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LOGARITHMIC (TRANSLATIONALLY) RAPIDLY VARYING SEQUENCES AND SELECTION PRINCIPLES

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Abstract. We introduce a proper subclass of the class of rapidly varying sequences (logarithmic (translationally) rapidly varying sequences), motivated by a notion in information theory (self-information of the system). We prove some of its basic properties. In the main result, we prove that Rothberger's and Kočinac's selection principles hold, when this class is on the second coordinate, and on the first coordinate we have the class of positive and unbounded sequences.

1. Introduction

A sequence $c = (c_n)$ of positive real numbers is said to be rapidly varying in the sense of de Haan (see e.g. [**1**]) of index of variability $+\infty$, if the following asymptotic condition is satisfied:

$$\lim_{n\to+\infty} \frac{c_{[\lambda n]}}{c_n} = +\infty, \qquad \text{for each } \lambda > 1.$$

The class of rapidly varying sequences we denote by $R_{\infty,s}$. These sequences are important objects in rapid variation theory in the sense of de Haan, which is very important in asymptotic analysis and its applications (see e.g. [**1**–**3**,**8**–**10**,**13**]). The theory of rapid variation is an important modification of the Karamata theory of regular variation [**17**], and their relations can be seen on the example of slow and rapid variations in terms of generalized inverse (see e.g. [**1**, **7**, **11**, **17**]).

Elements of the class $R_{\infty,s}$ are important objects in the dynamic systems theory [**14**, **15**, **19**], infinite topological games theory and selection principles theory [**3**–**6**, **13**].

A sequence $c = (c_n)$ of positive real numbers is said to be translationally rapidly varying [**5**], denoted by $c = (c_n) \in \mathrm{Tr}(R_{\infty,s})$, if the following asymptotic condition is satisfied:

$$\lim_{n\to+\infty} \frac{c_{[\lambda+n]}}{c_n} = +\infty, \qquad \text{for each } \lambda \geqslant 1.$$

It is known that $\mathrm{Tr}(R_{\infty,s}) \subsetneq R_{\infty,s}$ (see e.g. [**5**]).

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Let $\mathbb{S}$ be the set of sequences of positive real numbers [**3**], and $\bar{\mathbb{S}}^{\infty}$ the set of positive and unbounded sequences in $\mathbb{S}$. Let, also, $\mathcal{A}$ and $\mathcal{B}$ be nonempty subsets of the set $\mathbb{S}$. We identify $x \in \mathbb{S}$ with $Im(x)$.

Let us quote some important selection principles:

(1) Rothberger's selection principle $S_1(\mathcal{A}, \mathcal{B})$: for each sequence (A_n) of elements of $\mathcal{A}$, there is a sequence (b_n) element of $\mathcal{B}$ such that $b_n \in A_n$, for each $n \in \mathbb{N}$ (see e.g. [**16**]).
(2) Kočinac's selection principle $\alpha_2(\mathcal{A}, \mathcal{B})$: for each sequence (A_n) of elements of $\mathcal{A}$, there is a sequence (b_t) an element of $\mathcal{B}$ such that the set $(b_t) \cap A_n$ is infinite for each $n \in \mathbb{N}$ (see e.g. [**12**]).
(3) Kočinac's selection principle $\alpha_3(\mathcal{A}, \mathcal{B})$: for each sequence (A_n) of elements of $\mathcal{A}$, there is a sequence (b_t) in $\mathcal{B}$ such that the set $(b_t) \cap A_n$ is infinite for infinitely many $n \in \mathbb{N}$ (see e.g. [**12**]).
(4) Kočinac's selection principle $\alpha_4(\mathcal{A}, \mathcal{B})$: for each sequence (A_n) of elements of $\mathcal{A}$, there is a sequence $(b_t) \in \mathcal{B}$ such that the set $(b_t) \cap A_n$ is nonempty for infinitely many $n \in \mathbb{N}$ (see e.g. [**12**]).

Kočinac's selection principles $\alpha_i(\mathcal{A}, \mathcal{B})$, $i \in \{2, 3, 4\}$ (see e.g. [**18**]) are very important in selection principles theory. Evidently, $\alpha_2(\mathcal{A}, \mathcal{B}) \implies \alpha_3(\mathcal{A}, \mathcal{B}) \implies \alpha_4(\mathcal{A}, \mathcal{B})$.

2. Results

Let X be a countable set and let $p = (p_n)$ be distribution probability on the set X. Boltzmann's thermodynamic system (X, p) has self-information (quantity of information)

$$I[X] = \sum_{n=1}^{+\infty} p_n \cdot \log_2 p_n^{-1}. \tag{2.1}$$

Since $\sum_{n=1}^{+\infty} p_n = 1$, the equation

$$I[X] = \sum_{n=1}^{+\infty} \frac{\log_2 q_n}{q_n},$$

can be deduced by the change of variable $p_n = \frac{1}{q_n}$, $n \in \mathbb{N}$, in (2.1). We have that $(q_n) \in R_{\infty,s}$ and $(\log_2 q_n) \in R_{\infty,s}$ are very important for the rate of convergence for the sum represented by $I[X]$ (in the set of all possibilities which gives us condition $p_n \to 0$ $(n \to +\infty)$).

Remark 2.1. Since $\sum_{n=1}^{+\infty} p_n = 1$ holds, it follows $p_n \to 0$, as $n \to +\infty$. Also, for $q_n = \frac{1}{p_n}$, $n \in \mathbb{N}$, $\frac{\log_2 q_n}{q_n} \to 0$, as $n \to +\infty$ holds (we can notice that $\lim_{n\to+\infty} q_n = +\infty$). If $q = (q_n)_{n\in\mathbb{N}} \in R_{\infty,s}$, then $0 < \frac{\log_2 q_n}{\sqrt{q_n}} \cdot \frac{1}{\sqrt{q_n}} \leqslant \frac{1}{\sqrt{q_n}}$ (because $\lim_{n\to+\infty} \frac{\log_2 q_n}{\sqrt{q_n}} = 0$) for $n \in \mathbb{N}$ large enough, and the sum $\sum_{n=1}^{+\infty} \frac{1}{\sqrt{q_n}}$ converges (since $(\sqrt{q_n})_{n\in\mathbb{N}} \in R_{\infty,s}$ because of $\lim_{n\to+\infty} \frac{\sqrt{q_{[\lambda n]}}}{\sqrt{q_n}} = \sqrt{\lim_{n\to+\infty} \frac{q_{[\lambda n]}}{q_n}} = \infty$, for each $\lambda > 1$). Especially, this holds if $(q_n)_{n\in\mathbb{N}} \in \mathrm{Tr}(R_{\infty,s}) \subsetneq R_{\infty,s}$.

From previously mentioned it is natural to consider the operator

$$L_2\colon R_{\infty,s} \to \mathbb{S} \tag{2.2}$$

given by $L_2(c) = d$, where $d = (d_n)$, $d_n = 2^{c_n}$, for each $n \in \mathbb{N}$, whenever $c = (c_n)$ belongs to the class $R_{\infty,s}$.

Let also $L_2(R_{\infty,s}) = \{L_2(c) : c \in R_{\infty,s}\}$ and $L_2(\mathrm{Tr}(R_{\infty,s}))$ be the restriction of L_2 on $\mathrm{Tr}(R_{\infty,s})$.

Proposition 2.1. *$L_2(\mathrm{Tr}(R_{\infty,s})) \subsetneq L_2(R_{\infty,s}) \subsetneq R_{\infty,s}$ and $L_2(\mathrm{Tr}(R_{\infty,s})) \subsetneq \mathrm{Tr}(R_{\infty,s}) \subsetneq R_{\infty,s}$ hold.*

Proof. We prove only $L_2(R_{\infty,s}) \subsetneq R_{\infty,s}$. The inclusion $L_2(\mathrm{Tr}(R_{\infty,s})) \subsetneq \mathrm{Tr}(R_{\infty,s})$ can be proved analogously. A proof of other parts of the proposition is elementary. Let the sequence $c = (c_n) \in R_{\infty,s}$. Then, $L_2(c) = 2^{c_n}$ for $n \in \mathbb{N}$. Therefore,

$$\lim_{n\to+\infty} \frac{2^{c_{[\lambda n]}}}{2^{c_n}} = \lim_{n\to+\infty} 2^{c_n\left(\frac{c_{[\lambda n]}}{c_n}-1\right)} = +\infty$$

holds for $\lambda > 1$, because $c_n \to +\infty$, as $n \to +\infty$ (see e.g. [**3**]). Thus, $L_2(R_{\infty,s}) \subseteq R_{\infty,s}$.

Let us consider the sequence $d = (d_n)$ given with $d_n = 2^n$. This sequence is an element of the class $R_{\infty,s}$, because

$$\varliminf_{n\to+\infty} \frac{2^{[\lambda n]}}{2^n} \geqslant \varliminf_{n\to+\infty} 2^{(\lambda-1)n-1} = +\infty, \qquad \text{for } \lambda > 1.$$

The sequence d does not belong to the class $L_2(R_{\infty,s})$, since the sequence $\log_2 d_n = n$, for $n \in \mathbb{N}$, is regularly varying sequence in the sense of Karamata, with index of variability equal to 1. It means, $d \in R_{\infty,s} \setminus L_2(R_{\infty,s})$, so $L_2(R_{\infty,s}) \subsetneq R_{\infty,s}$. ☐

Elements of the set $L_2(R_{\infty,s})$ are called logarithmic rapidly varying sequences for the logarithm base 2. We consider rapidly varying sequences which are logarithmic rapidly varying sequences for the logarithm base k, $k > 1$, analogously. Similar observation can be applied to the class $L_2(\mathrm{Tr}(R_{\infty,s}))$.

Let us state an elementary, but very important property of elements from the class $L_2(\mathrm{Tr}(R_{\infty,s}))$. The symbol o below is one of the Landau symbols (see e.g. [**1**]).

Proposition 2.2. *Let $p = (p_n) \in \mathrm{Tr}(R_{\infty,s})$ and $q = (q_n) \in \mathrm{Tr}(R_{\infty,s})$.*

(a) *If $p_n = o(q_n)$ $(n \to +\infty)$, then $L_2(p_n) = o(L_2(q_n))$ $(n \to +\infty)$ holds.*
(b) *If $L_2(p_n) = o(L_2(q_n))$ $(n \to +\infty)$, then $\varlimsup_{n\to+\infty} \frac{p_n}{q_n} \leqslant 1$ holds.*

Proof. (a) Let the sequences $p = (p_n)$ and $q = (q_n)$ be elements of the class $\mathrm{Tr}(R_{\infty,s})$ and let $\lim_{n\to+\infty} \frac{p_n}{q_n} = 0$. Then

$$\lim_{n\to+\infty} \frac{L_2(p_n)}{L_2(q_n)} = \lim_{n\to+\infty} 2^{p_n-q_n} = \lim_{n\to+\infty} 2^{p_n\left(1-\frac{q_n}{p_n}\right)} = 0,$$

since the sequence p, as translationally rapidly varying, has the property $p_n \to +\infty$, as $n \to +\infty$. This means, $L_2(p_n) = o(L_2(q_n))$, as $n \to +\infty$ holds.

(b) Let $\lim_{n\to+\infty}\frac{L_2(p_n)}{L_2(q_n)}=0$. Then, by the construction of the operator L_2 in (2.2), $p=(p_n)\in \mathrm{Tr}(R_{\infty,s})$ and $q=(q_n)\in \mathrm{Tr}(R_{\infty,s})$ hold. Also, $\lim_{n\to+\infty} p_n(1-\frac{q_n}{p_n})=-\infty$. Hence $p_n\to+\infty$, as $n\to+\infty$, and therefore $\overline{\lim}_{n\to+\infty}\frac{p_n}{q_n}\leqslant 1$. $\square$

The class $L_2(\mathrm{Tr}(R_{\infty,s}))$ (also more general class $L_k(\mathrm{Tr}(R_{\infty,s}))$, for $k>1$) does not coincide with any subclass (nor contains it) of the class $R_{\infty,s}$ considered in the theory of selection principles (see e.g. [**2**, **5**, **6**]).

The following proposition contain improvements of analog results from [**3**].

Proposition 2.3. *The selection principle* $S_1(\bar{\mathbb{S}}^\infty, L_2(\mathrm{Tr}(R_{\infty,s})))$ *is satisfied.*

Proof. Let the sequence of sequences $(x_{n,m})$ be given, and for each $m^*\in\mathbb{N}$ the sequence $(x_{n,m^*})\in\bar{\mathbb{S}}^\infty$. Construct the sequence $y=(y_m)$ as follows:

1. Let us choose y_1 as an arbitrary element of the sequence $(x_{n,1})$.

2. Let the element y_m of the sequence y be chosen, for $m\in\mathbb{N}$, from the sequence $(x_{n,m})$. Let us choose the element y_{m+1} from the sequence $(x_{n,m+1})$ so that $y_{m+1}>y_m^m$. According to the construction, $y_m\in(x_{n,m})$ for each $m\in\mathbb{N}$, therefore y is the sequence of positive real numbers. Also,

$$\underline{\lim}_{m\to+\infty}\frac{\log_2(y_{[\lambda+m]})}{\log_2 y_m}\geqslant \underline{\lim}_{m\to+\infty}\Big(\frac{\log_2(y_{[\lambda+m]})}{\log_2(y_{[\lambda+m]-1})}\cdots\frac{\log_2(y_{m+1})}{\log_2(y_m)}\Big)\geqslant \underline{\lim}_{m\to+\infty} m^{[\lambda]}=+\infty,$$

for each $\lambda\geqslant 1$.

Therefore, $(\log_2 y_m)\in \mathrm{Tr}(R_{\infty,s})$ holds. It follows that $y_m=L_2(z_m)$ holds, for each $m\in\mathbb{N}$, where the sequence $z=(z_m)\in\mathrm{Tr}(R_{\infty,s})$, hence $y=(y_m)\in L_2(\mathrm{Tr}(R_{\infty,s}))$. $\square$

Proposition 2.4. *The selection principle* $\alpha_2(\bar{\mathbb{S}}^\infty, L_2(\mathrm{Tr}(R_{\infty,s})))$ *is satisfied.*

Proof. Let the sequence of sequences $(x_{n,m})$ be given, where for each $m^*\in\mathbb{N}$ the sequence $(x_{n,m^*})\in\bar{\mathbb{S}}^\infty$. Let us form the sequence $y=(y_t)$ as follows:

(1) Let $m\in\mathbb{N}$ be fixed. Consider the subsequence $(x_{k_m(n),m})$ of the sequence $(x_{n,m})$, so that $x_{k_m(n),m}\to+\infty$, for $n\to+\infty$. That subsequence represents a new sequence which will be denoted by $(\bar{x}_k^m)$. In the sequence $(\bar{x}_k^m)$ take disjoint countable subsequences (a sequence of subsequences) $(\bar{x}^m_{p_i^k})$, $i\in\mathbb{N}$, where p_i, i^{th} is a sequence of prime numbers in ascending order.

(2) Let us form, as in (1), the sequence of sequences

$$(\bar{x}^1_{p_1^k}),(\bar{x}^2_{p_1^k}),(\bar{x}^1_{p_2^k}),(\bar{x}^1_{p_3^k}),(\bar{x}^2_{p_2^k}),(\bar{x}^3_{p_1^k}),\dots$$

such that each of them belongs to the class $\bar{\mathbb{S}}^\infty$. Using the procedure from the previous proposition, we construct the sequence $y=(y_t)\in L_2(R_{\infty,s})$. By the construction of the sequence y we see that for each $m\in\mathbb{N}$ the set $y\cap(x_{n,m})$ is infinite. This means that the selection principle $\alpha_2(\bar{\mathbb{S}}^\infty, L_2(\mathrm{Tr}(R_{\infty,s}))$ is satisfied (which implies that the selection principles $\alpha_j(\bar{\mathbb{S}}^\infty, L_2(\mathrm{Tr}(R_{\infty,s})))$, $j\in\{3,4\}$ are also satisfied). $\square$

Notice that similarly one obtains that $\alpha_2(\bar{\mathbb{S}}^\infty, L_k(\mathrm{Tr}(R_{\infty,s})))$, $k>1$, is satisfied.

Now we define a relation on the class $\mathbb{S}$.

Definition 2.1. Let sequences $x = (x_n),\ y = (y_n) \in \mathbb{S}$ be given. Then x and y are said to be mutually logarithmic translationally rapidly equivalent (denoted by $x_n \overset{ltr}{\sim} y_n$, as $n \to +\infty$) if $\lim_{n\to+\infty} \frac{\log_2 x_{[\lambda+n]}}{\log_2 y_n} = +\infty$ and $\lim_{n\to+\infty} \frac{\log_2 y_{[\lambda+n]}}{\log_2 x_n} = +\infty$, for each $\lambda \geqslant 1$ hold.

Proposition 2.5. *Let the sequences $x = (x_n),\ y = (y_n) \in \mathbb{S}$ be given. If $x_n \overset{ltr}{\sim} y_n$, as $n \to +\infty$ holds, then $x \in L_2(\mathrm{Tr}(R_{\infty,s}))$ and $y \in L_2(\mathrm{Tr}(R_{\infty,s}))$.*

Proof. It holds

$$\begin{aligned}\lim_{n\to+\infty} \frac{\log_2 y_{n+2}}{\log_2 y_n} &= \lim_{n\to+\infty}\Big(\frac{\log_2 y_{n+2}}{\log_2 x_{n+1}}\cdot\frac{\log_2 x_{n+1}}{\log_2 y_n}\Big)\\ &= \lim_{n\to+\infty} \frac{\log_2 y_{n+2}}{\log_2 x_{n+1}}\cdot \lim_{n\to+\infty}\frac{\log_2 x_{n+1}}{\log_2 y_n} = (+\infty)\cdot(+\infty) = +\infty.\end{aligned}$$

Hence, we have that

$$+\infty = \lim_{n\to+\infty}\Big(\frac{\log_2 y_{n+2}}{\log_2 y_{n+1}}\cdot\frac{\log_2 y_{n+1}}{\log_2 y_n}\Big) = \lim_{s\to+\infty}\Big(\frac{\log_2 y_{s+1}}{\log_2 y_s}\Big)^2 = \Big(\lim_{s\to+\infty}\frac{\log_2 y_{s+1}}{\log_2 y_s}\Big)^2$$

holds.

Therefore, $\lim_{n\to+\infty}\frac{\log_2 y_{n+1}}{\log_2 y_n} = +\infty$, so that, for $\lambda \geqslant 1$,

$$\lim_{n\to+\infty}\frac{\log_2 y_{[\lambda+n]}}{\log_2 y_n} = \lim_{n\to+\infty}\frac{\log_2 y_{n+[\lambda]}}{\log_2 y_n} = \lim_{n\to+\infty}\Big(\frac{\log_2 y_{n+1}}{\log_2 y_n}\Big)^{[\lambda]} = +\infty.$$

It means, $y \in L_2(\mathrm{Tr}(R_{\infty,s}))$. Analogously we can prove that $x \in L_2(\mathrm{Tr}(R_{\infty,s}))$. □

Proposition 2.6. *The relation $\overset{ltr}{\sim}$ is symmetric, reflexive and need not to be transitive on $L_2(\mathrm{Tr}(R_{\infty,s}))$.*

Proof. 1. (Reflexivity) For $x = (x_n) \in L_2(\mathrm{Tr}(R_{\infty,s}))$ $\lim_{n\to+\infty}\frac{\log_2 x_{[\lambda+n]}}{\log_2 x_n} = +\infty$, for each $\lambda \geqslant 1$ holds, therefore $x_n \overset{ltr}{\sim} x_n$, as $n \to +\infty$ holds.

2. (Symmetry) According to the definition of $\overset{ltr}{\sim}$, symmetry holds.

3. (Non-transitivity) Transitivity does not hold, because for the sequences

$$x = (x_n),\ x_n = 2^{(n-1)!ln(n+1)};\quad y = (y_n),\ y_n = 2^{n!};\quad z = (z_n),\ z_n = 2^{\frac{(n+1)!}{ln(n+1)}}$$

it holds $x_n \overset{ltr}{\sim} y_n$, $y_n \overset{ltr}{\sim} z_n$, as $n \to +\infty$, and does not hold $x_n \overset{ltr}{\sim} z_n$, as $n \to +\infty$. □

For the sequence $c = (c_n) \in L_2(\mathrm{Tr}(R_{\infty,s}))$, we will define the set

$$[c]_{ltr} = \{x = (x_n) \in L_2(\mathrm{Tr}(R_{\infty,s})) \mid x_n \overset{ltr}{\sim} c_n,\ \text{as } n \to +\infty\}$$

on $L_2(\mathrm{Tr}(R_{\infty,s}))$, generated by the sequence c.

Recall the definition of an infinitely long game related to α_2 (see e.g. [**12**]).

Definition 2.2. Let $\mathcal{A}$ and $\mathcal{B}$ be nonempty subfamilies of the set $\mathbb{S}$. The symbol $G_{\alpha_2}(\mathcal{A},\mathcal{B})$ denotes the following infinitely long game for two players, who play a round for each natural number n. In the first round the first player plays an arbitrary element $(A_{1,j})_{j\in\mathbb{N}}$ from $\mathcal{A}$, and the second one chooses an element from the subsequence $y_{r_1} = (A_{1,r_1(j)})_{j\in\mathbb{N}}$ of the sequence A_1. At the k^{th} round, $k \geqslant 2$, the first player plays an arbitrary element $A_k = (A_{k,j})_{j\in\mathbb{N}}$ from $\mathcal{A}$ and the second one chooses an elements from the subsequence $y_{r_k} = (A_{k,r_k(j)})_{j\in\mathbb{N}}$ of the sequence A_k, such that $Im(r_{k(j)}) \cap Im(r_{p(j)}) = \varnothing$ is satisfied, for each $p \leqslant k-1$. The second player wins a play $A_1, y_{r_1}; \dots; A_k, y_{r_k}; \dots$ if and only if all elements from the $Y = \bigcup_{k\in\mathbb{N}} \bigcup_{j\in\mathbb{N}} A_k, r_{k(j)}$ form a subsequence of the sequence $y = (y_m)_{m\in\mathbb{N}} \in \mathcal{B}$.

Recall that a strategy of a player is a function σ from the set of all finite sequences of moves of the opponent into the set of moves of the strategy owner.

Proposition 2.7. *For each fixed element $c = (c_n) \in L_2(\mathrm{Tr}(R_{\infty,s}))$ the second player has a winning strategy in the game $G_{\alpha_2}([c]_{ltr}, [c]_{ltr})$.*

Proof. (m^{th} round, $m \geqslant 1$) The first player chooses the sequence $x_m = (x_{m,n}) \in [c]_{ltr}$ arbitrary. Then the second player chooses the subsequence $\sigma(x_m) = (x_{m,k_m(n)})_{n\in\mathbb{N}}$ of the sequence x_m, so that $Im(k_m)$ is the set of natural numbers greater than or equal to n_m, which are divisible by 2^m, and not divisible by 2^{m+1}, n_m belongs to $\mathbb{N}$, $\frac{\log_2 c_{n+1}}{\log_2 x_{m,n}} \geqslant 2^m$, and $\frac{\log_2 x_{m,n+1}}{\log_2 c_n} \geqslant 2^m$ for each $n \geqslant n_m$. Let $\lambda \geqslant 1$. Since $c \in L_2(\mathrm{Tr}(R_{\infty,s}))$, we have $\frac{\log_2 c_{n+1}}{\log_2 c_n} \geqslant 1$ for n large enough. Therefore,

$$\frac{\log_2 c_{[\lambda+n]}}{\log_2 x_{m,n}} = \frac{\log_2 c_{[\lambda+n]}}{\log_2 c_{[\lambda+n]-1}} \cdot \frac{\log_2 c_{[\lambda+n]-1}}{\log_2 c_{[\lambda+n]-2}} \cdots \frac{\log_2 c_{n+1}}{\log_2 x_{m,n}} \geqslant 2^m$$

for $n \geqslant n_m$. According to Proposition 2.5, $x_m \in L_2(\mathrm{Tr}(R_{\infty,s}))$. We can analogously prove $\frac{\log_2 x_{m,[\lambda+n]}}{\log_2 c_n} \geqslant 2^m$ for $n \geqslant n_m$.

Now, we form the set $Y = \bigcup_{m\in\mathbb{N}} \bigcup_{n\in\mathbb{N}} x_{m,k_m(n)}$ of positive real numbers indexed by two indexes. This set we can consider as the subsequence of the sequence $y = (y_i)_{i\in\mathbb{N}}$ given by:

$$y_i = \begin{cases} x_{m,k_m(n)}, & \text{if } i = k_m(n) \text{ for some } m, n \in \mathbb{N}; \\ c_i, & \text{otherwise.} \end{cases}$$

By the construction of the sequence y, we have that $y \in \mathbb{S}$. Also, the intersection between y and x_m ($m \in \mathbb{N}$) has infinitely many elements. Let us prove that $y_m \overset{ltr}{\sim} c_m$, as $m \to +\infty$. Let $M > 0$. Let us choose the smallest natural number m so that $2^m > M$. For each $k \in \{1, 2, \dots, m-1\}$ there is $n_k^* \in \mathbb{N}$, so that $\frac{\log_2 c_{[\lambda+n]}}{\log_2 x_{k,n}} \geqslant M$ and $\frac{\log_2 x_{k,[\lambda+n]}}{\log_2 c_n} \geqslant M$ for each $\lambda \geqslant 1$ and each $n \geqslant n_k^*$. Let $n^* = \max\{n_1^*, \dots, n_{m-1}^*\}$. Therefore, the inequalities $\frac{\log_2 c_{[\lambda+i]}}{\log_2 y_i} \geqslant M$ and $\frac{\log_2 y_{[\lambda+i]}}{\log_2 c_i} \geqslant M$ hold for each $\lambda \geqslant 1$ and each $i \geqslant n^*$. This means $y_i \overset{ltr}{\sim} c_i$, as $i \to +\infty$, because M was arbitrary, and thus $y \in [c]_{ltr}$. □

Corollary 2.1. *The selection principles $\alpha_i([c]_{ltr}, [c]_{ltr})$ hold for each fixed element $c = (c_n) \in L_2(\mathrm{Tr}(R_{\infty,s}))$ and each $i \in \{2, 3, 4\}$.*

Remark 2.2. From Propositions 2.5,2.6 and 2.7 and Corollary 2.1, it follows that propositions similar to Proposition 2.7 hold for translationally rapidly varying sequences and translational rapid equivalence.

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