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APPROXIMATION BETWEEN MODIFIED BASKAKOV AND SRIVASTAVA–GUPTA OPERATORS

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Abstract. We study the approximation of difference of operators. We find the quantitative estimate for the difference of modified Baskakov and the Srivastava–Gupta operators.

1. Introduction

On the difference of two linear positive several results have been compiled in the recent book [**4**]. Also, Aral et al. [**1**] established a result for the difference operators in weighted space recently.

Aral et al. [**1**] considered $F_k\colon D \to R$ be positive linear functional defined on a subspace D of $C[0,\infty)$, which contains $C_2[0,\infty)$ and the polynomials up to degree 6, such that $F_k(e_0) = 1$, $b^{F_k} := F_k(e_1)$, $\mu_r^{F_k} = F_k(e_1 - b^{F_k}e_0)^r$, $r \in \mathbb{N}$ and established the approximation result on polynomial weighted spaces. They considered the operators having the same basis function $p_k(x)$ given by

$$U(f,x) = \sum_{k=0}^{\infty} p_k(x)F_k(f), \quad V(f,x) = \sum_{k=0}^{\infty} p_k(x)G_k(f).$$

Let us consider $B_2[0,\infty) := \{f : |f(x)| \leqslant c_f(1+x^2) \text{ with } c_f > 0\}$. Also let $C_2[0,\infty)$ denotes the subspace of all continuous functions in $B_2[0,\infty)$. Further $C_2^*[0,\infty)$ denotes the closed subspace of $C_2[0,\infty)$ for which $\lim_{x\to\infty} |f(x)|(1+x^2)^{-1} < C$ for some constant C, and $\|.\|_2 = \sup_{x\in[0,\infty)} |f(x)|(1+x^2)^{-1}$. The main result given in [**1**] is the following:

Theorem A. [**1**] *Let $f \in C_2[0,\infty)$ with $f'' \in C_2^*[0,\infty)$ and $x \in [0,\infty)$, then for $n \in \mathbb{N}$, we have*

$$|(U-V)(f,x)| \leqslant \frac{\|f\|_2}{2}\beta(x) + 8\Omega(f'',\delta_1(x))(1+\beta(x)) + 16\Omega(f,\delta_2(x))(\gamma(x)+1),$$

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where

$$\beta(x)=\sum_{k=0}^{\infty}p_k(x)[(1+(b^{F_k})^2)\mu_2^{F_k}+(1+(b^{G_k})^2)\mu_2^{G_k}]$$

$$\gamma(x)=\sum_{k=0}^{\infty}p_k(x)(1+(b^{F_k})^2)$$

$$\delta_1(x)=\Bigg[\sum_{k=0}^{\infty}p_k(x)((1+(b^{F_k})^2)\mu_6^{F_k}+(1+(b^{G_k})^2)\mu_6^{G_k})\Bigg]^{1/4}$$

$$\delta_2(x)=\Bigg[\sum_{k=0}^{\infty}p_k(x)(1+(b^{F_k})^2)(b^{F_k}-b^{G_k})^4\Bigg]^{1/4}.$$

and weighted modulus of continuity is given by

$$\Omega(f,\delta)=\sup_{x\geqslant 0,|h|<\delta}\frac{|f(x+h)-f(x)|}{(1+x^2)(1+h^2)}.$$

2. Modified Baskakov operators and Srivastava–Gupta operators

Let us consider $F_k\colon D\to\mathbb{R}$, $G_k\colon D\to\mathbb{R}$ and define the operators

$$M_n(f,x)=\sum_{k=0}^{\infty}p_{n,k}(x)F_k(f),\quad V_n(f,x)=\sum_{k=0}^{\infty}p_{n,k}(x)G_k(f)$$

as modified Baskakov and Srivastava–Gupta operators respectively.

The modified Baskakov operators are defined as

$$M_n(f;x)=\sum_{k=0}^{\infty}p_{n,k}(x,c)F_k(f) \tag{2.1}$$

where

$$p_{n,k}(x,c)=\frac{(-x)^k}{k!}\phi_{n,c}^{(k)}(x), F_k(f)=f\Big(\frac{k}{n}\Big),$$

with the following special cases:

- If $c=0$ and $\phi_{n,c}(x)=e^{-nx}$ then we get $p_{n,k}(x,0)=e^{-nx}\frac{(nx)^k}{k!}$, and the operators M_n becomes Szász operators.
- If $c\in\mathbb{N}$ and $\phi_{n,c}(x)=(1+cx)^{-n/c}$, then we obtain

$$p_{n,k}(x,c)=\frac{(n/c)_k}{k!}\frac{(cx)^k}{(1+cx)^{\frac{n}{c}+k}},$$

  and operators (2.1) reduce to Baskakov operators.

The Srivastava–Gupta operators [**6**] are defined by

$$V_n(f,x)=\sum_{k=0}^{\infty}p_{n,k}(x,c)G_k(f)$$

where the basis function $p_{n,k}(x,c)$ are defined in (2.1) and

$$G_k(f)=n\int_0^\infty p_{n+c,k-1}(t,c)f(t)dt, 1\leqslant k<\infty,\quad G_0(f)=f(0).$$

Approximation properties of these operators were also discussed in [**2**, **3**, **5**, **7**] etc.

LEMMA 2.1. *The moments of modified Baskakov operators satisfy the following recurrence relation:*

$$M_n(e_{m+1},x)=\frac{x(1+cx)}{n}M_n'(e_m,x)+xM_n(e_m,x).$$

Few moments are given by

$$\begin{aligned}
M_n(e_0,x)&=1,\\
M_n(e_1,x)&=x,\\
M_n(e_2,x)&=\frac{x}{n}[x(n+c)+1],\\
M_n(e_3,x)&=\frac{x}{n^2}[x^2(n+c)(n+2c)+3x(n+c)+1],\\
M_n(e_4,x)&=\frac{x}{n^3}[x^3(n+c)(n+2c)(n+3c)+6x^2(n+c)(n+2c)+7x(n+c)+1],\\
M_n(e_5,x)&=\frac{x}{n^4}[x^4(n+c)(n+2c)(n+3c)(n+4c)+10x^3(n+c)(n+2c)(n+3c)\\
&\quad+25x^2(n+c)(n+2c)+15x(n+c)+1],\\
M_n(e_6,x)&=\frac{x}{n^5}[x^5(n+c)(n+2c)(n+3c)(n+4c)(n+5c)+15x^4(n+c)(n+2c)\\
&(n+3c)(n+4c)+65x^3(n+c)(n+2c)(n+3c)+90x^2(n+c)(n+2c)\\
&\quad+31x(n+c)+1],\\
M_n(e_7,x)&=\frac{x}{n^6}[x^6(n+c)(n+2c)(n+3c)(n+4c)(n+5c)(n+6c)+21x^5(n+c)\\
&(n+2c)(n+3c)(n+4c)(n+5c)+140x^4(n+c)(n+2c)(n+3c)(n+4c)\\
&\quad+350x^3(n+c)(n+2c)(n+3c)+301x^2(n+c)(n+2c)+63x(n+c)+1],\\
M_n(e_8,x)&=\frac{x}{n^7}[x^7(n+c)(n+2c)(n+3c)(n+4c)(n+5c)(n+6c)(n+7c)\\
&\quad+28x^6(n+c)(n+2c)(n+3c)(n+4c)(n+5c)(n+6c)+266x^5(n+c)\\
&(n+2c)(n+3c)(n+4c)(n+5c)+1050x^4(n+c)(n+2c)(n+3c)(n+4c)\\
&\quad+1701x^3(n+c)(n+2c)(n+3c)+966x^2(n+c)(n+2c)+127x(n+c)+1].
\end{aligned}$$

REMARK 2.1. For the modified Baskakov operators, we have $F_k(f)=f\big(\frac{k}{n}\big)$. Thus $b^{F_k}=F_k(e_1)=\frac{k}{n}$ and for $r\in\mathbb{N}$, we have

$$\mu_r^{F_k}:=F_k(e_1-b^{F_k}e_0)^r=0$$

REMARK 2.2. By simple computation, we have

$$G_k(e_r)=\frac{\Gamma(n/c-r)}{c^r\cdot\Gamma(n/c)}\cdot\frac{(k+r-1)!}{(k-1)!}.$$

Thus $b^{G_k} = G_k(e_1) = \frac{k}{n-c}$ and we have

$$\begin{aligned}\mu_2^{G_k} &:= G_k(e_1 - b^{G_k}e_0)^2 = G_k(e_2) + \Big(\frac{k}{n-c}\Big)^2 - 2G_k(e_1)\Big(\frac{k}{n-c}\Big)\\ &= \frac{k(k+1)}{(n-c)(n-2c)} - \Big(\frac{k}{n-c}\Big)^2 = \frac{ck^2+(n-c)k}{(n-c)^2(n-2c)}\end{aligned}$$

$$\begin{aligned}\mu_6^{G_k} &:= G_k(e_1 - b^{G_k}e_0)^6\\ &= G_k(e_6,x) - 6G_k(e_5,x)\Big(\frac{k}{n-c}\Big) + 15G_k(e_4,x)\Big(\frac{k}{n-c}\Big)^2\\ &\quad - 20G_k(e_3,x)\Big(\frac{k}{n-c}\Big)^3 + 15G_k(e_2,x)\Big(\frac{k}{n-c}\Big)^4\\ &\quad - 6G_k(e_1,x)\Big(\frac{k}{n-c}\Big)^5 + \Big(\frac{k}{n-c}\Big)^6\\ &= \frac{k(k+1)(k+2)(k+3)(k+4)(k+5)}{(n-c)(n-2c)(n-3c)(n-4c)(n-5c)(n-6c)}\\ &\quad - 6\frac{k(k+1)(k+2)(k+3)(k+4)}{(n-c)(n-2c)(n-3c)(n-4c)(n-5c)}\Big(\frac{k}{n-c}\Big)\\ &\quad + 15\frac{k(k+1)(k+2)(k+3)}{(n-c)(n-2c)(n-3c)(n-4c)}\Big(\frac{k}{n-c}\Big)^2\\ &\quad - 20\frac{k(k+1)(k+2)}{(n-c)(n-2c)(n-3c)}\Big(\frac{k}{n-c}\Big)^3\\ &\quad + 15\frac{k(k+1)}{(n-c)(n-2c)}\Big(\frac{k}{n-c}\Big)^4 - 5\Big(\frac{k}{n-c}\Big)^5\\ &= \frac{1}{(n-c)^6(n-2c)(n-3c)(n-4c)(n-5c)(n-6c)}[5k^6c^3(3n^2+80nc+37c^2)\\ &\quad + 15k^5c^2(n-c)(3n^2+80nc+37c^2) + 5k^4c(n-c)^2(9n^2+266nc+205c^2)\\ &\quad + 15k^3(n-c)^3(n^2+44nc+75c^2) + 10k^2(n-c)^4(13n+59c) + 120k(n-c)^5].\end{aligned}$$

For modified Baskakov and Srivastava–Gupta operatos, Theorem A takes the following form.

THEOREM 2.1. *Let $f \in C_2[0,\infty)$ with $f'' \in C_2^*[0,\infty)$ and $x \in [0,\infty)$, then for $n \in \mathbb{N}$, we have*

$$|(M_n - V_n)(f,x)| \leqslant \frac{\|f\|_2}{2}\beta(x) + 8\Omega(f'',\delta_1(x))(1+\beta(x)) + 16\Omega(f,\delta_2(x))(\gamma(x)+1),$$

where

$$\begin{aligned}\beta(x) = \frac{x}{n^3(n-c)^4(n-2c)}&[c(n+c)(n+2c)(n+3c)x^3 + (n+c)(n+2c)(n^2-nc+6c)x^2\\ &+ (n^5c - 2n^4c^2 + n^3c^3 + 3n^3 + n^2c - 3nc^2 + 7nc + 7c^2)x\\ &+ (n^6 - 3n^5c + 3n^4c^2 - n^3c^3 + 2n^2 + nc - c)],\end{aligned}$$

$$\gamma(x) = 1 + \frac{x((n+c)x+1)}{(n-c)^2},$$

$$
\begin{aligned}
\delta_1^4(x) = 5x\Big[&\frac{c^3}{(n-c)^2n^7}(3n^2+80nc+37c^2)[x^7(n+c)(n+2c)(n+3c)\\
&(n+4c)(n+5c)(n+6c)(n+7c)+28x^6(n+c)(n+2c)(n+3c)\\
&(n+4c)(n+5c)(n+6c)+266x^5(n+c)(n+2c)(n+3c)(n+4c)\\
&(n+5c)+1050x^4(n+c)(n+2c)(n+3c)(n+4c)+1701x^3(n+c)\\
&(n+2c)(n+3c)+966x^2(n+c)(n+2c)+127x(n+c)+1]\\
&\quad+\frac{3c^2}{(n-c)n^6}(3n^2+80nc+37c^2)[x^6(n+c)(n+2c)(n+3c)(n+4c)\\
&(n+5c)(n+6c)+21x^5(n+c)(n+2c)(n+3c)(n+4c)(n+5c)\\
&\quad+140x^4(n+c)(n+2c)(n+3c)(n+4c)+350x^3(n+c)(n+2c)\\
&(n+3c)+301x^2(n+c)(n+2c)+63x(n+c)+1]\\
&\quad+\frac{c}{n^5}(c^2(3n^2+205)+9n^2+80nc^3+266nc+37c^4)\\
&[x^5(n+c)(n+2c)(n+3c)(n+4c)(n+5c)+15x^4(n+c)(n+2c)\\
&(n+3c)(n+4c)+65x^3(n+c)(n+2c)(n+3c)+90x^2(n+c)(n+2c)\\
&\quad+31x(n+c)+1]+\frac{3(n-c)}{n^4}\\
&(3n^2c^2+n^2+80nc^3+44nc+37c^4+75c^2)[x^4(n+c)(n+2c)(n+3c)\\
&(n+4c)+10x^3(n+c)(n+2c)(n+3c)+25x^2(n+c)(n+2c)\\
&\quad+15x(n+c)+1]+\frac{(n-c)^2}{n^3}(9n^2c+266nc^2+26n+205c^3+118c)\\
&[x^3(n+c)(n+2c)(n+3c)+6x^2(n+c)(n+2c)+7x(n+c)+1]\\
&\quad+\frac{(n-c)^3}{n^2}(n^2+44nc+75c^2+8)[x^2(n+c)(n+2c)+3x(n+c)+1]\\
&\quad+\frac{2(n-c)^4}{n}(13n+59c)[x(n+c)+1]+24(n-c)^5\Big]
\end{aligned}
$$

and

$$
\begin{aligned}
\delta_2^4(x) = \frac{c^4x}{(n-c)^4n^{11}}[&x^5(n+c)(n+2c)(n+3c)(n+4c)(n+5c)+15x^4(n+c)\\
&(n+2c)(n+3c)(n+4c)+x^3(n+c)(n+2c)(n+3c)\\
&(n^4+65)+6x^2(n+c)(n+2c)(n^4+15)+x(n+c)\\
&(7n^4+31)+(n^4+1)].
\end{aligned}
$$

Proof. By using Remark 2.1, Remark 2.2 and applying Lemma 2.1, we have

$$
\beta(x):=\sum_{k=0}^{\infty}p_{n,k}(x,c)[(1+(b^{F_k})^2)\mu_2^{F_k}+(1+(b^{G_k})^2)\mu_2^{G_k}]
$$

$$
\begin{aligned}
&= \sum_{k=0}^{\infty} p_{n,k}(x,c)\Big(1+\frac{k^2}{(n-c)^2}\Big)\cdot\frac{ck^2+(n-c)k}{(n-c)^2(n-2c)}\\
&= \sum_{k=0}^{\infty} p_{n,k}(x,c)\Big(\frac{(n-c)^2+k^2}{(n-c)^2}\Big)\frac{ck^2+(n-c)k}{(n-c)^2(n-2c)}\\
&= \frac{x}{n^3(n-c)^4(n-2c)}[c(n+c)(n+2c)(n+3c)x^3\\
&\quad + (n+c)(n+2c)(n^2-nc+6c)x^2\\
&\quad + (n^5c-2n^4c^2+n^3c^3+3n^3+n^2c-3nc^2+7nc+7c^2)x\\
&\quad + (n^6-3n^5c+3n^4c^2-n^3c^3+2n^2+nc-c)],
\end{aligned}
$$

$$
\gamma(x) = \sum_{k=0}^{\infty} p_k(x)(1+(b^{F_k})^2) = \sum_{k=0}^{\infty} p_{n,k}(x,c)\Big(1+\frac{k^2}{(n-c)^2}\Big) = 1+\frac{x((n+c)x+1)}{(n-c)^2}.
$$

Also

$$
\begin{aligned}
\delta_1^4(x) &= \sum_{k=0}^{\infty} p_{n,k}(x,c)((1+(b^{F_k})^2)\mu_6^{F_k}+(1+(b^{G_k})^2)\mu_6^{G_k})\\
&= \sum_{k=0}^{\infty} \frac{p_{n,k}(x,c)\big(1+\frac{k^2}{(n-c)^2}\big)}{(n-c)^6(n-2c)(n-3c)(n-4c)(n-5c)(n-6c)}\\
&\quad [5k^6c^3(3n^2+80nc+37c^2)+15k^5c^2(n-c)(3n^2+80nc+37c^2)\\
&\qquad + 5k^4c(n-c)^2(9n^2+266nc+205c^2)+15k^3(n-c)^3(n^2+44nc+75c^2)\\
&\qquad + 10k^2(n-c)^4(13n+59c)+120k(n-c)^5]\\
&= 5x\Big[\frac{c^3}{(n-c)^2n^7}(3n^2+80nc+37c^2)\\
&\quad [x^7(n+c)(n+2c)(n+3c)(n+4c)(n+5c)(n+6c)(n+7c)\\
&\qquad + 28x^6(n+c)(n+2c)(n+3c)(n+4c)(n+5c)(n+6c)\\
&\qquad + 266x^5(n+c)(n+2c)(n+3c)(n+4c)(n+5c)\\
&\qquad + 1050x^4(n+c)(n+2c)(n+3c)(n+4c)\\
&\qquad + 1701x^3(n+c)(n+2c)(n+3c)+966x^2(n+c)(n+2c)\\
&\qquad + 127x(n+c)+1]+\frac{3c^2}{(n-c)n^6}(3n^2+80nc+37c^2)\\
&\quad [x^6(n+c)(n+2c)(n+3c)(n+4c)(n+5c)(n+6c)\\
&\qquad + 21x^5(n+c)(n+2c)(n+3c)(n+4c)(n+5c)\\
&\qquad + 140x^4(n+c)(n+2c)(n+3c)(n+4c)+350x^3(n+c)(n+2c)(n+3c)\\
&\qquad + 301x^2(n+c)(n+2c)+63x(n+c)+1]\\
&\qquad + \frac{c}{n^5}(c^2(3n^2+205)+9n^2+80nc^3+266nc+37c^4)
\end{aligned}
$$

$$
\begin{aligned}
&[x^5(n+c)(n+2c)(n+3c)(n+4c)(n+5c) \\
&\quad + 15x^4(n+c)(n+2c)(n+3c)(n+4c) + 65x^3(n+c)(n+2c)(n+3c) \\
&\quad + 90x^2(n+c)(n+2c) + 31x(n+c) + 1] \\
&\quad + \frac{3(n-c)}{n^4}(3n^2c^2 + n^2 + 80nc^3 + 44nc + 37c^4 + 75c^2) \\
&[x^4(n+c)(n+2c)(n+3c)(n+4c) + 10x^3(n+c)(n+2c)(n+3c) \\
&\quad + 25x^2(n+c)(n+2c) + 15x(n+c) + 1] \\
&\quad + \frac{(n-c)^2}{n^3}(9n^2c + 266nc^2 + 26n + 205c^3 + 118c) \\
&[x^3(n+c)(n+2c)(n+3c) + 6x^2(n+c)(n+2c) + 7x(n+c) + 1] \\
&\quad + \frac{(n-c)^3}{n^2}(n^2 + 44nc + 75c^2 + 8)[x^2(n+c)(n+2c) + 3x(n+c) + 1] \\
&\quad + \frac{2(n-c)^4}{n}(13n + 59c)[x(n+c) + 1] + 24(n-c)^5\Big]
\end{aligned}
$$

and

$$
\begin{aligned}
\delta_2^4(x) &= \sum_{k=0}^{\infty} p_{n,k}(x,c)(1 + (b^{F_k})^2)(b^{F_k} - b^{G_k})^4 \\
&= \sum_{k=0}^{\infty} p_{n,k}(x,c)\Big(1 + \Big(\frac{k}{n}\Big)^2\Big)\Big(\frac{k}{n} - \frac{k}{n-c}\Big)^4 \\
&= \frac{c^4 x}{(n-c)^4 n^{11}}[x^5(n+c)(n+2c)(n+3c)(n+4c)(n+5c) \\
&\quad + 15x^4(n+c)(n+2c)(n+3c)(n+4c) + x^3(n+c)(n+2c)(n+3c) \\
&(n^4 + 65) + 6x^2(n+c)(n+2c)(n^4 + 15) + x(n+c)(7n^4 + 31) + (n^4 + 1)].
\end{aligned}
$$

Combining the above estimates, we get the desired result. □

Remark 2.3. If we take $c = -1$ and $\phi_{n,c}(x) = (1-x)^n$ then $p_{n,k}(x,-1) = \binom{n}{k}x^k(1-x)^{n-k}$, and operators (2.1) reduce to Bernstein polynomials. As in the present paper we deal with the weighted modulus of continuity so this case may be discussed elsewhere.

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