34** (1969), 226–252.
3. M. Ben-Ari, *Mathematical Logic for Computer Science*, Springer, 2009.
4. M. Ferenczi, *Probabilities on first order models*, Publ. Inst. Math., Nouv. Sér. **78(92)** (2005), 107–115.
5. ______, *On conservative extensions in logics with infinitary predicates*, Stud. Log. **92** (2009), 121–135.
6. ______, *The polyadic generalization of the Boolean axiomatization of fields of sets*, Trans. Am. Math. Soc. **364** (2012), 867–886.
7. ______, *Cylindric algebras and finite polyadic algebras*, Algebra Univers. **79** (2018).
8. R. Goldblatt, *Lectures on the Hyperreals*, Springer, 1998.
9. C. R. Karp, *Languages with Expressions of Infinite Length*, North-Holland, 1964.
10. M. D. Rašković, *An application of nonstandard analysis for functional equations*, Publ. Inst. Math., Nouv. Sér. **37(51)** (1985), 23–24.
11. M. D. Rašković, R. S. Đorđević, *Cylindric probability algebras*, in H. Andréka, M. Ferenczi, I. Németi (eds.), *Cylindric-like Algebras and Algebraic Logic*, Springer, 2013, 303–320.
12. N. Vakil, *Real Analysis through Modern Infinitesimals*, Cambridge Univ. Press, 2011.
13. M. Väth, *Non-standard Analysis*, Birkhäuser, 2007.

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