

GENERALIZATIONS OF CERTAIN WELL KNOWN INEQUALITIES FOR POLYNOMIALS

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ABSTRACT. We obtain a generalization of Bernstein's result that if $p(z)$ and $q(z)$ are two polynomials with degree of $p(z)$ not exceeding that of $q(z)$ and $q(z)$ has all its zeros in $|z| \leq 1$, with $|p(z)| \leq |q(z)|$, $|z| = 1$, then $|p'(z)| \leq |q'(z)|$, $|z| = 1$, and use the generalization so obtained to obtain two more generalizations. Three generalizations together turn out to be generalizations of many well known inequalities for polynomials, including Bernstein's inequality and inequality of the well known Erdős–Lax theorem.

1. Introduction and statement of results

For a polynomial $p(z)$ of degree n we have the following well known result, known as Bernstein's Theorem, (with corresponding inequality being known as Bernstein's inequality) [10, 12].

THEOREM A. *If $p(z)$ is a polynomial of degree n and $\max_{|z|=1} |p(z)| = 1$ then*

$$(1.1) \quad |p'(z)| \leq n, \quad |z| = 1.$$

The result is the best possible and equality holds in (1.1) for $p(z) = \alpha z^n$ with $|\alpha| = 1$.

In 1930 Bernstein [3] proved the following interesting result, thereby suggesting a generalization of Theorem A.

THEOREM B. *Let $p(z)$ and $q(z)$ be two polynomials with degree of $p(z)$ not exceeding that of $q(z)$. If $q(z)$ has all its zeros in $|z| \leq 1$ and $|p(z)| \leq |q(z)|$, $|z| = 1$ then*

$$|p'(z)| \leq |q'(z)|, \quad |z| = 1.$$

In this paper we have firstly obtained the following generalization of Theorem B.

THEOREM 1.1. *Let $q(z)$ be a polynomial of degree n having all its zeros in $|z| \leq 1$ and $p(z)$ be a polynomial of degree not exceeding that of $q(z)$. If*

$$(1.2) \quad |p(z)| \leq |q(z)|, \quad |z| = 1$$

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then for $R \geq 1$, $0 \leq s \leq n$ and $|\beta| \leq 1$

$$\begin{aligned} & |z^s p^{(s)}(Rz) + \beta(R+1)^{n-s} \frac{s!C(n,s)}{2^n} p(z)| \\ & \leq |z^s q^{(s)}(Rz) + \beta(R+1)^{n-s} \frac{s!C(n,s)}{2^n} q(z)|, \quad |z| = 1. \end{aligned}$$

REMARK 1.1. Theorem 1.1 is also a generalization of the results [6, Lemma 2] and [9, Theorem 1].

By taking $q(z) = z^n$ in Theorem 1.1 we obtain

COROLLARY 1.1. If $p(z)$ is a polynomial of degree at most n and $\max_{|z|=1} |p(z)| \leq 1$, then for $R \geq 1$, $0 \leq s \leq n$ and $|\beta| \leq 1$

$$(1.3) \quad \left| z^s p^{(s)}(Rz) + \beta(R+1)^{n-s} \frac{s!C(n,s)}{2^n} p(z) \right| \leq s!C(n,s) \left| R^{n-s} + \beta \frac{(R+1)^{n-s}}{2^n} \right|, \quad |z| = 1.$$

Equality holds in (1.3) for $p(z) = z^n$,

thereby suggesting a generalization of Theorem A and the inequality [11, Part III, Chapter 6, Problem no. 269 with $r_2 = R$, $r_1 = 1$, $f(z) = p(z)$ and $\max_{|z|=1} |p(z)| = 1$], (for a polynomial $p(z)$ of degree n) $\max_{|z|=R} |p(z)| \leq R^n$, $R \geq 1$.

REMARK 1.2. Corollary 1.1 is also a generalization of the results ([6, Remark 2], [9, Corollary 2] and [7, Lemma 2]).

Secondly we have used Theorem 1.1 to obtain a generalization of the inequality [5, ineq. (3.2) with $M = 1$, $s = 1$ and $|z| = 1$], (for a polynomial $p(z)$ of degree n with $\max_{|z|=1} |p(z)| \leq 1$ and $q(z) = z^n p(1/\bar{z})$)

$$|p'(z)| + |q'(z)| \leq n, \quad |z| = 1$$

and the inequality [2, Lemma with $P(z) = p(z)$ and $Q(z) = q(z)$], (for a polynomial $p(z)$ of degree n with $q(z) = z^n p(1/\bar{z})$ and $\max_{|z|=1} |p(z)| = 1$)

$$|p(Re^{i\theta})| + |q(Re^{i\theta})| \leq R^n + 1, \quad R \geq 1 \text{ and } 0 \leq \theta \leq 2\pi.$$

More precisely we have proved

THEOREM 1.2. Let $p(z)$ be a polynomial of degree at most n with $\max_{|z|=1} |p(z)| \leq 1$. Further let $q(z) = z^n p(1/\bar{z})$. Then for $R \geq 1$, $0 \leq s \leq n$ and $|\beta| \leq 1$

$$(1.4) \quad \begin{aligned} & \left| z^s p^{(s)}(Rz) + \beta(R+1)^{n-s} \frac{s!C(n,s)}{2^n} p(z) \right| + \left| z^s q^{(s)}(Rz) \right. \\ & \quad \left. + \beta(R+1)^{n-s} \frac{s!C(n,s)}{2^n} q(z) \right| \\ & \leq s!C(n,s) \left\{ \left| \frac{d^s(1)}{dz^s} + \beta \frac{(R+1)^{n-s}}{2^n} \right| + \left| R^{n-s} + \beta \frac{(R+1)^{n-s}}{2^n} \right| \right\}, \quad |z| = 1. \end{aligned}$$

Equality holds in (1.4) for the polynomial $p(z) = z^n$.

REMARK 1.3. Theorem 1.2 is also a generalization of the results ([6, Theorem 1] and [7, Lemma 3]).

Lastly we have used both Theorem 1.1 and Theorem 1.2 to obtain the following generalization of well known Erdős–Lax theorem [8], (for a polynomial $p(z)$ of degree n , with $\max_{|z|=1} |p(z)| \leq 1$ and no zeros in $|z| < 1$) $|p'(z)| \leq \frac{n}{2}$, $|z| = 1$ and the inequality [1, Theorem 1], (for a polynomial $p(z)$ of degree n with $\max_{|z|=1} |p(z)| = 1$ and no zeros in $|z| < 1$)

$$\max_{|z|=R} |p(z)| \leq \frac{R^n + 1}{2}, \quad R \geq 1.$$

THEOREM 1.3. *Let $p(z)$ be a polynomial of degree at most n , having no zeros in $|z| < 1$, with $\max_{|z|=1} |p(z)| \leq 1$. Then for $R \geq 1$, $0 \leq s \leq n$ and $|\beta| \leq 1$*

$$(1.5) \quad \left| z^s p^{(s)}(Rz) + \beta(R+1)^{n-s} \frac{s!C(n,s)}{2^n} p(z) \right| \\ \leq \frac{s!C(n,s)}{2} \left\{ \left| \frac{d^s(1)}{dz^s} + \beta \frac{(R+1)^{n-s}}{2^n} \right| + \left| R^{n-s} + \beta \frac{(R+1)^{n-s}}{2^n} \right| \right\}, \quad |z| = 1.$$

Equality holds in (1.5) for $p(z) = \frac{1}{2}(z^n + 1)$.

REMARK 1.4. Theorem 1.3 is also a generalizaion of the result [7, Theorem 1].

2. Lemmas

For the proofs of the theorems we require the following lemmas.

LEMMA 2.1. *Let $p(z) = c \prod_{k=1}^n (z - z_k)$ be a polynomial of degree n , having all its zeros in $|z| \leq 1$. Then for $|z| = 1$*

$$|z p'(z)| \geq \left(\sum_{k=1}^n \frac{1}{1 + |z_k|} \right) |p(z)|.$$

This lemma is due to Giroux et al. [4, Proof of Theorem 5].

LEMMA 2.2. *Let $p(z)$ be a polynomial of degree n , having all its zeros in $|z| \leq 1$. Further let $1 \leq s \leq n$ and $p^{(j)}(z) = c_j \prod_{t_j=1}^{n-j} (z - z_{t_j})$, $j = 0, 1, \dots, s - 1$ with $p^{(0)}(z) = p(z)$. Then for $|z| = 1$*

$$|z^s p^{(s)}(z)| \geq \left\{ \left(\sum_{t_{s-1}=1}^{n-(s-1)} \frac{1}{1 + |z_{t_{s-1}}|} \right) \cdots \left(\sum_{t_0=1}^n \frac{1}{1 + |z_{t_0}|} \right) \right\} |p(z)|.$$

PROOF. By Lucas' theorem various derivatives of $p(z)$ will have all their zeros in $|z| \leq 1$. Therefore Lemma 2.2 follows by repeated applications of Lemma 2.1. \square

From Lemma 2.2 we easily obtain

LEMMA 2.3. *Let $p(z)$ be a polynomial of degree n , having all its zeros in $|z| \leq 1$. Then for $1 \leq s \leq n$*

$$|z^s p^{(s)}(z)| \geq \frac{n(n-1)\dots(n-s+1)}{2^s} |p(z)|, \quad |z| = 1.$$

REMARK 2.1. Lemma 2.3 is a generalization of the result [9, Lemma 2.1].

LEMMA 2.4. *If $p(z)$ is a polynomial of degree n having all its zeros in $|z| \leq 1$ then*

$$|p(Re^{i\theta})| \geq \left(\frac{R+1}{2}\right)^n |p(e^{i\theta})|, \quad R > 1 \text{ and } 0 \leq \theta \leq 2\pi.$$

This lemma is due to Jain [6].

By applying Lemma 2.4 to the polynomial $p^{(s)}(z)$, ($1 \leq s < n$), we obtain

LEMMA 2.5. *If $p(z)$ is a polynomial of degree n having all its zeros in $|z| \leq 1$ then for $1 \leq s \leq n$*

$$|p^{(s)}(Rz)| \geq \left(\frac{R+1}{2}\right)^{n-s} |p^{(s)}(z)|, \quad R \geq 1 \text{ and } |z| = 1.$$

On combining Lemma 2.5 and Lemma 2.3 we obtain

LEMMA 2.6. *If $p(z)$ is a polynomial of degree n having all its zeros in $|z| \leq 1$ then for $1 \leq s \leq n$*

$$|z^s p^{(s)}(Rz)| \geq (R+1)^{n-s} \frac{n(n-1)\dots(n-s+1)}{2^n} |p(z)|, \quad R \geq 1 \text{ and } |z| = 1$$

or equivalently

$$|p^{(s)}(Rz)| \geq (R+1)^{n-s} \frac{n(n-1)\dots(n-s+1)}{2^n} |p(z)|, \quad R \geq 1 \text{ and } |z| = 1.$$

Lemma 2.6, along with Lemma 2.4 leads to

LEMMA 2.7. *If $p(z)$ is a polynomial of degree n having all its zeros in $|z| \leq 1$ then for $0 \leq s \leq n$*

$$(2.1) \quad |z^s p^{(s)}(Rz)| \geq (R+1)^{n-s} \frac{s!C(n,s)}{2^n} |p(z)|, \quad R \geq 1 \text{ and } |z| = 1$$

or equivalently

$$(2.2) \quad |p^{(s)}(Rz)| \geq (R+1)^{n-s} \frac{s!C(n,s)}{2^n} |p(z)|, \quad R \geq 1 \text{ and } |z| = 1.$$

3. Proofs of the theorems

PROOF OF THEOREM 1.1. By using (1.2) and Rouché's theorem we can say that for $|\alpha| > 1$, $p(z) + \alpha q(z)$ is a polynomial of degree n , having all its zeros in $|z| \leq 1$. Therefore by Lemma 2.7 (ineq. (2.1)) we have

$$(3.1) \quad |z^s p^{(s)}(Rz) + \alpha z^s q^{(s)}(Rz)| \\ \geq (R+1)^{n-s} \frac{s!C(n,s)}{2^n} |p(z) + \alpha q(z)|, \quad R \geq 1 \text{ and } |z| = 1,$$

which helps us to write for $|\alpha| > 1$ and $|\beta| < 1$

$$(3.2) \quad z^s p^{(s)}(Rz) + \alpha z^s q^{(s)}(Rz) + \beta (R+1)^{n-s} \frac{s!C(n,s)}{2^n} \{p(z) + \alpha q(z)\} \neq 0 \\ \text{on } S = \{z : |z| = 1 \text{ and } q(z) \neq 0\}.$$

(On the contrary if

$$\xi^s p^{(s)}(R\xi) + \alpha \xi^s q^{(s)}(R\xi) + \beta(R+1)^{n-s} \frac{s!C(n,s)}{2^n} \{p(\xi) + \alpha q(\xi)\} = 0$$

for certain $\xi \in S$ then

$$|\xi^s p^{(s)}(R\xi) + \alpha \xi^s q^{(s)}(R\xi)| < (R+1)^{n-s} \frac{s!C(n,s)}{2^n} |p(\xi) + \alpha q(\xi)|,$$

contradicting the fact represented by (3.1). Hence we have for $|\beta| < 1$

$$\left| z^s p^{(s)}(Rz) + \beta(R+1)^{n-s} \frac{s!C(n,s)}{2^n} p(z) \right| \leq \left| z^s q^{(s)}(Rz) + \beta(R+1)^{n-s} \frac{s!C(n,s)}{2^n} q(z) \right|,$$

on S .

(On the contrary let

$$(3.3) \quad \left| \eta^s p^{(s)}(R\eta) + \beta(R+1)^{n-s} \frac{s!C(n,s)}{2^n} p(\eta) \right| > \left| \eta^s q^{(s)}(R\eta) + \beta(R+1)^{n-s} \frac{s!C(n,s)}{2^n} q(\eta) \right|$$

for certain $\eta \in S$. As for $|\beta| < 1$

$$\eta^s q^{(s)}(R\eta) + \beta(R+1)^{n-s} \frac{s!C(n,s)}{2^n} q(\eta) \neq 0,$$

similar to (3.2), we can take

$$\alpha_0 = - \frac{\eta^s p^{(s)}(R\eta) + \beta(R+1)^{n-s} \frac{s!C(n,s)}{2^n} p(\eta)}{\eta^s q^{(s)}(R\eta) + \beta(R+1)^{n-s} \frac{s!C(n,s)}{2^n} q(\eta)},$$

with $|\alpha_0| > 1$, by (3.3), thereby giving

$$\begin{aligned} \eta^s p^{(s)}(R\eta) + \beta(R+1)^{n-s} \frac{s!C(n,s)}{2^n} p(\eta) \\ + \alpha_0 (\eta^s q^{(s)}(R\eta) + \beta(R+1)^{n-s} \frac{s!C(n,s)}{2^n} q(\eta)) = 0, \end{aligned}$$

which contradicts the fact represented by (3.2). Now on using continuity with respect to β and z , Theorem 1.1 follows. \square

PROOF OF THEOREM 1.2. For $|\alpha| > 1$ the polynomial $P(z) = p(z) - \alpha$ is of degree at most n and has all its zeros in $|z| > 1$. Therefore the polynomial

$$Q(z) = z^n \overline{P(1/\bar{z})} = q(z) - \bar{\alpha} z^n$$

is of degree n and has all its zeros in $|z| < 1$, with $|P(z)| = |Q(z)|$, $|z| = 1$. Now on applying Theorem 1.1 we get for $R \geq 1$, $0 \leq s \leq n$ and $|\beta| \leq 1$

$$\begin{aligned} & \left| z^s p^{(s)}(Rz) + \beta(R+1)^{n-s} \frac{s!C(n,s)}{2^n} p(z) - \alpha s!C(n,s) \left\{ \frac{d^s(1)}{dz^s} + \beta \frac{(R+1)^{n-s}}{2^n} \right\} \right| \\ & \leq \left| z^s q^{(s)}(Rz) + \beta(R+1)^{n-s} \frac{s!C(n,s)}{2^n} q(z) - \bar{\alpha} C(n,s) s! z^n \left\{ R^{n-s} + \beta \frac{(R+1)^{n-s}}{2^n} \right\} \right|, \\ & \qquad \qquad \qquad |z| = 1, \end{aligned}$$

which for an appropriate choice of argument of α , implies

$$(3.4) \quad \left| z^s p^{(s)}(Rz) + \beta(R+1)^{n-s} \frac{s!C(n,s)}{2^n} p(z) \right| - |\alpha| s! C(n,s) \left| \frac{d^s(1)}{dz^s} + \beta \frac{(R+1)^{n-s}}{2^n} \right| \leq \left| \alpha |C(n,s)s! \left| R^{n-s} + \beta \frac{(R+1)^{n-s}}{2^n} \right| - \left| z^s q^{(s)}(Rz) + \beta(R+1)^{n-s} \frac{s!C(n,s)}{2^n} q(z) \right| \right|, \quad |z| = 1.$$

Finally on using Corollary 1.1 on the right hand side of (3.4) and then making $|\alpha| \rightarrow 1$, Theorem 1.2 follows. \square

PROOF OF THEOREM 1.3. Well $q(z) = z^n \overline{p(1/\bar{z})}$ is a polynomial of degree n , having all its zeros in $|z| \leq 1$, with $|p(z)| = |q(z)|$, $|z| = 1$. Therefore on combining Theorem 1.1 and Theorem 1.2, Theorem 1.3 follows. \square

REMARK 3.1. If we use Lemma 2.7 (2.2) instead of Lemma 2.7 (2.1), we can get the following results, similar to Theorem 1.1, Corollary 1.1, Theorem 1.2 and Theorem 1.3, in a similar manner.

THEOREM 3.1. *Under the same hypotheses as in Theorem 1.1, for $R \geq 1$, $0 \leq s \leq n$ and $|\beta| \leq 1$*

$$\left| p^{(s)}(Rz) + \beta(R+1)^{n-s} \frac{s!C(n,s)}{2^n} p(z) \right| \leq \left| q^{(s)}(Rz) + \beta(R+1)^{n-s} \frac{s!C(n,s)}{2^n} q(z) \right|, \quad |z| = 1.$$

COROLLARY 3.1. *Under the same hypotheses as in Corollary 1.1, for $R \geq 1$, $0 \leq s \leq n$ and $|\beta| \leq 1$*

$$\left| p^{(s)}(Rz) + \beta(R+1)^{n-s} \frac{s!C(n,s)}{2^n} p(z) \right| \leq s! C(n,s) \left| R^{n-s} + \beta \frac{(R+1)^{n-s}}{2^n} z^s \right|, \quad |z| = 1.$$

THEOREM 3.2. *Under the same hypotheses as in Theorem 1.2, for $R \geq 1$, $0 \leq s \leq n$ and $|\beta| \leq 1$*

$$\left| p^{(s)}(Rz) + \beta(R+1)^{n-s} \frac{s!C(n,s)}{2^n} p(z) \right| + \left| q^{(s)}(Rz) + \beta(R+1)^{n-s} \frac{s!C(n,s)}{2^n} q(z) \right| \leq s! C(n,s) \left\{ \left| \frac{d^s(1)}{dz^s} + \beta \frac{(R+1)^{n-s}}{2^n} \right| + \left| R^{n-s} + \beta \frac{(R+1)^{n-s}}{2^n} z^s \right| \right\}, \quad |z| = 1.$$

THEOREM 3.3. *Under the same hypotheses as in Theorem 1.3, for $R \geq 1$, $0 \leq s \leq n$ and $|\beta| \leq 1$*

$$\left| p^{(s)}(Rz) + \beta(R+1)^{n-s} \frac{s!C(n,s)}{2^n} p(z) \right| \leq \frac{s!C(n,s)}{2} \left\{ \left| \frac{d^s(1)}{dz^s} + \beta \frac{(R+1)^{n-s}}{2^n} \right| + \left| R^{n-s} + \beta \frac{(R+1)^{n-s}}{2^n} z^s \right| \right\}, \quad |z| = 1.$$

It may be further added that Theorem 3.1, like Theorem 1.1 is also a generalization of Theorem B, Corollary 3.1, as Corollary 1.1 is also a generalization of Theorem A, Theorem 3.2, as Theorem 1.2 is also a generalization of the inequality [5, ineq. (3.2) with $M = 1, s = 1$ and $|z| = 1$] and Theorem 3.3, as Theorem 1.3 is also a generalization of the well known Erdős–Lax theorem [8].

References

1. N. C. Ankeny, T. J. Rivlin, *On a theorem of S. Bernstein*, Pac. J. Math. **5** (1955), 849–852.
2. A. Aziz, Q. G. Mohammad, *Growth of polynomials with zeros outside a circle*, Proc. Am. Math. Soc. **81** (1981), 549–553.
3. S. N. Bernstein, *Sur la limitation des dérivées des polynômes*, C.R., Math., Acad. Sci. Paris **190** (1930), 338–340.
4. A. Giroux, Q. I. Rahman, G. Schmeisser, *On Bernstein's inequality*, Can. J. Math. **31** (1979), 347–353.
5. N. K. Govil, Q. I. Rahman, *Functions of exponential type not vanishing in a half plane and related polynomials*, Trans. Am. Math. Soc. **137** (1969), 501–517.
6. V. K. Jain, *On maximum modulus of polynomials*, Indian J. Pure Appl. Math. **23**(11) (1992), 815–819.
7. V. K. Jain, *Generalization of certain well known inequalities for polynomials*, Glas. Mat., III. Ser. **32**(52) (1997), 45–51.
8. P. D. Lax, *Proof of a conjecture of P. Erdős on the derivative of a polynomial*, Bull. Am. Math. Soc. **50** (1944), 509–513.
9. M. A. Malik, M. C. Vong, *Inequalities concerning the derivative of polynomials*, Rend. Circ. Mat. Palermo (2) **34** (1985), 422–426.
10. G. V. Milovanović, D. S. Mitrinović, Th. Rassias, *Topics in polynomials: Extremal properties, inequalities, zeros*, World Scientific, 1994.
11. G. Polya, G. Szegő, *Problems and Theorems in analysis*, Vol. 1, Springer-Verlag, Berlin–Heidelberg, 1972.
12. A. C. Schaeffer, *Inequalities of A. Markoff and S. N. Bernstein for polynomials and related functions*, Bull. Am. Math. Soc. **47** (1941), 565–579.

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