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DIVISIBILITY ORDERS IN βN

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Abstract. We consider five divisibility orders on the Stone–Čech compactification βN. We find some possible lengths of chains and antichains and number of maximal and minimal elements, as well as some other ordering properties of these relations.

1. Introduction

The book [**1**] considers extensions of semigroup operations on discrete spaces S to their Stone-Čech compatifications βS. We are interested in the extension of the multiplication on the set N of natural numbers in this way. The operation $\cdot$ on N is extended to βN as follows:

$$A \in p \cdot q \Leftrightarrow \{n \in N : A/n \in q\} \in p,$$

where $A/n = \{\frac{a}{n} : a \in A, n \mid a\}$. In particular, if $n \in N$ and $q \in \beta N$ then $A \in nq$ if and only if $A/n \in q$. The topology on βN is defined by taking $\bar{A} = \{p \in \beta N : A \in p\}$ (for $A \subseteq N$) as base sets (the set $\bar{A}$ is the closure of A so there is no abuse of notation).

Let us fix some notation. Identifying elements of N with the corresponding principal ultrafilters, we will denote $N^* = \beta N \setminus N$. The (unique) continuous extension of a function $f\colon N \to N$ to βN will be denoted by $\tilde{f}$. The symbol $\mid$ denotes the divisibility relation on N, and $\mid [A] = \{m \in N : \exists a \in A \ a \mid m\}$. Let also $\mathcal{U} = \{S \subseteq N : S \text{ is upward closed for } \mid\}$, $\mathcal{V} = \{S \subseteq N : S \text{ is downward closed for } \mid\}$ and $D(p) = \{A \subseteq N : \{n \in N : A/n = N\} \in p\}$. We also mention that, since almost all the results we use are contained in the book [**1**], for readers' convenience we chose to cite the book instead of various papers in which results may have appeared first.

In [**3**] we defined four possible extensions of the divisibility relation $\mid$ on N to βN. The eventual goal of investigating these relations is to try to translate problems from elementary number theory (of infinite character, i.e., problems dealing with

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infinity of certain subsets of N) into βN and use topological methods to approach them.

Definition 1.1. Let $p, q \in \beta N$.

(a) q is left-divisible by p, $p \mid_L q$, if there is $r \in \beta N$ such that $q = rp$.
(b) q is right-divisible by p, $p \mid_R q$, if there is $r \in \beta N$ such that $q = pr$.
(c) q is mid-divisible by p, $p \mid_M q$, if there are $r, s \in \beta N$ such that $q = rps$.
(d) $p \,\widetilde{\mid}\, q$ if for all $A \in p$, $\mid [A] \in q$ holds.

In semigroup theory $\mid_L$ and $\mid_R$ are known as the Green relations: they are equivalent to the inclusion relation on the sets of principal left (or right) ideals, for example $p \mid_L q$ if and only if $\beta N q \subseteq \beta N p$. Hence they have been considered before. The relation $\widetilde{\mid}$ was introduced by analogy with functions $\tilde{f}$, extending $\mid$ in such a way to satisfy certain continuity conditions, and in [**3**] it was proved that $\widetilde{\mid}$ is the maximal extension of $\mid$ which is continuous in that sense. In this paper we will investigate some ordering properties of these relations, adding one more, $\mid_{LN}$.

All the relations $\mid_L$, $\mid_R$, $\mid_M$ and $\widetilde{\mid}$ are preorders (reflexive and transitive), but none is antisymmetric (see [**3**, Section 4]). So for each of them we introduce another relation: $p =_L q$ if $p \mid_L q$ and $q \mid_L p$, and $=_R$, $=_M$ and $=_\sim$ are defined analogously. All these are equivalence relations, and all the divisibility relations can be viewed as partial orders on respective factor sets (we will use the same notation for orders on factor sets as for preorders above). Respective equivalence classes are denoted by $[p]_L$, $[p]_R$, $[p]_M$ and $[p]_\sim$.

Lemma 1.1. (a) *If p is right cancelable then $[p]_L = \{p\}$.* (b) *If p is left cancelable then $[p]_R = \{p\}$.*

Proof. (a) Assume the opposite, that there is $q \neq p$ such that $p =_L q$. This means that $p = xq$ and $q = yp$ for some $x, y \in \beta N$. Then $p = xyp$, so since p is right cancelable, $xy = 1$. But N^* is an ideal of βN [**1**, Theorem 4.36], which means that $x = y = 1$, so $p = q$.

(b) is proven analogously. □

Note also that the sets of right cancelable and left cancelable elements are downward closed in $\mid_R$ and $\mid_L$ respectively.

In the next proposition we collect several useful facts concerning elements of N.

Proposition 1.1. (a) *[**1**, Theorem 6.10] Elements of N commute with all elements of βN.*

(b) *[**1**, Lemma 6.28] If $m, n \in N$ and $p \in \beta N$, then $mp = np$ implies $m = n$.*

(c) *[**3**, Lemma 5.1] If $n \in N$, each of the following statements are equivalent*

(i) $n \mid_L p$, (ii) $n \mid_R p$, (iii) $n \mid_M p$, (iv) $n \,\widetilde{\mid}\, p$, (v) $nN \in p$.

The previous lemma allows us to drop subscripts and write only $n \mid p$ for $n \in N$.

Lemma 1.2. *The following conditions are equivalent:*

(i) $p \,\widetilde{\mid}\, q$; (ii) $D(p) \subseteq q$; (iii) $p \cap \mathcal{U} \subseteq q \cap \mathcal{U}$; (iv) $q \cap \mathcal{V} \subseteq p \cap \mathcal{V}$.

Proof. The equivalence of (i), (ii) and (iii) was proved in [**3**, Theorem 6.2]. (iii)$\Leftrightarrow$(iv) follows easily from the fact that $A \in \mathcal{U}$ iff $A^c \in \mathcal{V}$. $\square$

Lemma 1.3. (a) *For each $p \in \beta N$ the set $p \uparrow_L = \{q : p \mid_L q\}$ is closed;*

(b) *for each $p \in \beta N$ the sets $p \uparrow_\sim = \{q : p \,\widetilde{\mid}\, q\}$, $p \downarrow_\sim = \{q : q \,\widetilde{\mid}\, p\}$ and $[p]_\sim$ are closed.*

Proof. (a) $p \uparrow_L = \beta N p = \overline{Np}$, which is clearly closed.

(b) By Lemma 1.2(ii) $q \in p \uparrow_\sim$ iff $q \in \bigcap_{A \in D(p)} \bar{A}$, which is a closed set. By Lemma 1.2(iv) $p \downarrow_\sim = \bigcap_{B \in p \cap \mathcal{V}} \bar{B}$, also a closed set. $[p]_\sim = p \uparrow_\sim \cap p \downarrow_\sim$, so it is also closed. $\square$

In [**1**, Definition 1.34], orders $\geqslant_L$, $\geqslant_R$ and $\geqslant$ were defined on the set $E(\beta N)$ of idempotents of $(\beta N, \cdot)$: $p \geqslant_L q$ if $pq = q$, $p \geqslant_R q$ if $qp = q$, and $p \geqslant q$ if both $p \geqslant_L q$ and $p \geqslant_R q$.

Lemma 1.4. *$\geqslant_R$ is the restriction of $\mid_L$ to $E(\beta N)$, and $\geqslant_L$ is the restriction of $\mid_R$ to $E(\beta N)$.*

Proof. We prove the result for $\mid_L$; the proof for $\mid_R$ is analogous. Clearly, for $p, q \in E(\beta N)$ $p \geqslant_R q$ implies $p \mid_L q$. On the other hand, if $p \mid_L q$ then there is $x \in \beta N$ such that $q = xp$. Then $qp = xpp = xp = q$ because p is an idempotent. $\square$

Clearly, if for two of the considered relations τ and σ holds $\tau \subseteq \sigma$, then equivalence classes of $=_\sigma$ are unions of equivalence classes of $=_\tau$. We will show that the following diagram holds:

$$\begin{array}{ccccc} & & \mid_{LN} & & \\ & \rotatebox{45}{$\subset$} & & & \\ \mid_L & & & & \\ & \rotatebox{-45}{$\subset$} & & & \\ & & \mid_M & \subset & \widetilde{\mid} \\ & \rotatebox{45}{$\subset$} & & & \\ \mid_R & & & & \end{array}$$

The part of the diagram concerning $\mid_L$, $\mid_R$, $\mid_M$ and $\widetilde{\mid}$ was explained in [**3**]. The inclusion $\mid_L \subset \mid_{LN}$ will be clear from the definition of $\mid_{LN}$. Why $\mid_R$, $\mid_M$ and $\widetilde{\mid}$ are incomparable with $\mid_{LN}$ will be explained at the end of Section 4.

2. The preorder $\mid_{LN}$

The following result suggests introduction of another relation on βN, representing divisibility "up to elements of N".

Proposition 2.1. (a) *[**1**, Theorem 3.40] If A and B are countable subsets of βN such that $\bar{A} \cap \bar{B} \neq \emptyset$, then $A \cap \bar{B} \neq \emptyset$ or $\bar{A} \cap B \neq \emptyset$.*

(b) *[**1**, Theorem 6.19] If $xp = yq$ for $p, q, x, y \in \beta N$, then there are $n \in N$ and $z \in \beta N$ such that either $np = zq$ or $zp = nq$.*

Definition 2.1. $p \mid_{LN} q$ if there is $n \in N$ such that $p \mid_L nq$.

To see that this relation is strictly stronger than $|_L$, note that for any $p \in N^*$ not divisible by 2, we have $2p \mid_{LN} p$, but not $2p \mid_L p$.

The relation $|_{LN}$ is also a preorder, so we introduce $=_{LN}$ and $[p]_{LN}$ as for other relations.

LEMMA 2.1. *For each $p \in \beta N$, we have*

$$[p]_{LN} = \{mr : m \in N, r \in \beta N, \exists n \in N(r \mid_L p \wedge p \mid_L nr)\}.$$

PROOF. It is obvious that, if there is $n \in N$ such that $r \mid_L p$ and $p \mid_L nr$, then $mr =_{LN} p$. So let $p \mid_{LN} q$ and $q \mid_{LN} p$ for some $q \in \beta N$. $q \mid_{LN} p$ means that there is $m \in N$ such that $q \mid_L mp$, i.e., $xq = mp$ for some $x \in \beta N$. We can assume that m is minimal such element of N; let us show that this implies $m \mid q$.

If not, there are a prime k and $s \in N \cup \{0\}$ such that $k^s \mid q$, $k^{s+1} \nmid q$ and $k^{s+1} \mid m$, so since prime numbers are also irreducible in βN [**3**, Lemma 7.3], and $k^{s+1} \mid xq$, we would have $k \mid x$, i.e., $x = kx_1$. Then we could cancel out k and get $x_1 q = \frac{m}{k} p$, which is a contradiction with the minimality of m.

So $q = mr$ for some $r \in \beta N$, and $r \mid_L p$. That $p \mid_L nr$ for some $n \in N$ follows directly from $p \mid_{LN} q$. $\square$

The following lemma is what may make this new relation useful.

LEMMA 2.2. *For every $q \in \beta N$ the set $\{[p]_{LN} : p \mid_{LN} q\}$ is linearly ordered by $|_{LN}$.*

PROOF. Let $p_1 \mid_{LN} q$ and $p_2 \mid_{LN} q$. This means that there are $n_1, n_2 \in N$ and $x_1, x_2 \in \beta N$ such that $n_1 q = x_1 p_1$ and $n_2 q = x_2 p_2$. Then $n_2 x_1 p_1 = n_1 n_2 q = n_1 x_2 p_2$. By Proposition 2.1(b) one of the elements p_1 and p_2 must be $|_{LN}$-divisible by the other. $\square$

3. Chains and antichains

In this section we investigate possible lengths of chains and antichains in our partial orders.

PROPOSITION 3.1. *[**1**, Theorem 6.73] There is an infinite strictly $|_R$-descending chain in βN.*

LEMMA 3.1. *There is an infinite strictly $|_L$-descending chain of right cancelable elements in βN.*

PROOF. We construct the wanted chain as follows: let $p_0 \in \bigcap_{n \in N} \overline{2^n N}$ be right cancelable (by [**3**, Theorem 5.2], this set is a nonempty G_δ set, thus it contains an open subset, and by [**1**, Theorem 8.10] it contains a right cancelable element). Let $p_n = 2p_{n+1}$ for $n \in \omega$. By induction it is easy to prove that all p_n are right cancelable: for example $xp_1 = yp_1 \Rightarrow xp_0 = yp_0 \Rightarrow x = y$. These elements are also distinct (by Proposition 1.1(b)), belong to different $|_L$-classes and $\dots p_2 \mid_L p_1 \mid_L p_0$. $\square$

LEMMA 3.2. *There is a strictly $\widetilde{|}$-descending chain of length $\omega + 1$.*

Proof. Let P be the set of prime numbers, and let $\langle P_n : n < \omega\rangle$ be a sequence of sets such that $P_0 = P$, $P_{n+1} \subset P_n$ and $P_n \setminus P_{n+1}$ is infinite for all $n \in \omega$. For each $n < \omega$ let $X_n = \{k \in N : \text{ all prime divisors of } k \text{ belong to } P_n\}$.

For each $n < \omega$ the set $\{X_n\} \cup \{A \in \mathcal{U} : A \cap X_n \neq \emptyset\}$ has the finite intersection property: let $A_1, A_2, \dots, A_k \in \mathcal{U}$ be given with nonempty intersections with X_n. For every $i = 1, 2, \dots, k$ choose an element $a_i \in A_i \cap X_n$; then $LCM(a_1, a_2, \dots, a_k) \in A_1 \cap A_2 \cap \dots \cap A_k \cap X_n$. Note that $A \cap X_n \neq \emptyset$ for $A \in \mathcal{U}$ actually means that $A \cap X_n$ is infinite.

Hence we can pick ultrafilters p_n so that $p_n \cap \mathcal{V} = \{A \in \mathcal{V} : P_n \subseteq A\}$. Hence $p_m \cap \mathcal{V} \subset p_n \cap \mathcal{V}$ for $m < n < \omega$ so, by Lemma 1.2, $p_n \mathrel{\widetilde{\mid}} p_m$ and $p_m \mathrel{\widetilde{\nmid}} p_n$.

Finally, the family $\bigcup_{n<\omega}(p_n \cap \mathcal{V})$ has the finite intersection property, so there is an ultrafilter containing $\bigcup_{n<\omega}(p_n \cap \mathcal{V})$, which is below all p_n for $n < \omega$. $\square$

In the next two lemmas we adapt ideas from the proof of [**1**, Lemma 9.22].

Lemma 3.3. *Every $|_L$-ascending chain of length ω has an upper bound in βN.*

Proof. Let $\langle q_n : n \in \omega\rangle$ be a $|_L$-ascending chain. Let $q \in \mathrm{cl}\{q_n : n \in \omega\} \setminus \{q_n : n \in \omega\}$ be arbitrary. Then, for each $m \in \omega$, $q_n \in \beta N q_m$ for all $n \geqslant m$ so, since $\beta N q_m$ is closed, $q \in \mathrm{cl}\{q_n : n \geqslant m\} \subseteq \beta N q_m$ i.e., $q_m \mid_L q$. $\square$

Lemma 3.4. *Every strictly $|_{LN}$-ascending chain of length ω has an upper bound q in βN that is right cancelable.*

Proof. Let $\langle r_n : n \in \omega\rangle$ be a strictly $|_{LN}$-ascending sequence, and for each $n \in \omega$ let $k_n \in N$ be such that $r_i \mid_L k_n r_n$ for all $i < n$. Let $q_n = k_n r_n$. Then $\langle q_n : n \in \omega\rangle$ is a $|_L$-ascending sequence which is also strictly $|_{LN}$-ascending. As in Lemma 3.3 we can find $q \in \mathrm{cl}\{q_n : n \in \omega\}$ which is a $|_L$-upper bound of $\langle q_n : n \in \omega\rangle$, and hence a $|_{LN}$-upper bound of $\langle r_n : n \in \omega\rangle$ as well.

Suppose q is not right cancelable. Then, by [**1**, Theorem 8.11], there are $x \in N^*$ and $a \in N$ such that $xq = aq$. Hence, $aq \in \mathrm{cl}\{aq_n : n \in \omega\} \cap \mathrm{cl}((N \setminus \{a\})q)$ so, by Proposition 2.1(a), we have one of the following two possibilities: either $\{aq_n : n \in \omega\} \cap \mathrm{cl}((N \setminus \{a\})q) \neq \emptyset$ or $\mathrm{cl}\{aq_n : n \in \omega\} \cap (N \setminus \{a\})q \neq \emptyset$. The first one leads to contradiction right away, since $aq_n = yq$ would mean that $q \mid_{LN} q_n$ and thus $q_{n+1} \mid_{LN} q_n$ as well.

So $aq' = bq$ for some $q' \in \mathrm{cl}\{q_n : n \in \omega\}$ and some $b \in N \setminus \{a\}$. This means that $\mathrm{cl}\{aq_n : n \in \omega\} \cap \mathrm{cl}\{bq_n : n \in \omega\} \neq \emptyset$ so, again by Proposition 2.1(a), either $aq_m = bq''$ or $aq'' = bq_m$ for some $m \in N$ and some $q'' \in \mathrm{cl}\{q_n : n \in \omega\}$. ($q'' \notin \{q_n : n < \omega\}$ because $q'' = q_n$ would imply that $q_m =_{LN} q_n$.) Without loss of generality assume the first possibility. It follows that $q'' \mid_{LN} q_m$, so $q_{m+1} \mid_{LN} q_m$, a contradiction again. $\square$

For divisibility relations we will consider the following notion of compatibility.

Definition 3.1. If ρ is a preorder on βN we will say that two elements x and y are ρ-compatible if there is $z \neq 1$ such that $z \rho x$ and $z \rho y$.

We define the compatibility relation C_L on N^* as follows: pC_Lq if there is $r \in N^*$ such that $r \mid_L p$ and $r \mid_L q$. The relation $\sim_L$ is its transitive closure: $p \sim_L q$ if there are $x_1, x_2, \dots, x_k \in N^*$ such that $pC_Lx_1, x_1C_Lx_2, \dots, x_kC_Lq$.

Clearly, $p \mid_L q$ implies pC_Lq (and hence $p \sim_L q$). $\sim_L$ is an equivalence relation.

In [**1**, Definition 6.48], a relation R on N^* is defined by $pRq \Leftrightarrow \beta Np \cap \beta Nq \neq \emptyset$; hence pRq if and only if p and q are $\mid_L^{-1}$-compatible. Then $\sim_L$ is also the transitive closure of R. The graphs of relations $\mid_L$, $\mid_{LN}$ (more precisely, of their symmetric closures), R, C_L and $\sim_L$ on N^* have the same connected components. The following result (a straightforward consequence of [**1**, Theorem 6.53] and the considerations above) shows that these graphs are far from being connected.

Proposition 3.2. *There are $2^{\mathfrak{c}}$ equivalence classes $[p]_{\sim_L} = \{q \in N^* : p \sim_L q\}$. Each of these classes is nowhere dense in N^* and it is a left ideal of βN.*

For the relation $\widetilde{\mid}$ (and consequently for $\mid_R$ and $\mid_M$) we have a weaker result. We remind the reader that sets A and B are almost disjoint if $A \cap B$ is finite, and that on any infinite set there exists an almost disjoint family of cardinality $\mathfrak{c}$ (see, for example, [**2**, Lemma 3.1.2]).

Theorem 3.1. *There is a family $\{p_\alpha : \alpha < \mathfrak{c}\}$ of $\widetilde{\mid}$-incompatible elements in βN.*

Proof. Let $\{A_\alpha : \alpha < \mathfrak{c}\}$ be an almost disjoint family of infinite subsets of the set P of prime numbers. For $\alpha < \mathfrak{c}$ let $p_\alpha \in N^*$ be an ultrafilter containing A_α. We will prove that any two ultrafilters p_α and p_β for $\alpha \neq \beta$ are $\widetilde{\mid}$-incompatible. First, since each of the sets nN for $n \in N \setminus \{1\}$ is almost disjoint with A_α, it follows that $nN \notin p_\alpha$ so by Proposition 1.1(c) p_α is not divisible by any $n \in N \setminus \{1\}$.

Now assume there is $r \in N^*$ such that $r \,\widetilde{\mid}\, p_\alpha$ and $r \,\widetilde{\mid}\, p_\beta$. Let $B_\alpha = \mid [P \setminus A_\alpha]$ and $B_\beta = \mid [P \setminus A_\beta]$; then $B_\alpha, B_\beta \in \mathcal{U}$. Since B_α is disjoint from A_α, $B_\alpha \notin p_\alpha$ so by Lemma 1.2 $B_\alpha \notin r$ as well. In the same way we conclude $B_\beta \notin r$. Also, $S = N \setminus (P \cup \{1\})$ is a set in $\mathcal{U}$ disjoint from A_α, so again $S \notin r$ and $P \in r$. This means that $P \setminus (B_\alpha \cup B_\beta) = A_\alpha \cap A_\beta$ must be in r, so r must be a principal ultrafilter; a contradiction. □

4. Maximal and minimal elements

In [**1**, Theorems 1.51 and 1.64], it is shown that $(\beta N, \cdot)$ has the smallest ideal, denoted by $K(\beta N)$ and that $K(\beta N) = \bigcup\{L : L \text{ is a minimal left ideal of } \beta N\} = \bigcup\{R : R \text{ is a minimal right ideal of } \beta N\}$. Clearly, every minimal left ideal L is principal and moreover generated by any element $p \in L$.

Theorem 4.1. (a) *$\mid_L$ has $2^{\mathfrak{c}}$ maximal classes, they are exactly minimal left ideals of βN, and for every $q \in \beta N$ there is p such that $q \mid_L p$ and $[p]_L$ is maximal.*

(b) *$\mid_R$ has $2^{\mathfrak{c}}$ maximal classes, they are exactly minimal right ideals of βN, and for every $q \in \beta N$ there is p such that $q \mid_R p$ and $[p]_R$ is maximal.*

(c) *For every $q \in \beta N$ there is p such that $q \mid_{LN} p$ and $[p]_{LN}$ is maximal.*

(d) *$(\beta N/_{=_M}, \mid_M)$ has the greatest element, and it is exactly the class $K(\beta N)$.*

(e) *$(\beta N/_{=_\sim}, \widetilde{\mid})$ has the greatest element, $\bigcap_{A \in \mathcal{U}} \bar{A}$, containing $K(\beta N)$.*

Proof. (a) By [**1**, Theorem 6.44] there are $2^{\mathfrak{c}}$ minimal left ideals (and each of them has exactly $2^{\mathfrak{c}}$ elements). But $[p]_L$ is maximal if and only if βNp is a minimal left ideal. Since every principal left ideal contains a minimal left ideal [**1**, Corollary 2.6], there is a $|_L$-maximal element above every $q \in \beta N$.

(b) is proved analogously to (a), using [**1**, Corollary 6.41].

(c) If for some $p, q \in \beta N$ we have $p \mid_{LN} q$ and $p \nmid_L q$, then there is $r =_{LN} q$ such that $p \mid_L r$. We conclude that $|_L$-maximal elements are also $|_{LN}$-maximal so, by (a), above every $q \in \beta N$ there is a $|_{LN}$-maximal element.

(d) We first prove that all elements of $K(\beta N)$ are in the same $=_M$-equivalence class. Let $p, q \in K(\beta N)$. By [**1**, Theorem 2.7(d)] the left ideal $\beta N p$ intersects the right ideal $q \beta N$; let $r \in \beta N p \cap q \beta N$. Then $p =_L r$ (because $\beta N p = \beta N r$) and $r =_R q$, so $p =_M r =_M q$.

It remains to prove that no element $p \notin K(\beta N)$ is in this maximal class (or above it). Assume the opposite, that there is p such that $q \mid_M p$ for $q \in K(\beta N)$. But this means that $p = aqb$ for some $a, b \in \beta N$, and since $K(\beta N)$ is an ideal, it follows that $p \in K(\beta N)$ as well.

(e) Since $\mathcal{U}$ has the finite intersection property, there are ultrafilters containing all the sets from $\mathcal{U}$, and by Lemma 1.2 they are clearly maximal for $\widetilde{\mid}$. Since $|_M \subseteq \widetilde{\mid}$, by (d) all the ultrafilters from $K(\beta N)$ are among them. □

Lemma 4.1. *If $p \in \beta N$ is right cancelable, then it has $2^{\mathfrak{c}}$ incomparable $|_L$-successors (and $2^{\mathfrak{c}}$ incomparable $|_R$-successors).*

Proof. Let q_α (for $\alpha < 2^{\mathfrak{c}}$) be $|_L$-maximal elements such that $q_\alpha \beta N$ are different minimal left ideals. Then $q_\alpha p \nmid_L q_\gamma p$ for $\alpha \neq \gamma$: if $q_\gamma p = x q_\alpha p$, by cancelability we would have $q_\gamma = x q_\alpha$, so $q_\gamma \in \beta N q_\alpha$, and by minimality $\beta N q_\alpha$ and $\beta N q_\gamma$ would be the same minimal left ideals. □

Each of the orders we are investigating clearly has the smallest element: the one-element class $[1] = \{1\}$ is the smallest in $|_L$, $|_R$, $|_M$ and $\widetilde{\mid}$, and the equivalence class $[1] = N$ is the smallest in the order $|_{LN}$. It is more interesting to ignore the class $[1]$ and define, for each relation ρ, the set of minimal elements M_ρ to be the union of minimal classes in $(\beta N/_{=_\rho} \setminus [1], \rho)$.

Lemma 4.2. (a) *$M_{|_L} = M_{|_R} = M_{|_M}$: it is the set of irreducible ultrafilters (those that can not be written as pq for $p, q \in \beta N \setminus \{1\}$).*

(b) *For $\rho \in \{|_L, |_R, |_M, \widetilde{\mid}\}$, $M_\rho \cap N = P$.*

(c) *For $\rho \in \{|_L, |_R, |_M\}$, $P^* \subset M_\rho \cap N^*$.*

(d) *There are $|_{LN}$-minimal elements.*

Proof. (a) is obvious.

(b) is obvious for $\rho \in \{|_L, |_R, |_M\}$. But $\widetilde{\mid} \restriction_{N^2} = |_L \restriction_{N^2}$, so the result holds for $\widetilde{\mid}$ too.

(c) $P^* \subseteq M_\rho \cap N^*$ follows from [**3**, Theorem 7.3] and the strict inclusion from [**3**, Theorem 7.5].

(d) follows from [**1**, Theorem 8.22]. □

Now $\mid_M \not\subseteq \mid_{LN}$ is clear since by Theorem 4.1 the order $\mid_M$ has the greatest element and $\mid_{LN}$ does not. But then $\mid_R \not\subseteq \mid_{LN}$ as well, since $\mid_R \subseteq \mid_{LN}$ along with $\mid_L \subseteq \mid_{LN}$ would imply that the transitive closure of $\mid_L \cup \mid_R$ (which is $\mid_M$) would be contained in $\mid_{LN}$ too.

Finally, $\mid_{LN} \not\subseteq \widetilde{\mid}$ because $2 \mid_{LN} 1$ but not $2 \,\widetilde{\mid}\, 1$.

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