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APPROXIMATION BY MODIFIED PĂLTĂNEA OPERATORS

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ABSTRACT. We discuss some approximation properties of hybrid genuine operators. We find central moments using the concept of moment generating function. A quantitative Voronovskaya and Grüss–Voronovskaya type theorem are proven. Also, we obtain the degree of approximation of the considered operators by means of the second order Ditzian–Totik modulus of smoothness.

1. Introduction

Păltănea [**21**] (see also [**23**]) proposed a modification of the well known Phillips operators based on a non-negative parameter ρ. Actually in the operators of [**23**] there is Szász–Mirakyan basis in summation, which may further be generalized by taking general form in summation containing Baskakov as well Szász–Mirakyan basis and one can define the modification based on the parameters $\rho > 0$ and $c \in (0,1]$ (see [**13**]) as

$$B_{\alpha}^{\rho}(f;x,c) = \sum_{k=1}^{\infty} p_{\alpha,k}(x,c) \int_0^{\infty} \theta_{\alpha,k}^{\rho}(t) f(t)dt + p_{\alpha,0}(x,c)f(0), \tag{1.1}$$

where

$$p_{\alpha,k}(x,c) = \frac{(\alpha/c)_k}{k!} \frac{(cx)^k}{(1+cx)^{\alpha/c+k}}, \quad \theta_{\alpha,k}^{\rho}(t) = \frac{\alpha\rho}{\Gamma(k\rho)} e^{-\alpha\rho t} (\alpha\rho t)^{k\rho-1}$$

and $(n)_k = n(n+1)(n+2)\cdots(n+k-1)$. For special cases of the operators (1.1), we refer the readers to [**13**].

After the Durrmeyer type modification of the Bernstein polynomials, in the last few decades several new operators of integral type have been constructed and studied by several researchers. Milovanović et al. [**17**] (see also [**18, 19**]) in their book covered some topics on the behavior of some polynomial operators in real and complex domains. Very recently, Gupta and Agarwal [**11**] and Gupta and Tachev

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[**14**] also presented convergence estimates of many operators in real and complex domains.

Here we obtain the moments using the concept of moment generating functions. We obtain quantitative asymptotic formula and a global direct result for the hybrid operators. We also discuss the Grüss–Voronovskaya theorem for the operator defined by (1.1). Grüss inequality [**16**] measures the difference of integral of two functions with the product of integral of the two functions. Initially, Acu et al. [**4**] showed the application of Grüss inequality in approximation theory. In [**15**], Gonska and Tachev discussed Grüss-type inequality using second order modulus of smoothness. Gal and Gonska [**10**], proved Grüss–Voronovskaya estimates for the first time using Grüss inequality for Bernstein operators and for a class of Bernstein–Durrmeyer polynomials of real and complex variables. Recently, Tariboon and Ntouyas [**24**] introduced Grüss inequality in q-calculus. After that in [**25**] and [**1**] authors investigated q-Grüss–Voronovskaya theorem for q-Baskakov operators and q-Szász operators respectively. In [**2**], Acar et al. discussed new forms of Voronovskaya type theorem in weighted spaces. Very recently, Deniz [**7**], obtained Grüss–Voronovskaya theorem for Jain–Kantorovich operators.

2. Moment Generating Functions and Moments

First we have

$$\int_0^\infty \theta^\rho_{\alpha,k}(t)e^{At}dt = \Big(\frac{\alpha\rho}{\alpha\rho - A}\Big)^{k\rho},$$

Thus, by using the well known binomial series $\sum_{k=0}^\infty \frac{(a)_k}{k!}z^k = (1-z)^{-a}, \qquad |z|<1,$ we have

$$\begin{aligned} B^\rho_\alpha(e^{At};x,c) &= \sum_{k=0}^\infty \frac{(\alpha/c)_k}{k!}\frac{(cx)^k}{(1+cx)^{\alpha/c+k}}\Big(\frac{\alpha\rho}{\alpha\rho - A}\Big)^{k\rho} \\ &= (1+cx)^{-\alpha/c}\sum_{k=0}^\infty \frac{(\alpha/c)_k}{k!}\frac{(cx)^k}{(1+cx)^{k}}\Big(\frac{\alpha\rho}{\alpha\rho - A}\Big)^{k\rho} \\ &= (1+cx)^{-\alpha/c}\Bigg[1 - \frac{cx\big(\frac{\alpha\rho}{\alpha\rho-A}\big)^\rho}{1+cx}\Bigg]^{-\alpha/c}, \end{aligned}$$

which implies that

$$B^\rho_\alpha(e^{At};x,c) = \big\{1 + cx\left[1-(\alpha\rho)^\rho(\alpha\rho - A)^{-\rho}\right]\big\}^{-\alpha/c}.$$

By expanding the right-hand side in powers of A, we have

$$\begin{aligned} B^\rho_\alpha(e^{At};x,c) &= 1 + xA + \{\rho x(\alpha+c)+\rho+1\}\frac{xA^2}{2\alpha\rho} \\ &\quad + \{\rho(x(\alpha+c)(\rho(2cx+\alpha x+3)+3)+\rho+3)+2\}\frac{xA^3}{6\alpha^2\rho^2} \\ &\quad + B_4\frac{xA^4}{24\alpha^3\rho^3} + B_5\frac{xA^5}{120\alpha^4\rho^4} + B_6\frac{xA^6}{720\alpha^5\rho^5} + O[A^7], \end{aligned}$$

where

$$\begin{aligned}B_4 &= \rho^3[x(\alpha+c)(x(\alpha+2c)(3cx+\alpha x+6)+7)+1]\\&\quad+\rho^2[6x(\alpha+c)(2cx+\alpha x+3)+6]+\rho[11x(\alpha+c)+11]+6,\\B_5 &= \rho^4[x(\alpha+c)(x(\alpha+2c)(x(\alpha+3c)(4cx+\alpha x+10)+25)+15)+1]\\&\quad+10\rho^3[x(\alpha+c)(x(\alpha+2c)(3cx+\alpha x+6)+7)+1]\\&\quad+35\rho^2[x(\alpha+c)(2cx+\alpha x+3)+1]+50\rho[x(\alpha+c)+1]+24,\\B_6 &= \rho^5[x(\alpha+c)(x(\alpha+2c)(x(\alpha+3c)(x(\alpha+4c)(5cx+\alpha x+15)+65)+90)+31)+1]\\&\quad+15\rho^4[x(\alpha+c)(x(\alpha+2c)(x(\alpha+3c)(4cx+\alpha x+10)+25)+15)+1]\\&\quad+85\rho^3[x(\alpha+c)(x(\alpha+2c)(3cx+\alpha x+6)+7)+1]\\&\quad+225\rho^2[x(\alpha+c)(2cx+\alpha x+3)+1]+274\rho[x(\alpha+c)+1]+120.\end{aligned}$$

Thus, $B_\alpha^\rho(e^{At};x,c)$ is the moment generating function of B_α^ρ and the m-th order moment $B_\alpha^\rho(t^m;x,c)$ is the coefficient of $A^m/m!$. Alternatively, we can find the moments by the equivalence

$$B_\alpha^\rho(t^m;x,c)=\left[\frac{\partial^m}{\partial A^m}\left(1+c\left(1-(\alpha\rho)^\rho(-A+\alpha\rho)^{-\rho}\right)x\right)^{-\frac{\alpha}{c}}\right]_{A=0}.$$

Further by change of origin property of the moment generating function, the central moments are

$$\begin{aligned}\mu_{\alpha,m}(x)&=B_\alpha^\rho((t-x)^m;x,c)\\&=\left[\frac{\partial^m}{\partial A^m}e^{-Ax}\left(1+c\left(1-(\alpha\rho)^\rho(-A+\alpha\rho)^{-\rho}\right)x\right)^{-\frac{\alpha}{c}}\right]_{A=0}.\end{aligned}\tag{2.1}$$

Remark 2.1. Expanding (2.1) the central moments of order m can be obtained by collecting the coefficient of $A^m/m!$. Some of the central moments are:

$$\mu_{\alpha,1}(x)=0,\quad \mu_{\alpha,2}(x)=\frac{x[1+\rho(1+cx)]}{\alpha\rho},$$

$$\begin{aligned}\mu_{\alpha,4}(x)=\frac{1}{\alpha^3\rho^3}\Big(&6x+11\rho x+6\rho^2x+\rho^3x+11c\rho x^2+3\alpha\rho x^2+18c\rho^2x^2\\&+6\alpha\rho^2x^2+7c\rho^3x^2+3\alpha\rho^3x^2+12c^2\rho^2x^3\\&+6c\alpha\rho^2x^3+12c^2\rho^3x^3+6c\alpha\rho^3x^3+6c^3\rho^3x^4+3c^2\alpha\rho^3x^4\Big),\end{aligned}$$

$$\begin{aligned}\mu_{\alpha,6}(x)=\frac{1}{\alpha^5\rho^5}\Big(&120x+274\rho x+225\rho^2x+85\rho^3x+15\rho^4x+\rho^5x+274c\rho x^2\\&+130\alpha\rho x^2+675c\rho^2x^2+375\alpha\rho^2x^2+595c\rho^3x^2+385\alpha\rho^3x^2\\&+225c\rho^4x^2+165\alpha\rho^4x^2+31c\rho^5x^2+25\alpha\rho^5x^2+450c^2\rho^2x^3\\&+375c\alpha\rho^2x^3+15\alpha^2\rho^2x^3+1020c^2\rho^3x^3+900c\alpha\rho^3x^3\\&+45\alpha^2\rho^3x^3+750c^2\rho^4x^3+705c\alpha\rho^4x^3+45\alpha^2\rho^4x^3\\&+180c^2\rho^5x^3+180c\alpha\rho^5x^3+15\alpha^2\rho^5x^3+510c^3\rho^3x^4\end{aligned}$$

$$+ 515c^2\alpha\rho^3x^4 + 45c\alpha^2\rho^3x^4 + 900c^3\rho^4x^4 + 930c^2\alpha\rho^4x^4$$
$$+ 90c\alpha^2\rho^4x^4 + 390c^3\rho^5x^4 + 415c^2\alpha\rho^5x^4 + 45c\alpha^2\rho^5x^4$$
$$+ 360c^4\rho^4x^5 + 390c^3\alpha\rho^4x^5 + 45c^2\alpha^2\rho^4x^5 + 360c^4\rho^5x^5 + 390c^3\alpha\rho^5x^5$$
$$+ 45c^2\alpha^2\rho^5x^5 + 120c^5\rho^5x^6 + 130c^4\alpha\rho^5x^6 + 15c^3\alpha^2\rho^5x^6\Big).$$

REMARK 2.2. For $c \in (0,1)$, if the m-th order central moment $\mu_{\alpha,m}(x)$ is defined by

$$\mu_{\alpha,m}(x) := B_\alpha^\rho((t-x)^m; x, c) = \sum_{k=1}^{\infty} p_{\alpha,k}(x,c) \int_0^\infty \theta_{\alpha,k}^\rho(t)(t-x)^m dt + p_{\alpha,0}(x,c)(-x)^m,$$

then, $\mu_{\alpha,0}(x) = 1$, $\mu_{\alpha,1}(x) = 0$ and the following recurrence relation holds:

$$\alpha\mu_{\alpha,m+1}(x) = x(1+cx)\mu'_{\alpha,m}(x) + mx\left[\frac{1}{\rho} + (1+cx)\right]\mu_{\alpha,m-1}(x) + \frac{m}{\rho}\mu_{\alpha,m}(x).$$

3. Direct Results

Let $f \in B_2[0,\infty)$ be the space of all functions f defined on $[0,\infty)$, satisfying the condition $|f(x)| \leqslant M(1+x^2), M > 0$. By $C_2^*[0,\infty)$ we mean the subspace of all functions $f \in B_2[0,\infty) \cap C[0,\infty)$ for which $\lim_{x\to\infty} \frac{f(x)}{1+x^2}$ is finite. In order to study the approximation of functions in the weighted space $C_2^*[0,\infty)$, Ispir and Atakut [**6**], introduced the following weighted modulus of continuity

$$\Omega(f;\delta) = \sup_{x\in[0,\infty),|h|\leqslant\delta} \frac{|f(x+h)-f(x)|}{(1+h^2)(1+x^2)},$$

and proved that $\lim_{\delta\to 0}\Omega(f;\delta) = 0$ and for each $\lambda > 0$,

$$\Omega(f,\lambda\delta) \leqslant 2(1+\lambda)\left(1+\delta^2\right)\Omega(f,\delta). \tag{3.1}$$

THEOREM 3.1. *If $f, f'' \in C_2^*[0,\infty)$, then we have for $x \in [0,\infty)$ that*

$$\left|B_\alpha^\rho(f;x,c) - f(x) - \frac{x[1+\rho(1+cx)]}{2\alpha\rho} f''(x)\right| = 8(1+x^2)\,\mathcal{O}(\alpha^{-1})\,\Omega\big(f'', 1/\sqrt{\alpha}\big).$$

PROOF. By Taylor's formula, there exists ξ lying between x and t such that

$$f(t) = f(x) + f'(x)(t-x) + \frac{f''(x)}{2}(t-x)^2 + h(t,x)(t-x)^2,$$

where $h(t,x) := \frac{f''(\xi)-f''(x)}{2}$. Applying the operator B_α^ρ to the above equality and using Remark 2.1, we can write

$$\left|B_\alpha^\rho(f;x,c) - f(x) - \frac{x[1+\rho(1+cx)]}{2\alpha\rho} f''(x)\right| \leqslant B_\alpha^\rho\left(|h(t,x)|(t-x)^2; x, c\right).$$

From inequality (3.1) and the definition of $\Omega(f,\delta)$, we get

$$|f(t)-f(x)| \leqslant 2(1+x^2)\big[1+(t-x)^2\big](1+|t-x|/\delta)\big(1+\delta^2\big)\Omega(f,\delta) \tag{3.2}$$

for every $f \in C_2^*[0,\infty)$ and $x,t \in [0,\infty)$. Using (3.2) and the inequality $|\xi - x| \leqslant |t-x|$, we can write

$$|h(t,x)| \leqslant [1+(t-x)^2](1+x^2)(1+|t-x|/\delta)(1+\delta^2)\,\Omega(f'',\delta).$$

Also,

$$|h(t,x)| \leqslant \begin{cases} 2(1+x^2)(1+\delta^2)^2\,\Omega(f'',\delta), & |t-x|<\delta,\\ [1+(t-x)^2](1+x^2)(1+|t-x|/\delta)(1+\delta^2)\,\Omega(f'',\delta), & |t-x|\geqslant\delta. \end{cases}$$

Now choosing $\delta < 1$, we have

$$\begin{aligned} |h(t,x)| &\leqslant 2\ (1+x^2)\Big(1+\frac{(t-x)^4}{\delta^4}\Big)(1+\delta^2)^2\ \Omega(f'',\delta)\\ &\leqslant 8\ (1+x^2)\Big(1+\frac{(t-x)^4}{\delta^4}\Big)\ \Omega(f'',\delta). \end{aligned}$$

Using Remark 2.1, we deduce that

$$\begin{aligned} &B_\alpha^\rho(|h(t,x)|(t-x)^2;x,c)\\ &\qquad= 8(1+x^2)\,\Omega(f'',\delta)\Big[B_\alpha^\rho((t-x)^2;x,c)+\frac{1}{\delta^4}B_\alpha^\rho((t-x)^6;x,c)\Big]\\ &\qquad= 8(1+x^2)\,\Omega(f'',\delta)\Big[\mathcal{O}(\alpha^{-1})+\frac{1}{\delta^4}\mathcal{O}(n^{-3})\Big]. \end{aligned}$$

Choosing $\delta = 1/\sqrt{\alpha}$, we have

$$B_\alpha^\rho\big(|h(t,x)|(t-x)^2;x,c\big) = 8(1+x^2)\ \mathcal{O}\left(\alpha^{-1}\right)\Omega\big(f'',1\sqrt{\alpha}\,\big).$$

Hence the result follows. □

In the following result we discuss the Grüss–Voronovskaya theorem for operator defined by (1.1) which show the non-multiplicativity of positive linear operator $B_\alpha^\rho(.;x,c)$.

THEOREM 3.2. *Let f, g, $f'(x)$, $g'(x)$, $f''(x)$, $g''(x)$, $(fg)'(x)$, and $(fg)''(x)$ belong to $C_2^*[0,\infty)$. Then we have*

$$\lim_{\alpha\to\infty}\alpha\{B_\alpha^\rho(fg;x,c)-B_\alpha^\rho(f;x,c)B_\alpha^\rho(g;x,c)\} = \frac{x[1+\rho(1+cx)]}{\rho}f'(x)g'(x).$$

PROOF. One can easily obtain the following expression

$$\begin{aligned} &\alpha\{B_\alpha^\rho(fg;x,c)-B_n^\rho(f;x,c)B_\alpha^\rho(g;x,c)\}\\ &\quad=\alpha\Big\{B_\alpha^\rho(fg;x,c)-f(x)g(x)-\frac{(fg)''}{2!}B_\alpha^\rho((t-x)^2;x,c)\\ &\quad+g(x)\Big(B_\alpha^\rho(f;x,c)-f(x)-\frac{f''(x)}{2!}B_\alpha^\rho((t-x)^2;x,c)\Big)\\ &\quad+B_\alpha^\rho(f;x,c)\Big(B_\alpha^\rho(g;x,c)-g(x)-\frac{g''(x)}{2!}B_\alpha^\rho((t-x)^2;x,c)\Big)\\ &\quad+2\frac{B_\alpha^\rho((t-x)^2;x,c)}{2!}f'(x)g'(x)+g''(x)\frac{B_\alpha^\rho((t-x)^2;x)}{2!}\big(f(x)-B_\alpha^\rho(f;x,c)\big). \end{aligned}$$

Now, using Remark 2.1 and Theorem 3.1, we obtain

$$\lim_{\alpha\to\infty} B_\alpha^\rho((t-x)^2;x,c) = \frac{x[1+\rho(1+cx)]}{\rho}, \quad \text{and}$$

$$\left\{\lim_{\alpha\to\infty} B_\alpha^\rho(f;x,c) - f(x) - f''(x)\frac{B_\alpha^\rho((t-x)^2;x,c)}{2!}\right\} \to 0,$$

as $\alpha\to\infty$ for $f''\in C_2^*[0,\infty)$ at any $x\in[0,\infty)$. Hence the desired result. □

Finally, in this section we obtain a direct result in terms of the weighted Ditzian–Totik modulus of smoothness. The second order Ditzian–Totik modulus of smoothness is defined by

$$\omega_\varphi^2(f,\delta) = \sup_{0<h\leqslant\delta}\ \sup_{x\pm h\varphi(x)\in[0,\infty)} |f(x+h\,\varphi(\alpha,x)) - 2f(x) + f(x-h\,\varphi(\alpha,x))|,$$

$\varphi(c,\rho,x)=\sqrt{x(1+\rho^{-1}+cx)}$, $x>0$. The corresponding K-functional is

$$K_{2,\varphi}(f,\delta^2) = \inf_{h\in W_\infty^2(\varphi)}\{\|f-h\| + \delta^2\|\varphi^2 h''\|\},$$

where $W_\infty^2(\varphi) = \{h\in C_B[0,\infty) : h'\in AC_{\rm loc}[0,\infty) : \varphi^2h''\in C_B[0,\infty)\}$. By Theorem 2.1.1 of [**8**], it follows that

$$C^{-1}\omega_\varphi^2(f,\delta) \leqslant K_{2,\varphi}(f,\delta^2) \leqslant C\omega_\varphi^2(f,\delta)$$

for some absolute constant $C>0$.

Theorem 3.3. *If $f\in C_B[0,\infty)$ and $\alpha>0$, then for each $x\in(0,\infty)$, we have*

$$|B_\alpha^\rho(f;x,c) - f| \leqslant 2\,\omega_\varphi^2\big(f,1/\sqrt{\alpha}\,\big).$$

Proof. For $g\in W_\infty^2(\varphi)$, by Taylor's formula, we have

$$g(t) = g(x) + (t-x)\,g'(x) + \int_x^t (t-u)\,g''(u)\,du.$$

Applying the operator B_α^ρ to the above equality and then taking the modulus, we get

$$\begin{aligned}|B_\alpha^\rho(g;x,c) - g(x)| &\leqslant B_\alpha^\rho\bigg(\,\bigg|\int_x^t (t-u)\,g''(u)\,du\bigg|;x,c\bigg)\\ &\leqslant \|\varphi^2 g''\|\ \frac{B_\alpha^\rho((t-x)^2;x,c)}{x(1+\rho^{-1}+cx)}\\ &\leqslant \|\varphi^2 g''\|\ \frac{1}{\alpha}.\end{aligned}$$

Now for $f\in C_B[0,\infty)$ we have

$$\begin{aligned}|B_\alpha^\rho(f;x,c) - f(x)| &= |B_\alpha^\rho(f-g;x,c) - (f-g)(x)| + |B_\alpha^\rho(g;x,c) - g(x)|\\ &\leqslant 2\,\|f-g\| + \frac{1}{\alpha}\|\varphi^2 g''\|.\end{aligned}$$

Hence, by the definition of $K_{2,\varphi}(f,\delta^2)$, we obtain the inequality

$$|B_\alpha^\rho(f;x,c) - f| \leqslant 2\,K_{2,\varphi}(f,1/\alpha) \leqslant 2\,\omega_\varphi^2\big(f,1\sqrt{\alpha}\,\big).$$ □

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