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SOME INEQUALITIES FOR SELF-INVERSIVE RATIONAL FUNCTIONS WITH PRESCRIBED POLES

Abdullah Mir

Abstract. We establish some inequalities for self-inversive rational functions with prescribed poles in the sup-norm on the unit circle in the complex plane. Generalizations of polynomial inequalities of Malik and O'Hara and Rodriguez are obtained for such rational functions.

1. Introduction

Let $\mathbb{P}_n$ denote the class of all complex algebraic polynomials $P(z)$ of degree at most n and $P'(z)$ is the derivative of $P(z)$. For $a_j \in \mathbb{C}$ with $j = 1, 2, \ldots, n$, let

$$W(z) := \prod_{j=1}^{n} (z - a_j)$$

and let

$$B(z) := \prod_{j=1}^{n} \Big(\frac{1 - \bar{a}_j z}{z - a_j} \Big), \quad \mathbb{R}_n := \mathbb{R}_n(a_1, a_2, \ldots, a_n) = \Big\{ \frac{P(z)}{W(z)} : \ P \in \mathbb{P}_n \Big\}.$$

Then $\mathbb{R}_n$ is the set of rational functions with poles $a_1, a_2, \ldots, a_n$ at most and with finite limit at ∞. Note that $B(z) \in \mathbb{R}_n$ and $|B(z)| = 1$ for $|z| = 1$.

1.1. Definition.

(i) For $P \in \mathbb{P}_n$, the conjugate transpose P^* of P is defined as $P^*(z) = z^n \overline{P\big(\frac{1}{\bar{z}}\big)}$.

(ii) For $r(z) = \frac{P(z)}{W(z)} \in \mathbb{R}_n$, the conjugate transpose r^* of r is defined as $r^*(z) = B(z)\overline{r\big(\frac{1}{\bar{z}}\big)}$.

(iii) The polynomial $P \in \mathbb{P}_n$ is called self-inversive, if $P^*(z) = \zeta P(z)$ for some $|\zeta| = 1$.

(iv) The rational function $r \in \mathbb{R}_n$ is called self-inversive, if $r^*(z) = \zeta r(z)$ for some $|\zeta| = 1$.

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Note that if $r \in \mathbb{R}_n$ and $r(z) = \frac{P(z)}{W(z)}$, then $r^*(z) = \frac{P^*(z)}{W(z)}$ and hence $r^* \in \mathbb{R}_n$. So $r(z) = \frac{P(z)}{W(z)}$ is self-inversive if and only if $P(z)$ is self-inversive.

If $P \in \mathbb{P}_n$, then concerning the estimate of $|P'(z)|$ on $|z| = 1$, we have

$$|P'(z)| \leqslant n \sup_{|z|=1} |P(z)|. \tag{1.1}$$

Inequality (1.1) is a famous result due to Bernstein [**1**], who proved it in 1912. Later, in 1969 (see [**9**]), Malik improved (1.1) and established that if $P \in \mathbb{P}_n$, then for $|z| = 1$, we have

$$|P'(z)| + |Q'(z)| \leqslant n \sup_{|z|=1} |P(z)|, \tag{1.2}$$

where $Q(z) = z^n \overline{P(\frac{1}{\bar{z}})}$.

As an application of (1.2), it follows that for $P \in \mathbb{P}_n$ to be self-inversive, then for $|z| = 1$, we have

$$|P'(z)| \leqslant \frac{n}{2} \sup_{|z|=1} |P(z)|. \tag{1.3}$$

In 1974, a stronger conclusion was given by O'Hara and Rodriguez [**5**](see also Saff and Shiel-Small [**12**]) who proved that if $P \in \mathbb{P}_n$ is self-inversive, then

$$|P'(z)| = \frac{n}{2} \sup_{|z|=1} |P(z)|. \tag{1.4}$$

Although the literature on polynomial inequalities is vast and growing, the interested readers can consult the books of Milovanović et al. [**10**] and Borwein and Erdélyi [**2**]. In the past few years, several papers pertaining to Bernstein-type inequalities for rational functions have appeared in the study of rational approximation problems. In fact in 1995, Li, Mohapatra and Rodriguez [**8**] proved some inequalities similar to (1.1) and (1.2) for rational functions with poles outside the unit circle. They extended Bernstein's inequality (1.1) to rational functions by proving that if $r \in \mathbb{R}_n$, then for $|z| = 1$,

$$|r'(z)| \leqslant |B'(z)| \sup_{|z|=1} |r(z)|. \tag{1.5}$$

As an improvement of (1.5) and an extension of (1.2), they also proved that if $r \in \mathbb{R}_n$ and $|z| = 1$, then

$$|r'(z)| + |(r^*(z))'| \leqslant |B'(z)| \sup_{|z|=1} |r(z)|. \tag{1.6}$$

From (1.6), they obtained the following extension of (1.3) to self-inversive rational functions.

THEOREM 1.1. *If $r \in \mathbb{R}_n$ is self-inversive and $|z| = 1$, then*

$$|r'(z)| \leqslant \frac{|B'(z)|}{2} \sup_{|z|=1} |r(z)|. \tag{1.7}$$

These inequalities have their own significance and importance in Approximation Theory and the latest development of further results along this line can be found in the papers [**3**, **4**, **6**, **11**]. Our main aim is to obtain some inequalities for self-inversive rational functions with poles outside the unit circle as considered by Li, Mohapatra and Rodriguez [**8**].

2. Lemmas

In this section, we provide the following lemmas that are used in the later sections for proving our main results.

LEMMA 2.1. *If $r \in \mathbb{R}_n$ has n zeros all lie in $|z| \leqslant 1$, then*

$$|r'(z)| \geqslant \frac{1}{2}|B'(z)||r(z)| \quad \textit{for } |z| = 1. \tag{2.1}$$

Equality holds in (2.1) for $r(z) = \mu B(z) + \zeta$ with $|\mu| = |\zeta| = 1$.
The above lemma is due to Li, Mohapatra and Rodriguez [**8**].

LEMMA 2.2. *Let A and B be any two complex numbers. Then*

(i) *if $|A| \geqslant |B|$ and $B \neq 0$, then $A \neq \delta B$ for all complex numbers δ satisfying $|\delta| < 1$.*
(ii) *Conversely, if $A \neq \delta B$ for all complex numbers δ satisfying $|\delta| < 1$ then $|A| \geqslant |B|$.*

The above lemma is due to Li [**7**].

LEMMA 2.3. *If r, $s \in R_n$ and all the zeros of $s(z)$ lie in $|z| \leqslant 1$ and $|r(z)| \leqslant |s(z)|$ for $|z| = 1$. Then for every $\beta \in \mathbb{C}$ with $|\beta| \leqslant 1$ and $|z| = 1$, we have*

$$\Big|B(z)r'(z) + \frac{\beta}{2}B'(z)r(z)\Big| \leqslant \Big|B(z)s'(z) + \frac{\beta}{2}B'(z)s(z)\Big|. \tag{2.2}$$

The result is sharp and equality holds in (2.2) for $r(z) = \mu s(z)$ with $|\mu| = 1$.

PROOF. First assume that no zeros of $s(z)$ are on the unit circle $|z| = 1$ and therefore that all the zeros of $s(z)$ are in $|z| < 1$. By Rouche's Theorem, the rational function $\lambda r(z) + s(z)$ has all its zeros in $|z| < 1$ for $|\lambda| < 1$ and has no poles in $|z| \leqslant 1$. On applying Lemma 2.1 to $\lambda r(z) + s(z)$, we get on $|z| = 1$,

$$2|B(z)||\lambda r'(z) + s'(z)| \geqslant |B'(z)||\lambda r(z) + s(z)|. \tag{2.3}$$

Now, note that $B'(z) \neq 0$ (e.g, see [**8**, formula (14)]). So, the right-hand side of (2.3) is nonzero. Thus, by using (i) of Lemma 2.2, we have for all $\beta \in \mathbb{C}$ with $|\beta| < 1$,

$$2B(z)\big(\lambda r'(z) + s'(z)\big) \neq -\beta B'(z)\big(\lambda r(z) + s(z)\big) \quad \text{for } |z| = 1.$$

Equivalently, for $|z| = 1$,

$$\lambda(2B(z)r'(z) + \beta B'(z)r(z)) \neq -(2B(z)s'(z) + \beta B'(z)s(z)) \qquad \text{for } |z| = 1,\ |\lambda| < 1 \text{ and } |\beta| < 1.$$

Now using (ii) of Lemma 2.2, we have

$$|2B(z)r'(z)+\beta B'(z)r(z)| \leqslant |2B(z)s'(z)+\beta B'(z)s(z)| \quad \text{for } |z|=1 \text{ and } |\beta|<1. \tag{2.4}$$

Now using the continuity in the zeros and β, we can obtain inequality (2.4), when some zeros of $s(z)$ lie on the unit circle $|z|=1$ and $|\beta| \leqslant 1$. □

Applying Lemma 2.3 to the rational functions $r(z)$ and $B(z)\sup_{|z|=1}|r(z)|$, we get the following:

LEMMA 2.4. *If $r \in R_n$, then for all $\beta \in \mathbb{C}$ with $|\beta| \leqslant 1$ and $|z|=1$, we have*

$$\Big|B(z)r'(z)+\frac{\beta}{2}B'(z)r(z)\Big| \leqslant |B'(z)|\Big|1+\frac{\beta}{2}\Big| \sup_{|z|=1}|r(z)|. \tag{2.5}$$

Equality holds in (2.5) for $r(z)=\mu s(z)$ with $|\mu|=1$.

3. Main Results and Proofs

From now on, we shall always assume that all the poles $a_1, a_2, \ldots, a_n$ lie in $|z|>1$. Our first result that is presented below provides a generalization of (1.6).

THEOREM 3.1. *If $r \in \mathbb{R}_n$ and $|z|=1$, then for every β with $|\beta| \leqslant 1$,*

$$\Big|B(z)r'(z)+\frac{\beta}{2}B'(z)r(z)\Big| + \Big|B(z)(r^*(z))'+\frac{\beta}{2}B'(z)r^*(z)\Big| \leqslant |B'(z)|\Big\{\Big|1+\frac{\beta}{2}\Big|+\Big|\frac{\beta}{2}\Big|\Big\} \sup_{|z|=1}|r(z)|. \tag{3.1}$$

PROOF. Let $M := \sup_{|z|=1}|r(z)|$. Therefore, for every λ with $|\lambda|>1$, $|r(z)| < |\lambda M B(z)|$ for $|z|=1$. By Rouche's theorem, all the zeros of $G(z)=r(z)+\lambda M B(z)$ lie in $|z|<1$. If $H(z)=B(z)\overline{G(\frac{1}{\bar{z}})}$, then $|H(z)|=|G(z)|$ for $|z|=1$ and hence for any γ with $|\gamma|<1$, the rational function $\gamma H(z)+G(z)$ has all zeros in $|z|<1$. By applying Lemma 2.1 to $\gamma H(z)+G(z)$, we have

$$2|B(z)(\gamma H'(z)+G'(z))| \geqslant |B'(z)||\gamma H(z)+G(z)| \quad \text{for } |z|=1. \tag{3.2}$$

Since $B'(z)\neq 0$, so the right-hand side of (3.2) is nonzero. Thus, by using (i) of Lemma 2.2, we have for all $\beta \in \mathbb{C}$ with $|\beta|<1$,

$$2B(z)(\gamma H'(z)+G'(z)) \neq -\beta B'(z)(\gamma H(z)+G(z)) \quad \text{for } |z|=1.$$

Equivalently, for $|z|=1$,

$$-\gamma(2B(z)H'(z)+\beta B'(z)H(z)) \neq (2B(z)G'(z)+\beta B'(z)G(z)) \quad \text{for } |\gamma|<1 \text{ and } |\beta|<1. \tag{3.3}$$

Using (ii) of Lemma 2.2 in (3.3), we have

$$|2B(z)G'(z)+\beta B'(z)G(z)| \geqslant |2B(z)H'(z)+\beta B'(z)H(z)| \quad \text{for } |z|=1 \text{ and } |\beta|<1. \tag{3.4}$$

Now by putting $G(z) = r(z) + \lambda M B(z)$ and $H(z) = r^*(z) + \bar{\lambda} M$ in (3.4), we get for $|z| = 1$ and $|\beta| < 1$,

$$|2B(z)(r^*(z))' + \beta B'(z)r^*(z) + \bar{\lambda}\beta M B'(z)| \leqslant |2B(z)r'(z) + \beta B'(z)r(z) + \lambda B(z)B'(z)(2+\beta)M|. \tag{3.5}$$

By choosing a suitable argument of λ and applying Lemma 2.4 on the right hand side of (3.5), we get for $|z| = 1$ and $|\beta| < 1$,

$$|2B(z)(r^*(z))' + \beta B'(z)r^*(z)| - |\lambda||\beta B'(z)|M \leqslant |\lambda||B(z)B'(z)(2+\beta)|M - |2B(z)r'(z) + \beta B'(z)r(z)|. \tag{3.6}$$

Note that $|B(z)| = 1$ for $|z| = 1$. Making $|\lambda| \to 1$ and using continuity for $|\beta| = 1$ in (3.6), we get the desired result. □

Remark 3.1. For $\beta = 0$, (3.1) reduces to (1.6).

By using Theorem 3.1, we prove the following generalization of Theorem 1.1.

Theorem 3.2. *If $r \in \mathbb{R}_n$ is self-inversive, then for every β with $|\beta| \leqslant 1$ and $|z| = 1$, we have*

$$\Big|B(z)r'(z) + \frac{\beta}{2}B'(z)r(z)\Big| \leqslant \frac{|B'(z)|}{2}\Big\{\Big|1 + \frac{\beta}{2}\Big| + \Big|\frac{\beta}{2}\Big|\Big\} \sup_{|z|=1} |r(z)|. \tag{3.7}$$

Proof. Since $r(z)$ is self-inversive, therefore, we have $r^*(z) = \lambda r(z)$ with $|\lambda| = 1$. Hence for all $\beta \in \mathbb{C}$,

$$\Big|B(z)r'(z) + \frac{\beta}{2}B'(z)r(z)\Big| = \Big|B(z)(r^*(z))' + \frac{\beta}{2}B'(z)r^*(z)\Big|. \tag{3.8}$$

Combining Theorem 3.1 and equation (3.8), we have for every β with $|\beta| \leqslant 1$ and $|z| = 1$,

$$\begin{aligned} 2\Big|B(z)r'(z) + \frac{\beta}{2}B'(z)r(z)\Big| &= \Big|B'(z)r'(z) + \frac{\beta}{2}B'(z)r(z)\Big| + \Big|B(z)(r^*(z))' + \frac{\beta}{2}B'(z)r^*(z)\Big| \\ &\leqslant |B'(z)|\Big\{\Big|1 + \frac{\beta}{2}\Big| + \Big|\frac{\beta}{2}\Big|\Big\} \sup_{|z|=1} |r(z)|, \end{aligned}$$

which proves Theorem 3.2 completely. □

Remark 3.2. For $\beta = 0$, (3.7) reduces to (1.7).

Finally, we present the following extension of (1.4) to rational functions with prescribed poles.

Theorem 3.3. *If $r \in \mathbb{R}_n$ is self-inversive, then for every $|\beta| \leqslant 1$, we have*

$$\frac{1}{2}(1 - |\beta|) \sup_{|z|=1} |r(z)| \leqslant \sup_{|z|=1} \Big|\frac{r'(z)}{B'(z)} + \frac{\beta}{2}\frac{r(z)}{B(z)}\Big| \leqslant \frac{1}{2}(1 + |\beta|) \sup_{|z|=1} |r(z)|. \tag{3.9}$$

Proof. Since $r \in \mathbb{R}_n$ is self-inversive, therefore, we have $r^*(z) = \zeta r(z)$ with $|\zeta| = 1$, where $r^*(z) = B(z)\overline{r\big(\frac{1}{\bar{z}}\big)}$. Now

$$z(r^*(z))' = zB'(z)\overline{r\Big(\frac{1}{\bar{z}}\Big)} - \frac{B(z)}{z}\overline{r'\Big(\frac{1}{\bar{z}}\Big)},$$

which gives for $|z| = 1$,

$$|(r^*(z))'| = |zB'(z)\overline{r(z)} - B(z)\overline{zr'(z)}| = |B(z)|\Big|\frac{zB'(z)}{B(z)}\overline{r(z)} - \overline{zr'(z)}\Big|. \tag{3.10}$$

Since [**8**, Lemma 1] the quantity $\frac{zB'(z)}{B(z)}$ is real, it follows from (3.10) that for $|z| = 1$,

$$|(r^*(z))'| = |B(z)|\Big|\frac{zB'(z)}{B(z)}r(z) - zr'(z)\Big| = |B'(z)r(z) - r'(z)B(z)|.$$

This gives for $|z| = 1$,

$$|(r^*(z))'| \geqslant |B'(z)||r(z)| - |r'(z)||B(z)| = |B'(z)||r(z)| - |r'(z)|,$$

or

$$|(r^*(z))'| + |r'(z)| \geqslant |B'(z)||r(z)| \text{ for } |z| = 1. \tag{3.11}$$

Also

$$\begin{aligned} 2\Big|B(z)r'(z) + \frac{\beta}{2}B'(z)r(z)\Big| &= \Big|B(z)r'(z) + \frac{\beta}{2}B'(z)r(z)\Big| \\ &\quad + \Big|B(z)(r^*(z))' + \frac{\beta}{2}B'(z)r^*(z)\Big| \\ &\geqslant |B(z)r'(z)| - \frac{|\beta|}{2}|B'(z)r(z)| \\ &\quad + |B(z)(r^*(z))'| - \frac{|\beta|}{2}|B'(z)r^*(z)| \\ &= |r'(z)| + |(r^*(z))'| - |\beta||B'(z)||r(z)| \\ &\geqslant |B'(z)|(1 - |\beta|)|r(z)|, \ \text{(by (3.11))}, \end{aligned}$$

which implies for $|z| = 1$,

$$\frac{1}{2}(1 - |\beta|)|r(z)| \leqslant \Big|\frac{r'(z)}{B'(z)} + \frac{\beta}{2}\frac{r(z)}{B(z)}\Big|,$$

from which it follows for every β with $|\beta| \leqslant 1$, that

$$\frac{1}{2}(1 - |\beta|)\sup_{|z|=1}|r(z)| \leqslant \sup_{|z|=1}\Big|\frac{r'(z)}{B'(z)} + \frac{\beta}{2}\frac{r(z)}{B(z)}\Big|. \tag{3.12}$$

Again for $|z|=1$ and $|\beta|\leqslant 1$, we have

$$\begin{aligned}2\Big|B(z)r'(z)+\frac{\beta}{2}B'(z)r(z)\Big| &= \Big|B(z)r'(z)+\frac{\beta}{2}B'(z)r(z)\Big| \\ &\quad + \Big|B(z)(r^*(z))'+\frac{\beta}{2}B'(z)r^*(z)\Big| \\ &\leqslant |B(z)r'(z)|+\frac{|\beta|}{2}|B'(z)r(z)| \\ &\quad +|B(z)(r^*(z))'|+\frac{|\beta|}{2}|B'(z)r^*(z)| \\ &= |r'(z)|+|(r^*(z))'|+|\beta||B'(z)||r(z)| \\ &\leqslant |B'(z)|\sup_{|z|=1}|r(z)|+|\beta||B'(z)|\sup_{|z|=1}|r(z)|, \ \text{(by (1.6))},\end{aligned}$$

which gives for $|z|=1$,

$$\Big|\frac{r'(z)}{B'(z)}+\frac{\beta}{2}\frac{r(z)}{B(z)}\Big|\leqslant\frac{1}{2}(1+|\beta|)\sup_{|z|=1}|r(z)|.$$

This implies for every $|\beta|\leqslant 1$,

$$\sup_{|z|=1}\Big|\frac{r'(z)}{B'(z)}+\frac{\beta}{2}\frac{r(z)}{B(z)}\Big|\leqslant\frac{1}{2}(1+|\beta|)\sup_{|z|=1}|r(z)|. \tag{3.13}$$

From (3.12) and (3.13), we get (3.9) and this completes the proof. □

For $\beta=0$, the above theorem reduces to the following corollary which extends (1.4) to self-inversive rational functions.

Corollary 3.1. *If $r\in\mathbb{R}_n$ is self-inversive, then*

$$\sup_{|z|=1}\Big|\frac{r'(z)}{B'(z)}\Big|=\frac{1}{2}\sup_{|z|=1}|r(z)|.$$

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Department of Mathematics
University of Kashmir
Srinagar
India
mabdullah_mir@yahoo.co.in

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