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# STRONG CONVERGENCE OF MODIFIED ISHIKAWA ITERATIVE-TYPE ALGORITHM IN CAT(0) SPACES

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ABSTRACT. We introduce a modified Ishikawa iterative type scheme in CAT(0) space and prove strong convergence theorem for total asymptotically demicontractive mappings. Our result improve and extend many results in the literature.

## 1. Introduction

Metric spaces are important tools used in the modelling of day-to-day life problems. Of course, the structure of a metric space is sometimes far too general in order to apply existing theories used in the study of such processes. In order to assure a certain regularity, some properties are normally considered on the metric space. Some of these properties provide sufficient information which allows the development and extension of mathematical theories that play an essential role in solving such problems. The existence of distance-preserving curves between any two points of the space is one of the most important properties that can be imposed in a metric space since it endows the space with a structure that resembles in some way the linear structure of a normed space. Such spaces are called geodesic metric spaces. More precisely, having a metric space $(X, d)$, a geodesic path from $x \in X$ to $y \in X$ is a distance-preserving mapping $c\colon [0; l] \subseteq \mathbb{R} \to X$ such that $c(0) = x$ and $c(l) = y$. The image $c([0, l])$ of $c$ forms a geodesic segment which joins $x$ and $y$ and is not necessarily unique. A metric space is geodesic if every two points of its can be joined by a geodesic path. A comprehensive treatment of geodesic metric spaces can be found, for instance, in [**1**, **4**].

A metric space $(X, d)$ is a CAT(0) space if it is geodesically connected and if every geodesic triangle in $X$ is at least as thin as its comparison triangle in the Euclidean plane. Complete CAT(0) spaces are often called Hadamard spaces. Let

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$x, y \in X$ and $\lambda \in [0,1]$. We write $\lambda x \oplus (1-\lambda)y$ for the unique point $z$ in the geodesic segment joining from $x$ to $y$ such that

$$d(z,x) = (1-\lambda)d(x,y) \quad \text{and} \quad d(z,y) = \lambda\, d(x,y).$$

We also denote by $[x, y]$ the geodesic segment joining from $x$ to $y$, that is, $[x,y] = \{\lambda x \oplus (1-\lambda)y : \lambda \in [0,1]\}$. A subset $C$ of a CAT(0) space is convex if $[x,y] \subseteq C$ for all $x, y \in C$. Berg and Nikolaev [**2**] introduced the concept of *quasilinearization* in a metric space $X$. Let denote a pair $(a,b) \in X \times X$ by $\overrightarrow{ab}$ and call it a vector. The quasilinearization is a map $\langle .\,,.\rangle\colon (X\times X)\times(X\times X) \to R$ defined by

$$\langle \overrightarrow{ab}, \overrightarrow{cd}\rangle = \tfrac{1}{2}(d^2(a,d) + d^2(b,c) - d^2(a,c) - d^2(b,d)), \quad \forall a,b,c,d \in X.$$

It is easily seen that

$$\langle \overrightarrow{ab}, \overrightarrow{cd}\rangle = \langle \overrightarrow{cd}, \overrightarrow{ab}\rangle, \langle \overrightarrow{ab}, \overrightarrow{cd}\rangle = -\langle \overrightarrow{ba}, \overrightarrow{cd}\rangle \;\text{ and }\; \langle \overrightarrow{ax}, \overrightarrow{cd}\rangle + \langle \overrightarrow{xb}, \overrightarrow{cd}\rangle = \langle \overrightarrow{ab}, \overrightarrow{cd}\rangle$$

for all $a, b, c, d \in X$. We say that $X$ satisfies the Cauchy–Schwarz inequality if $\langle \overrightarrow{ab}, \overrightarrow{cd}\rangle \leqslant d(a,b)d(c,d)$ for all $a, b, c, d \in X$. It is known that a geodesically connected metric space is a CAT(0) space if and only if it satisfies the Cauchy–Schwarz inequality (see [**2**]).

Fixed point theory is a mathematical domain that has enjoyed a prosperous development in the last fifty years. This theory was extended in various directions, classical instruments were generalized, new notions and results have been given and constantly improved. Fixed point theorems are used in many branches of mathematics such as analysis, geometry or dynamical systems. In addition, they are important tools applied in other domains with no immediate connection with mathematics at first glance. Many stability and equilibrium problems can be modelled using fixed points. Such examples can be found in economics, game theory, compiler theory and many others. Metric fixed point theory was born with the well-known Banach–Caccioppoli Contraction Principle that was initially published in 1922. This result states that every self-contraction defined on a complete metric space has a unique fixed point and any sequence of Picard iterates converges to the unique fixed point. A speed of convergence of the Picard iterates to the unique fixed point is also established. Since then, this principle was constantly improved and extended in many directions. Particularly, fixed point theorems in a CAT(0) space have attracted the attention of many researchers see [**6**–**8**, **12**, **14**, **15**, **21**, **22**]. It can be applied to graph theory, biology and computer sciences (see eg, [**3**, **9**, **13**, **20**]).

Let $C$ be a nonempty subset of a complete CAT(0) space $X$. Let $T$ from $C$ into itself be a mapping. Then, a point $x \in C$ is called a fixed point of $T$ if $Tx = x$. We denote by $F(T)$ the set of all the fixed points of $T$. In 2013, Kim [**11**] defined the following mappings

Definition 1.1. Let $C$ be a nonempty subset of a metric space $(X,d)$ and $F(T)$ denote the fixed point set of $T$ with $F(T) \neq \emptyset$.

(1) A mapping $T\colon C \to C$ is said to be $k$-strict asymptotically pseudocontractive with sequence $\{u_n\}$ if $\lim_{n\to\infty} u_n = 1$ for some constant $k$, $0 \leqslant k < 1$ and $d^2(T^n x, T^n y) \leqslant u_n^2 d^2(x,y) + k(d(x,T^n x) - d(y,T^n y))^2$ for all $x, y \in C$, $n \in \mathbb{N}$.

(2) A mapping $T : C \to C$ is said to be $k$-strict asymptotically demicontractive with sequence $\{u_n\}$ if $\lim_{n\to\infty} = 1$ for some constant $k$, $0 \leqslant k < 1$ and

$$d^2(T^n x, p) \leqslant u_n^2 d^2(x, p) + kd^2(x, T^n x)$$

for all $p \in F(T)$, $x \in C$, $n \in \mathbb{N}$.

Using the following Ishikawa-type algorithm:

$$\begin{aligned} y_n &= (1-\beta_n)x_n \oplus \beta_n T^n x_n, \\ x_{n+1} &= (1-\alpha_n)x_n \oplus \alpha_n T^n y_n, \end{aligned} \tag{1.1}$$

under some assumption on sequences of $\{\alpha_n\}$ and $\{\beta_n\}$ in $(0, 1)$. The author in [**11**] establishes the theorems of strong convergence, for the Ishikawa-type iteration scheme, to a fixed point of a completely continuous uniformly $L$-Lipschitzian asymptotically demicontractive mapping in a bounded convex subset of CAT(0) space.

In 2014, using the concept of a quasilinearization, Liu and Chang [**17**] defined total asymptotically demicontractive mappings in CAT(0) spaces as follows:

Definition 1.2. Let $C$ be a nonempty subset of a CAT(0) space. A mapping $T: C \to C$ is a said to be demicontractive if $F(T) \neq \emptyset$ and there exists a constant $k \in [0, 1)$ such that

$$\langle \overrightarrow{Txp}, \overrightarrow{xp} \rangle \leqslant d^2(x, p) - kd^2(x, Tx), \quad \forall\ x \in C,\ p \in F(T).$$

It is easy to show that the above inequality is equivalent to

$$d^2(Tx, p) \leqslant d^2(x, p) + (1 - 2k)d^2(x, Tx).$$

Definition 1.3. Let $X$ be a CAT(0) space, $C$ be a nonempty subset of $X$. A mapping $T: C \to C$ with $F(T) \neq \emptyset$ is said to be:

(1) *an asymptotically demicontractive mapping*, if there exists a constant $k \in [0, 1)$ and a nonnegative sequence $\{u_n\} \subset [0, \infty)$ with $u_n \to 0$ such that

$$d^2(T^n x, p) \leqslant (1 + u_n)d^2(x, p) + kd^2(x, T^n x),$$

for all $n \geqslant 1$, $x \in C$, $p \in F(T)$.

(2) $(\{u_n\}, \{v_n\}, \{\phi\})$-*total asymptotically demicontractive mapping* if there exist a constant $k \in [0, 1)$ and nonnegative sequences $\{u_n\}, \{v_n\} \subset [0, \infty)$ with $u_n \to 0$, $v_n \to 0$ and strictly continuous function $\phi$: $[0, \infty) \to [0, \infty)$ with $\phi(0) = 0$ such that

$$d^2(T^n x, p) \leqslant d^2(x, p) + u_n \phi(d(x, p)) + kd^2(x, T^n x) + v_n \tag{1.2}$$

for all $n \geqslant 1$, $x \in C$, $p \in F(T)$.

The authors in [**17**] proved the following convergence theory using an Ishikawa type algorithm.

Theorem 1.1. [**17**] *Let $X$ be a complete* CAT(0) *space and $C$ be a nonempty closed convex subset of $X$. Let $T: C \to C$ be completely continuous, uniformly $L$-Lipschitzian and total asymptotically demicontractive mappings with $\{u_n\}, \{v_n\}$ and mappings $\phi_{i(n)}: [0, \infty) \to [0, \infty)$ satisfying $\sum_{n=1}^{\infty} u_n < \infty$ and $\sum_{n=1}^{\infty} v_n < \infty$, such* *that $F(T) \neq \emptyset$. Let $\{x_n\}_{n=1}^{\infty}$ be a sequence generated by* (1.1) *where $\{\alpha_n\}_{n=1}^{\infty}, \{\beta_n\} \subset (0,1)$, $\epsilon \leqslant k \leqslant \alpha_n \leqslant \beta_n \leqslant b$, $\forall n \geqslant 1$ for some $\epsilon > 0$, $k \in [0,1)$ and $b \in (0, L^{-2}[\sqrt{1+L^2}-1])$. Assume there exist positive constants $M^*$ and $M$, such that $\phi(r) \leqslant Mr^2, \forall r \geqslant M^*$. Then $\{x_n\}_{n=1}^{\infty}$ converges strongly to $F(T)$.*

In 2016, Pakkaranang and Kumam [**16**], define the concept of an asymptotically $k$-strictly pseudo-contractive mapping in a CAT(0) as follows: Let $C$ be a nonempty subset of a CAT(0) space $X$. A mapping $T\colon C \to C$ is said to be asymptotically $k$-strictly pseudo-contractive if there exists a constant $k \in [0,1)$ and sequence $k_n \in [1,\infty)$ with $\lim_{n\to\infty} k_n = 1$ such that

$$d^2(T^n x, T^n y) \leqslant k_n d^2(x,y) + k(d(x,T^n x) + d(y,T^n y))^2, \quad \forall x,y \in C. \tag{1.3}$$

Using (1.3) with the assumption that $T$ is semicompact, they proved convergence theorems of the modified Mann iteration for asymptotically k-strictly pseudo-contracttive mappings in CAT(0) spaces.

On the other hand, in 2013, Osilike et al. [**19**] introduced a modified Ishikawa iterative scheme in the Hilbert space as follows: For an arbitrary $x \in C$, the sequence $\{x_n\}$, given by

$$\begin{aligned} z_n &= P_C((1-t_n)x_n), \\ y_n &= (1-\beta_n)z_n + \beta_n T^n z_n, \\ x_{n+1} &= (1-\alpha_n)z_n + \alpha_n T^n y_n,\ n \geqslant 1. \end{aligned} \tag{1.4}$$

Under some assumption on the $t_n, \alpha_n, \beta_n \in (0,1)$, they proved the strong convergence of (1.4) for uniformly $L$-Lipschitzian and asymptotically pseudocontractive mapping.

Question: Can we study modified Ishikawa iterative type scheme (1.4) in CAT(0) space and establish its strong convergence without any assumption of completely continuous or compactness on the operator?

In this paper, we study modified Ishikawa iterative type scheme (1.4) and prove strong convergence result for uniformly $L$-Lipschitzian and total asymptotically demicontractive mappings in CAT(0) space. Our result improve and extend the results of Liu and Chang [**17**], and Pakkaranang and Kumam [**16**] respectively without assuming completely continuous on the operator and results of Kim [**11**] without assuming completely continuous on operator and boundedness on the space. Our result also extends and generalizes the results of Osilike et al. [**19**].

## 2. Preliminaries

In the sequel, we shall need the following definition and results.

Lemma 2.1. [**5**, **8**] *Let $X$ be a* CAT(0) *space, $x,y,z \in X$ and $\lambda \in [0,1]$. Then*

(i) $d(\lambda x \oplus (1-\lambda)y, z) \leqslant \lambda\, d(x,z) + (1-\lambda)d(y,z)$.
(ii) $d^2(\lambda x \oplus (1-\lambda)y, z) \leqslant \lambda\, d^2(x,z) + (1-\lambda)d^2(y,z) - \lambda(1-\lambda)d^2(x,y)$.
(iii) $d^2(\lambda x \oplus (1-\lambda)y, z) \leqslant \lambda^2 d^2(x,z) + (1-\lambda)^2 d^2(y,z) + 2\lambda(1-\lambda)\langle \overrightarrow{xz}, \overrightarrow{yz} \rangle$.

Let $\{x_n\}$ be a bounded sequence in a complete CAT(0) space $X$. For $x \in X$, we set

$$r(x, \{x_n\}) = \limsup_{n\to\infty} d(x, x_n).$$

The asymptotic radius $r(\{x_n\})$ of $\{x_n\}$ is given by

$$r(\{x_n\}) = \inf\{r(x, \{x_n\}) : x \in X\},$$

and the asymptotic center $A(\{x_n\})$ of $\{x_n\}$ is the set

$$A(\{x_n\}) = \{x \in X : r(x, \{x_n\}) = r(\{x_n\})\}.$$

It is well known that in a CAT(0) space, $A(\{x_n\})$ consists of exactly one point [**7**]. A sequene $\{x_n\}$ in $X$ is called $\Delta$-convergent to $x \in X$, denoted by $\Delta - \lim_n x_n = x$ if $x$ is the unique asymptotic center of $\{u_n\}$, for every subsequence $\{u_n\}$ of $\{x_n\}$.

Lemma 2.2. [**15**] *Every bounded sequence in a complete* CAT(0) *space always has a* $\Delta$*-convergent subsequence.*

Lemma 2.3. [**6**] *If* $\{x_n\}$ *is a bounded sequence in a closed and convex subset* $C$ *of a complete* CAT(0) *space, then the asymptotic center of* $\{x_n\}$ *is in* $C$.

Let $\{x_n\}$ be a bounded sequence in a complete CAT(0) space $X$, and $C$ be a closed and convex subset of $X$ which contains $\{x_n\}$. We employ the notation

$$\{x_n\} \rightharpoonup w \iff \Phi(w) = \inf_{x\in C} \Phi(x)$$

where $\Phi(x) = \limsup_{n\to\infty} d(x_n, x)$. We note that $\{x_n\} \rightharpoonup w$ if and only if $A(\{x_n\}) = \{w\}$ (see [**18**]).

Lemma 2.4. [**18**] *If* $\{x_n\}$ *is a bounded sequence in a closed and convex subset* $C$ *of a complete* CAT(0) *space, then* $\Delta - \lim_{n\to\infty} x_n = p$ *implies that* $\{x_n\} \rightharpoonup p$.

Lemma 2.5. [**10**] *Let* $\{x_n\}$ *be a sequence in a complete* CAT(0) *space* $X$*, and* $x \in X$*. Then* $\{x_n\}$ *is* $\Delta$*-convergent to* $x$ *if and only if* $\limsup_{n\to\infty}\langle \overrightarrow{xx_n}, \overrightarrow{xy}\rangle \leqslant 0$ *for all* $y \in C$.

Lemma 2.6. [**24**] *If* $\{a_n\}$ *is a sequence of nonnegative real numbers satisfying*

$$a_{n+1} \leqslant (1-\alpha_n)a_n + \alpha_n\sigma_n + \gamma_n, \ \ n \geqslant 0,$$

*where,* (i) $\{\alpha_n\} \subset [0,1]$, $\sum \alpha_n = \infty$*;* (ii) $\limsup \sigma_n \leqslant 0$*;* (iii) $\gamma_n \geqslant 0$*;* $(n \geqslant 0)$ *and* $\sum \gamma_n < \infty$*. Then,* $a_n \to 0$ *as* $n \to \infty$.

## 3. Main Result

Theorem 3.1. *Let* $X$ *be a complete* CAT(0) *space and* $C$ *be a nonempty closed convex subset of* $X$*. Let* $T\colon C \to C$ *be uniformly* $L$*-Lipschitzian and total asymptotically demicontractive mappings with* $\{u_n\}, \{v_n\}$ *and mappings* $\phi\colon [0,\infty) \to [0,\infty)$ *satisfying* $\sum_{n=1}^{\infty} u_n < \infty$ *and* $\sum_{n=1}^{\infty} v_n < \infty$*, such that* $T$ *is* $\Delta$*-demiclosed at* 0 *and* $F(T) \neq \emptyset$*. Let* $\{x_n\}_{n=1}^{\infty}$ *be a sequence generated by* $x_1 = x \in C$,

$$z_n = P_C((1-t_n)x_n),$$

$$y_n = (1-\beta_n)z_n \oplus \beta_n T^n z_n, \tag{3.1}$$

$$x_{n+1} = (1-\alpha_n)z_n \oplus \alpha_n T^n y_n,$$

*where* $\{t_n\}_{n=1}^{\infty} \subset (0,1)$, $\epsilon \leqslant k \leqslant \alpha_n \leqslant \beta_n \leqslant b$, $\forall n \geqslant 1$ *for some* $\epsilon > 0$, $k \in [0,1)$ *and* $b \in (0, L^{-2}[\sqrt{1+L^2}-1])$ *with* $\lim_{n\to\infty} t_n = 0$, $\sum_{n=1}^{\infty} t_n = \infty$ *and* $\lim_{n\to\infty} \frac{u_n}{t_n} = 0$, $\lim_{n\to\infty} \frac{v_n}{t_n} = 0$. *Assume there exist positive constants* $M^*$ *and* $M$, *such that* $\phi(r) \leqslant Mr^2, \forall r \geqslant M^*$. *Then* $\{x_n\}_{n=1}^{\infty}$ *converges strongly to* $F(T)$.

Proof. Letting $\delta_n := [1 + \alpha_n u_n M(1+\beta_n(1+u_n M))]t_n$. Since there exists $N_0 > 0$ such that

$$\frac{u_n}{t_n} \leqslant \frac{\epsilon[1+\alpha_n\beta_n(1+\beta_n(1+u_nM))]}{\alpha_n M(1+\beta_n(1+u_nM))}, \quad \frac{v_n}{t_n} \leqslant \frac{[1+\alpha_n u_n M(1+\beta_n(1+u_nM))]}{\alpha_n(1+\beta_n(1+u_nM))},$$

for all $n \geqslant N_0$ and for some $\epsilon > 0$ satisfying $0 \leqslant (1-\epsilon)\delta_n \leqslant 1$. Now, from (3.1), set $G_n z_n := T^n((1-\beta_n)z_n \oplus \beta_n T^n z_n)$. Since, $T$ is uniformly Lischitzian, we obtain

$$\begin{aligned} d(T^n z_n, G_n z_n) &= d(T^n z_n, T^n((1-\beta_n)z_n \oplus \beta_n T^n z_n)) \\ &\leqslant Ld(z_n, (1-\beta_n)z_n \oplus \beta_n T^n z_n) \\ &\leqslant \beta_n L d(z_n, T^n z_n). \end{aligned}$$

For any point $p \in F(T)$ and $n \geqslant N_0$, from (1.2), we obtain

$$d^2(T^n x_n, p) \leqslant d^2(x_n,p) + u_n\phi(d(x_n,p)) + kd^2(x_n,p) + v_n,$$

since $\phi$ is an increasing function, we have the result that $\phi(t) \leqslant \phi(M^*)$ if $t \leqslant M^*$ and $\phi(t) \leqslant Mt^2$ if $t \geqslant M^*$. In either case, we obtain

$$\phi(d(x_n,p)) \leqslant \phi(M^*) + Md^2(x_n,p).$$

It follows that

$$d^2(T^n x_n, p) \leqslant (1+u_nM)d^2(x_n,p) + kd^2(x_n,p) + v_n.$$

Then from (3.1), we obtain

$$\begin{aligned} d^2(G_n z_n, p) &= d^2(T^n((1-\beta_n)z_n \oplus \beta_n z_n), p) \\ &\leqslant (1+u_nM)d^2((1-\beta_n)z_n \oplus \beta_n T^n z_n, p) \\ &\quad + kd^2((1-\beta_n)z_n \oplus \beta_n T^n z_n, G_n z_n) + v_n \\ &\leqslant (1+u_nM)(1-\beta_n)d^2(z_n,p) + (1+u_nM)\beta_n d^2(T^n z_n, p) \\ &\quad - \beta_n(1-\beta_n)(1+u_nM)d^2(z_n, T^n z_n) + (1-\beta_n)kd^2(z_n, G_n z_n) \\ &\quad + \beta_n kd^2(T^n z_n, G_n z_n) - \beta_n(1-\beta_n)kd^2(z_n, T^n z_n) + v_n \\ &\leqslant (1+u_nM)(1-\beta_n)d^2(z_n,p) - \beta_n(1-\beta_n)(1+u_nM)d^2(z_n, T^n z_n) \\ &\quad + (1+u_nM)\beta_n[(1+u_nM)d^2(z_n,p) + kd^2(z_n, T^n z_n) + v_n] \\ &\quad + (1-\beta_n)kd^2(z_n, G_n z_n) + \beta_n^3 L^2 kd^2(z_n, T^n z_n) \\ &\quad - \beta_n(1-\beta_n)kd^2(z_n, T^n z_n) + v_n \\ &= [1+u_nM(1+\beta_n(1+u_nM))]d^2(z_n,p) + (1+\beta_n(1+u_nM))v_n \\ &\quad - \beta_n[k(1-\beta_n-\beta_n^2L^2) + (1+u_nM)(1-\beta_n-k)]d^2(z_n, T^n z_n) \end{aligned} \tag{3.2}$$

$$(1-\beta_n)kd^2(z_n, G_nz_n).$$

Observe from the assumption that $k(1-\beta_n)-(1-\alpha_n) \leqslant 0$ and we can see that $[k(1-\beta_n-\beta_n^2L^2)+(1+u_nM)(1-\beta_n-k)] > 0$. >From (3.1) and (3.2), we obtain

$$\begin{aligned}
d^2(x_{n+1},p) &= d^2((1-\alpha_n)z_n \oplus \alpha_nG_nz_n, p)\\
&\leqslant (1-\alpha_n)d^2(z_n,p)+\alpha_nd^2(G_nz_n,p)-\alpha_n(1-\alpha_n)d^2(z_n,G_nz_n)\\
&\leqslant [1+\alpha_nu_nM[1+\beta_n(1+u_nM)]]d^2(z_n,p)\\
&\quad +\alpha_n(1+\beta_n(1+u_nM))v_n+\alpha_n[k(1-\beta_n)-(1-\alpha_n)]d^2(z_n,G_nz_n)\\
&\quad -\alpha_n\beta_n[k(1-\beta_n-\beta_n^2L^2)+(1+u_nM)(1-\beta_n-k)]d^2(z_n,T^nz_n)\\
&\leqslant [1+\alpha_nu_nM[1+\beta_n(1+u_nM)]]d^2(z_n,p)\\
&\quad -\alpha_n\beta_n[k(1-\beta_n-\beta_n^2L^2)+(1+u_nM)(1-\beta_n-k)]d^2(z_n,T^nz_n)\\
&\quad +\alpha_n(1+\beta_n(1+u_nM))v_n
\end{aligned} \tag{3.3}$$

Also, from (3.1) and for any zero point denoted by $0 \in X$, we obtain

$$\begin{aligned}
d^2(z_n,p) &= d^2(P_C(t_n(0)\oplus t_nx_n),p)\\
&\leqslant d^2(t_n(0)\oplus t_nx_n,p)\\
&\leqslant t_nd^2(0,p)+(1-t_n)d^2(x_n,p)-t_n(1-t_n)d^2(0,x_n)\\
&\leqslant t_nd^2(0,p)+(1-t_n)d^2(x_n,p).
\end{aligned} \tag{3.4}$$

Then from (3.3) and (3.4), we obtain

$$\begin{aligned}
d^2(x_{n+1},p) &\leqslant [1+\alpha_nu_nM[1+\beta_n(1+u_nM)]](1-t_n)d^2(x_n,p)\\
&\quad +[1+\alpha_nu_nM(1+\beta_n(1+u_nM))]t_nd^2(0,p)\\
&\quad +\alpha_n[1+\beta_n(1+u_nM)]v_n-\alpha_n\beta_n[k(1-\beta_n-\beta_n^2L^2)\\
&\quad +(1+u_nM)(1-\beta_n-k)]d^2(z_n,T^nz_n)\\
&\leqslant [1+\alpha_nu_nM[1+\beta_n(1+u_nM)]](1-t_n)d^2(x_n,p)\\
&\quad +[1+\alpha_nu_nM(1+\beta_n(1+u_nM))]t_nd^2(0,p)\\
&\quad +\alpha_n[1+\beta_n(1+u_nM)]v_n\\
&\leqslant [1-\delta_n(1-\epsilon)]d^2(x_n,p)+\delta_n(d^2(0,p)+1)\\
&\leqslant \max\{d^2(x_n,p),(1-\epsilon)^{-1}[d^2(0,p)+1]\}.
\end{aligned} \tag{3.5}$$

Thus, by induction

$$d^2(x_n,p) \leqslant \max\{d^2(x_{N_0},p),(1-\epsilon)^{-1}[d^2(0,p)+1]\}, \quad \forall n \geqslant N_0.$$

It follows that $\{x_n\}$ is bounded. Hence $\{z\}, \{y_n\}, \{Tx_n\}, \{Tz_n\}$ and $\{Ty_n\}$ are also bounded. Furthermore, from (3.1)

$$d(z_n,x_n) \leqslant t_nd(0,p)+(1-t_n)d(x_n,x_n) \to 0 \quad \text{as} \quad n\to\infty. \tag{3.6}$$

Also, from (3.1) and Lemma 2.1, we obtain

$$d^2(z_n,p) \leqslant d^2(t_n(0)\oplus(1-t_n)x_n,p) \tag{3.7}$$

$$\leqslant t^2 d^2(0,p) + (1-t_n)d^2(x_n,p) + 2t_n(1-t_n)\langle\overrightarrow{0p}, \overrightarrow{x_np}\rangle.$$

From (3.5) and (3.7), letting $\theta_n := \alpha_n u_n M[1+\beta_n u_n M(1+u_nM)]$ and $\rho_n := [k(1-\beta_n-\beta_n^2L^2) + (1+u_nM)(1-\beta_n-k)] > 0$. Since $\{x_n\}$ is bounded, then $d(x_n,p) \leqslant D$, for all $n \geqslant 1$, for some $D > 0$, we get

$$\begin{aligned} d^2(x_{n+1},p) &\leqslant (1-t_n)d^2(x_n,p) + \theta_n(1-t_n)D^2 \\ &\quad + (1+\theta_n)t_n^2 d^2(0,p) + 2t_n(1-t_n)(1+\theta_n)\langle\overrightarrow{0p}, \overrightarrow{x_np}\rangle \\ &\quad + \alpha_n[1+\beta_n(1+u_nM)]v_n - \alpha_n\beta_n\rho_n d^2(z_n, T^nz_n) \\ &\leqslant (1-t_n)d^2(x_n,p) + \theta_n(1-t_n)D^2 \\ &\quad + t_n[(1+\theta_n)[t_nd^2(0,p) + 2(1-t_n)\langle\overrightarrow{0p}, \overrightarrow{x_np}\rangle]] \\ &\quad + \alpha_n[1+\beta_n(1+u_nM)]v_n. \end{aligned} \tag{3.8}$$

Also, using the boundedness of $\{x_n\}$, there exists $D_1 > 0$ such that

$$[(1+\theta_n)[t_nd^2(0,p) + 2(1-t_n)\langle\overrightarrow{0p}, \overrightarrow{x_np}\rangle]] \leqslant D_1, \quad \forall n \geqslant 1. \tag{3.9}$$

From (3.8) and (3.9), we obtain

$$\begin{aligned} \alpha_n\beta_n\rho_n d^2(z_n, T^nz_n) &\leqslant (d^2(x_{n+1},p) - d^2(x_n,p)) \\ &\quad + t_n\Big[\frac{\theta_n}{t_n}(1-t_n)D + D_1 + \alpha_n[1+\beta_n(1+u_nM)]\frac{v_n}{t_n}\Big]. \end{aligned} \tag{3.10}$$

Now, we consider two cases to complete the proof.

CASE 3.1. Assume that there exists $N_0 > 0$, such that $\{d^2(x_n,p)\}$ is non-increasing sequence. Then, since $\{d^2(x_n,p)\}$ is bounded, then $\lim_{n\to\infty} d^2(x_n,p)$ exist. So from (3.10), for all $n \geqslant 1$ we obtain

$$\lim_{n\to\infty} \alpha_n\beta_n\rho_n d^2(z_n, T^nz_n) = 0,$$

which implies

$$\lim_{n\to\infty} d(z_n, T^nz_n) = 0. \tag{3.11}$$

Also from (3.1) and (3.11), we obtain

$$d(y_n, z_n) \leqslant (1-\beta_n)d(z_n,z_n) + \beta_n d(z_n, T^nz_n) \to 0 \quad \text{as} \quad n\to\infty \tag{3.12}$$

and

$$\begin{aligned} d(y_n, T^nz_n) &\leqslant (1-\beta_n)d(z_n, T^nz_n) + \beta_n d(T^nz_n, T^nz_n) \\ &\leqslant (1-\beta_n)d(z_n, T^nz_n) + \beta_n L d(z_n,z_n) \to 0 \quad \text{as} \quad n\to\infty. \end{aligned}$$

Now from (3.1), (3.11) and (3.12), we obtain

$$\begin{aligned} d(y_n, T^ny_n) &\leqslant (1-\beta_n)d(z_n, T^ny_n) + \beta_n d(T^nz_n, T^ny_n) \\ &\leqslant (1-\beta_n)[d(z_n, T^nz_n) + d(T^nz_n, T^ny_n)] + \beta_n d(T^nz_n, T^ny_n) \\ &\leqslant (1-\beta_n)d(z_n, T^nz_n) + Ld(z_n, y_n) \to 0 \end{aligned} \tag{3.13}$$

as $n \to \infty$. >From (3.1) and (3.13), we obtain

$$d(x_{n+1}, y_n) \leqslant (1-\alpha_n) d(z_n, y_n) + \alpha_n d(T^n y_n, y_n) \to 0 \quad \text{as} \quad n \to \infty \tag{3.14}$$

and from (3.6), (3.12) and (3.14), we get

$$d(x_{n+1}, x_n) \leqslant d(x_{n+1}, y_n) + d(y_n, z_n) + d(z_n, x_n) \to 0 \quad \text{as} \quad n \to \infty. \tag{3.15}$$

Furthermore

$$\begin{aligned} d(x_n, T^n x_n) &\leqslant d(x_n, z_n) + d(z_n, T^n z_n) + d(T^n z_n, T^n x_n) \\ &\leqslant (1+L) d(x_n, z_n) + d(z_n, T^n z_n), \end{aligned}$$

then from (3.6) and (3.11), we obtain

$$\lim_{n\to\infty} d(x_n, T^n x_n) = 0. \tag{3.16}$$

Observe that since $T$ is uniformly $L$-Lipschitzian, we obtain

$$\begin{aligned} d(x_n, Tx_n) &\leqslant d(x_n, T^n x_n) + d(T^n x_n, Tx_n) \\ &\leqslant d(x_n, T^n x_n) + L d(T^{n-1} x_n, x_n) \\ &\leqslant d(x_n, T^n x_n) + L d(T^{n-1} x_n, T^{n-1} x_{n-1}) \\ &\quad + L d(T^{n-1} x_{n-1}, x_{n-1}) + L d(x_{n-1}, x_n) \\ &\leqslant d(x_n, T^n x_n) + L d(T^{n-1} x_n, x_{n-1}) \\ &\quad + L(1+L) d(x_n, x_{n-1}). \end{aligned}$$

Then from (3.15) and (3.16), we obtain

$$\lim_{n\to\infty} d(x_n, Tx_n) = 0. \tag{3.17}$$

Since $\{x_n\}$ is bounded and $X$ is a complete CAT(0) space, then from Lemma 2.2, there exists a subsequence $\{x_{n_i}\}$ of $\{x_n\}$ such that $\Delta - \lim x_{n_i} = q \in C$. It follows from (3.17) and the fact that $T$ is $\Delta$-demiclosed at 0, we get $q \in F(T)$. Furthermore from Lemma (2.5), we obtain $\limsup_{n\to\infty} \langle \overrightarrow{0q}, \overrightarrow{x_n q} \rangle \leqslant 0$.

Finally letting $q := p$ in (3.8), we obtain

$$\begin{aligned} d^2(x_{n+1}, q) \leqslant{}& (1-t_n) d^2(x_n, q) + \theta_n (1-t_n) D + \alpha_n [1 + \beta_n (1 + u_n M)] v_n \\ &+ t_n [(1+\theta_n)[t_n d^2(0,q) + 2(1-t_n) \langle \overrightarrow{0q}, \overrightarrow{x_n q} \rangle]]. \end{aligned} \tag{3.18}$$

Consequently

$$d^2(x_{n+1}, q) \leqslant (1-t_n) d^2(x_n, q) + t_n \gamma_n + \lambda_n$$

where $\gamma_n := [(1+\theta_n)[t_n d^2(0,q) + 2(1-t_n)\langle \overrightarrow{0q}, \overrightarrow{x_n q} \rangle]]$ and $\lambda_n := \theta_n (1-t_n) D + \alpha_n [1 + \beta_n (1 + u_n M)] v_n$. It now follows from Lemma 2.6 that $d^2(x_n, q) \to 0$ as $n \to \infty$, hence $x_n \to q$ as $n \to \infty$.

CASE 3.2. Suppose that $\{d^2(x_n, q)\}_{n=1}^{\infty}$ is not a monotone decreasing sequence, then set $\Gamma_n := d^2(x_n, q)$, and let $\tau \colon N \to N$ be a mapping defined for all $n \geqslant N_0$ for some sufficiently large $N_0$ by

$$\tau(n) := \max\{k \in N : k \leqslant n, \Gamma_k \leqslant \Gamma_{k+1}\}.$$

Then $\{\tau(n)\}$ is a non-decreasing sequence such that $\tau(n) \to \infty$ as $n \to \infty$ and $\Gamma_{\tau(n)} \leqslant \Gamma_{\tau(n)+1}$ for $n \geqslant N_0$. Following the same argument as in Case 3.1, we obtain

$$d(x_{\tau(n)}, Tx_{\tau(n)}) \to 0 \quad \text{as} \quad n \to \infty.$$

As in Case 3.1, we also obtain that $x_{\tau(n)}$ $\triangle$-converges to some $p \in F(T)$. Furthermore, for all $n \geqslant N_0$, we obtain from (3.18) that

$$\begin{aligned}(3.19)\quad 0 \leqslant\ & (d^2(x_{\tau(n)}, q) - d^2(x_{\tau(n)+1}, q)) \\ & + \theta_{\tau(n)}(1 - t_{\tau(n)})D + \alpha_{\tau(n)}[1 + \beta_{\tau(n)}(1 + u_{\tau(n)}M)]v_{\tau(n)} \\ & + t_{\tau(n)}[(1 + \theta_{\tau(n)})[t_{\tau(n)}d^2(0, q) + 2(1-t_{\tau(n)})\langle\overrightarrow{0q}, \overrightarrow{x_{\tau(n)}q}\rangle] - d^2(x_{\tau(n)}, q)].\end{aligned}$$

It follows from (3.19) that

$$\begin{aligned}d^2(x_{\tau(n)}, p) \leqslant\ & \frac{\theta_{\tau(n)}}{t_{\tau(n)}}(1 - t_{\tau(n)})D + \alpha_{\tau(n)}[1 + \beta_{\tau(n)}(1 + u_{\tau(n)}M)]\frac{v_{\tau(n)}}{t_{\tau(n)}} \\ & + [(1 + \theta_{\tau(n)})[t_{\tau(n)}d^2(0, q) + 2(1 - t_{\tau(n)})\langle\overrightarrow{0q}, \overrightarrow{x_{\tau(n)}q}\rangle]].\end{aligned}$$

Thus $\lim_{n\to\infty} \Gamma_{\tau(n)} = \lim_{n\to\infty} \Gamma_{\tau(n)+1}$. Furthermore, for $n \geqslant N_0$, we have $\Gamma_n \leqslant \Gamma_{\tau(n)+1}$ if $n \neq \tau(n)$ (i.e., $\tau(n) < n$), because $\Gamma_j > \Gamma_{j+1}$ for $\tau(n) \leqslant j \leqslant n$. It then follows that for all $n \geqslant N_0$, we have $0 \leqslant \Gamma_n \leqslant \max\{\Gamma_{\tau(n)}, \Gamma_{\tau(n)+1}\} = \Gamma_{\tau(n)+1}$. This implies $\lim_{n\to\infty} \Gamma_n = 0$, and hence $\{x_n\}$ converges strongly to $q \in F(T)$. This complete the proof. □

When $T$ is uniformly $L$-Lipschitzain and generalize asymptotically demicontractive mapping in Theorem 3.1, we obtain the following corollary.

Corollary 3.1. *Let $X$ be a complete* CAT(0) *space and $C$ be a nonempty closed convex subset of $X$. Let $T\colon C \to C$ be uniformly $L$-Lipschitzian and generalize asymptotically demicontractive mappings with $\{u_n\}, \{v_n\}$ satisfying $\sum_{n=1}^{\infty} u_n < \infty$ and $\sum_{n=1}^{\infty} v_n < \infty$, such that $T$ is $\Delta$-demiclosed at* 0 *and $F(T) \neq \emptyset$. Let $\{x_n\}_{n=1}^{\infty}$ be a sequence generated by $x_1 = x \in C$,*

$$\begin{aligned} z_n &= P_C((1 - t_n)x_n), \\ y_n &= (1 - \beta_n)z_n \oplus \beta_n T^n z_n, \\ x_{n+1} &= (1 - \alpha_n)z_n \oplus \alpha_n T^n y_n, \end{aligned}$$

*where $\{t_n\}_{n=1}^{\infty} \subset (0, 1)$, $\epsilon \leqslant k \leqslant \alpha_n \leqslant \beta_n \leqslant b$, $\forall n \geqslant 1$ for some $\epsilon > 0$, $k \in [0, 1)$ and $b \in (0, L^{-2}[\sqrt{1 + L^2} - 1])$ with $\lim_{n\to\infty} t_n = 0$, $\sum_{n=1}^{\infty} t_n = \infty$ and $\lim_{n\to\infty} \frac{u_n}{t_n} = 0$, $\lim n \to \infty \frac{v_n}{t_n} = 0$. Then $\{x_n\}_{n=1}^{\infty}$ converges strongly to $F(T)$.*

If $k = 0$ in Theorem 3.1, then $T$ becomes uniformly $L$-Lipschitzian and totally quasi nonexpansive mapping. Hence, we obtain the following corollary.

Corollary 3.2. *Let $X$ be a complete* CAT(0) *space and $C$ be a nonempty closed convex subset of $X$. Let $T\colon C \to C$ be uniformly $L$-Lipschitzian and total asymptotically quasi nonexpansive mappings with $\{u_n\}, \{v_n\}$ and mappings $\phi\colon [0, \infty)$*

$\to [0,\infty)$ *satisfying* $\sum_{n=1}^{\infty} u_n < \infty$ *and* $\sum_{n=1}^{\infty} v_n < \infty$*, such that* $T$ *is* $\Delta$*-demiclosed at* $0$ *and* $F(T) \neq \emptyset$*. Let* $\{x_n\}_{n=1}^{\infty}$ *be a sequence generated by* $x_1 = x \in C$*,*

$$z_n = P_C((1-t_n)x_n),$$
$$y_n = (1-\beta_n)z_n \oplus \beta_n T^n z_n,$$
$$x_{n+1} = (1-\alpha_n)z_n \oplus \alpha_n T^n y_n,$$

*where* $\{t_n\}_{n=1}^{\infty} \subset (0,1)$*,* $\epsilon \leqslant \alpha_n \leqslant \beta_n \leqslant b < 1$*,* $\forall n \geqslant 1$ *for some* $\epsilon > 0$ *with* $\lim_{n\to\infty} t_n = 0$*,* $\sum_{n=1}^{\infty} t_n = \infty$ *and* $\lim_{n\to\infty} \frac{u_n}{t_n} = 0, \lim n \to \infty \frac{v_n}{t_n} = 0$*. Assume there exist positive constants* $M^*$ *and* $M$*, such that* $\phi(r) \leqslant Mr^2, \forall r \geqslant M^*$*. Then* $\{x_n\}_{n=1}^{\infty}$ *converges strongly to* $F(T)$*.*

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