CLASSIFICATION THEOREM FOR STRONG TRIANGLE BLOCKING ARRANGEMENTS

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ABSTRACT. A strong triangle blocking arrangement is a geometric arrangement of some line segments in a triangle with certain intersection properties. It turns out that they are closely related to blocking sets. We prove a classification theorem for strong triangle blocking arrangements. As an application, we obtain a new proof of the result of Ackerman, Buchin, Knauer, Pinchasi and Rote which says that n points in general position cannot be blocked by $n - 1$ points, unless $n = 2, 4$. We also conjecture an extremal variant of the blocking points problem.

1. Introduction

We start with some background. In [**3**], Erdős and Purdy posed the following problem. Given a set P of n points in the plane, not all on a line, how many new points (different from points in P) have to be chosen so that every line spanned by P meets a new point? They conjectured that the answer is at least $(1 + o(1))n$ lines. We refer to the set P as the *initial points* and to the newly chosen points as the *blocking points*. We also say that a blocking point x *blocks* a line l if $x \in l$. The current record is due to Pinchasi [**7**].

THEOREM 1.1 (Pinchasi [**7**]). *Given a set P of n points in the plane, not all on a line, we need at least $\frac{n-1}{3}$ blocking points to block all lines spanned by P.*

It is worth mentioning that the first bounds of the form $\Omega(n)$ were due to Szemerédi and Trotter [**9**] and Beck [**2**]. Those papers are in fact about Dirac's conjecture, which states that for a set $\mathcal{L}$ of n nonconcurrent lines in the plane, there is a line $l \in \mathcal{L}$ that forms at least $\frac{n}{2} - O(1)$ different intersection points with lines in $\mathcal{L}$. Szemerédi and Trotter, and Beck, prove that there is a line with $\Omega(n)$ different intersection points, which implies in particular that at least $\Omega(n)$ blocking

points are necessary. Let us also remark that in [**9**], Szemerédi and Trotter prove their celebrated theorem on the number of incidence between lines and points, and they use this theorem to deduce the weak form of Dirac's conjecture. On the other hand, Pinchasi's proof is purely combinatorial and avoids any use of the incidence theorem.

In the same paper [**3**], Erdős and Purdy propose a variant of the question where P has to be in general position (i.e., no three points are collinear), but remark that Grünbaum pointed out to them that $2\lfloor\frac{n}{2}\rfloor$ new points suffice. It is easy to see that we need at least n new points when n is odd, and $n-1$ new points when n is even, so this remark actually has a typo, and should probably be $2\lceil\frac{n}{2}\rceil-1$. However, with this obvious typo corrected, what Erdős and Purdy actually imply is that there are examples where $n-1$ new points are sufficient, for infinitely many n. Therefore, the right question is: when do $n-1$ new points suffice? This was answered by Ackerman, Buchin, Knauer, Pinchasi and Rote in [**1**].

THEOREM 1.2 (Ackerman, Buchin, Knauer, Pinchasi and Rote [**1**]). *Let P be a set of $n \geqslant 5$ points in general position and let B be a set of some other points in the plane, such that every line determined by two points in P meets a point in B. Then $|B| \geqslant n$.*

Observe immediately that the theorem is trivial when n is odd. Indeed, if n is odd, each point in B can meet at most $\lfloor\frac{n}{2}\rfloor = \frac{n-1}{2}$ lines spanned by P, so at least n points are needed. (If n is even, the same counting argument gives only the bound of $n-1$.) On the other hand, the fact that there is a nontrivial example for $n = 4$ shows that we cannot hope for such a short argument in the general case. Similarly, the theorem may naturally be compared to the Sylvester–Gallai theorem (posed by Sylvester [**8**] and solved by Gallai [**4**]), but once again, the $n = 4$ case tells us that we can expect the proof to be more involved than, for example, looking at the minimal height of a triangular region formed in the dual (as in the usual proof of the Sylvester–Gallai theorem).

Let us also remark that the regular n-gon in the projective plane with n points on the line at infinity corresponding to directions of diagonals, show that for every n, n blocking points suffice for certain configurations. Of course, using suitable transformations, we can make such examples affine.

Our proof of Theorem 1.2 is based on the following classification theorem (Theorem 1.3), which is the main result of this paper. At this stage, we state this theorem informally, as otherwise the definitions would occupy significant part of the introduction. For an arbitrary convex polygon R, we write ∂R for the *boundary* of R, which is the union of its edges.

Before we state the classification theorem, let us very briefly explain what the theorem is about. Namely, we consider a triangular region which has two sets of segments $\mathcal{S}$ and $\mathcal{B}$ inside it, with vertices on the boundary of the triangle. These have the property that whenever two segments in $\mathcal{S}$ intersect, then there is a unique segment in $\mathcal{B}$ passing through their intersection. Also, when a segment in $\mathcal{S}$ meets a segment in $\mathcal{B}$, then there is a unique other segment in $\mathcal{S}$ passing through their intersection. These two conditions come from considering the dual of the

hypothetical extremal arrangement of points in general position and their blocking points, and are requirements (i) and (ii) in the classification theorem. Our aim is to classify all such collections of segments that satisfy an additional condition. This is the condition (iii) in the statement. Let us remark here that although this condition looks somewhat artificial compared to the other two, it actually develops naturally in the proof of Theorem 1.2.

THEOREM 1.3. *Let T be a triangle, with edges e_1, e_2, e_3, and let $\mathcal{S}$ and $\mathcal{B}$ be collections of segments inside T, with endpoints on the edges of T, but no internal point of a segment meets ∂T. Write $\overline{\mathcal{S}} = \mathcal{S} \cup \{e_1, e_2, e_3\}$. Suppose that*

(i) *No three segments of $\overline{\mathcal{S}}$ are concurrent, and for any two such segments that intersect at a point p, there is a unique segment $\beta(p) \in \mathcal{B}$ that passes through p, except possibly when the two segments are edges of T, in which case there might not be any such segment in $\mathcal{B}$.*

(ii) *For every intersection p of a segment in $\overline{\mathcal{S}}$ and a segment in $\mathcal{B}$, there is a unique second segment in $\overline{\mathcal{S}}$ that passes through p.*

(iii) *In every minimal $\overline{\mathcal{S}}$-region R, for any consecutive vertices v_1, v_2, v_3, v_4, v_5 appearing in this order on ∂R, we have that, if $l(v_1 v_2)$ and $\beta(v_3)$ intersect in T, and $\beta(v_3)$ crosses the interior of R, then $l(v_1 v_2), \beta(v_3), l(v_4 v_5)$ are concurrent.*

Then the configuration formed by $T, \mathcal{S}$ and $\mathcal{B}$ must have one of the structures shown in Figure 1.

We call an arrangement satisfying conditions (i) and (ii) a *triangle blocking arrangement*. If a triangle blocking arrangement additionally satisfies the condition (iii), then we call it a *strong triangle blocking arrangement*.

Given that there is such a strong structure theorem in this setting, it is plausible that an extremal result could hold. Recall that Theorem 1.2 resembles the Sylvester–Gallai theorem, which has its extremal version in the following theorem of Green and Tao. For a given set of points in the plane, we say that a line is *ordinary* if it passes through exactly two points in the set.

THEOREM 1.4 (Green and Tao [**5**]). *There is an n_0 such that, whenever we have $n \geqslant n_0$ points in the plane that span at most $\frac{n}{2}$ ordinary lines, there is a cubic curve containing the given points.*

With this in mind, we formulate the following conjecture.

CONJECTURE 1.1. *Suppose that P is a set of n points in the plane in general position, and let B be a set of blocking points for P. If $|B| = n$, then $P \cup B$ lie on a cubic curve.*

We postpone the discussion of the connection between proof of Theorem 1.4 and the classification theorem (Theorem 1.3) we prove here to the concluding remarks. There we also discuss why the condition (iii) is necessary in the classification theorem.

The plan of the paper is as follows. In the next section we describe the classification theorem. Then, in Section 3, we see how to deduce Theorem 1.2 from the
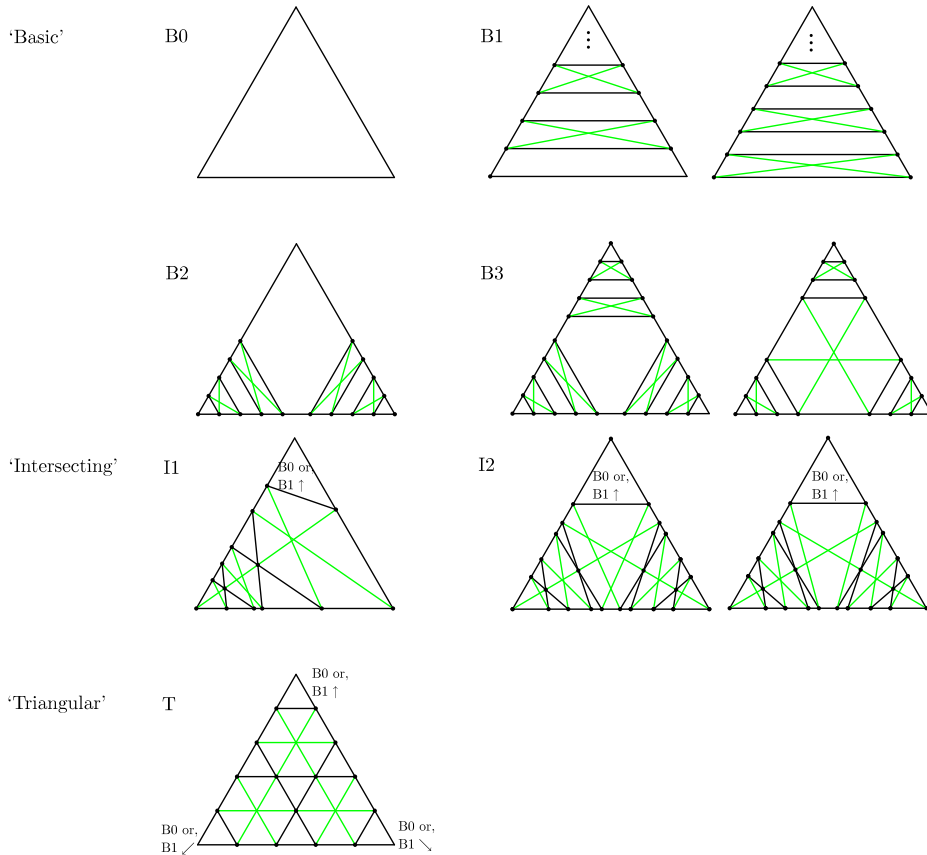
FIGURE 1. Types of triangle blocking arrangements

classification theorem. In Section 4, we prove the classification theorem. This is actually the main part of the paper.

2. Detailed description of the structural theorem

Before stating the classification result, we first need to introduce some terminology. Recall that *triangle blocking arrangement* is a triple $\Delta = (T, \mathcal{S}, \mathcal{B})$ consisting of a triangle T, with vertices x_1, x_2 and x_3, a two collections of segments $\mathcal{S}$ and $\mathcal{B}$ such that the endpoints of each segment lie on the boundary of ∂T (possibly coinciding with some vertex x_i), and no interior point of a segment lies on the boundary ∂T, and the following intersection condition is satisfied: for every pair of segments in $\overline{\mathcal{S}} := \mathcal{S} \cup \{x_1 x_2, x_2 x_3, x_3 x_1\}$, except possibly the pairs of sides of T, if they intersect, there is a unique segment in $\mathcal{B}$ that passes through their intersection, and for every intersecting pair of segments, where one segment is in $\overline{\mathcal{S}}$ and the other in $\mathcal{B}$, there is a unique second segment in $\overline{\mathcal{S}}$ that passes through their intersection. We
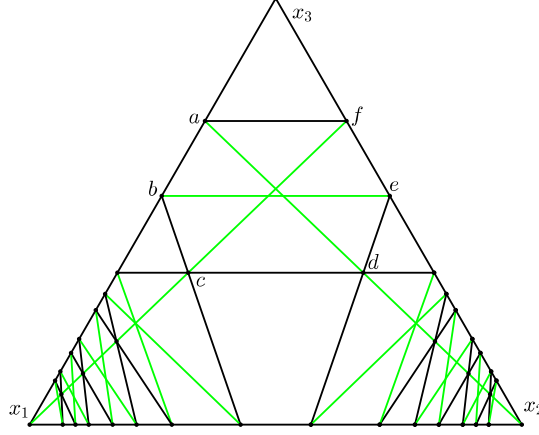
FIGURE 2. Example of a triangle blocking arrangement not among definitions

call the elements of $\overline{\mathcal{S}}$ the *initial segments*, the elements of $\mathcal{S}$ the *proper initial segments* when we have to distinguish them from $\overline{\mathcal{S}}$, and the elements of $\mathcal{B}$ the *blocking segments*. Furthermore, if x, y are two points on a segment $s \in \overline{\mathcal{S}} \cup \mathcal{B}$, we also say that xy is *initial segment* if $s \in \overline{\mathcal{S}}$, that xy is *proper initial segment* if $s \in \mathcal{S}$, and that xy is *blocking segment* if $s \in \mathcal{B}$. We refer to intersections of initial segments as *vertices* in T. We will write $\beta(v)$ for the unique blocking segment through the vertex v, and in general, for any two points x, y we write $l(xy)$ for the line through x and y.

Given a triangle T', whose vertices lie in T (possibly on the edges of T), and whose edges are subsets of segments in $\overline{\mathcal{S}}$, we define $\overline{\mathcal{S}_{T'}} := \{T' \cap s : s \in \overline{\mathcal{S}}\}$, $\mathcal{B}_{T'} := \{T' \cap s : s \in \mathcal{B}\}$ and $\mathcal{S}_{T'} := \overline{\mathcal{S}_{T'}} \smallsetminus \{\text{edges of } T'\}$. We say that $\Delta' = (T', \mathcal{S}_{T'}, \mathcal{B}_{T'})$ is a *sub-triangle blocking arrangement* of Δ, *induced by T'*. One can check that sub-triangle blocking arrangement is itself a triangle blocking arrangement.

We now define some special types of triangle blocking arrangements. Let us stress that these do not include all possible triangle blocking arrangements. An example not included in definitions is shown in Figure 2.

Basic types $\mathbf{B_0}, \mathbf{B_1}, \mathbf{B_2}, \mathbf{B_3}$. We say a triangle blocking arrangement $\Delta = (T, \mathcal{S}, \mathcal{B})$ is of type $\mathbf{B_0}$ if $\mathcal{S} = \mathcal{B} = \emptyset$.

We say a triangle blocking arrangement $\Delta = (T, \mathcal{S}, \mathcal{B})$ is of type $\mathbf{B_1}$ if it has the following structure. There is an ordering y_1, y_2, y_3 of x_1, x_2, x_3 such that the vertices of $\mathcal{S} \cup \mathcal{B}$ are $v_1, v_2, \ldots, v_n \in y_1 y_2$ and $u_1, u_2, \ldots, u_n \in y_1 y_3$, with ordering $y_1, v_1, v_2, \ldots, v_n, y_2$ on $y_1 y_2$ and ordering $y_1, u_1, u_2, \ldots, u_n, y_3$ on $y_1 y_3$, such that $\mathcal{S} = \{v_1 u_1, v_2 u_2, \ldots, v_n u_n\}$ and $\mathcal{B} = \{v_1 u_2, v_2 u_1, v_3 u_4, v_4 u_3, \ldots, v_{t-1} u_t, v_t u_{t-1}\}$, where $t = n$ if n is even, and $t = n + 1$, if n is odd, and in that case, we set $v_{n+1} = y_2$, $u_{n+1} = y_3$. We say that y_1 is the *first vertex* of Δ. Also, when n is even, we say that Δ is *even*, and if n is odd, we say that Δ is *odd*.

We say a triangle blocking arrangement $\Delta = (T, \mathcal{S}, \mathcal{B})$ is of type $\mathbf{B_2}$ if it has the following structure. There is an ordering y_1, y_2, y_3 of x_1, x_2, x_3 such that the

vertices of $\mathcal{S} \cup \mathcal{B}$ are $v_1, v_2, \ldots, v_n \in y_1y_2$, then $u_1, u_2, \ldots, u_n, u'_m, \ldots, u'_1 \in y_1y_3$ and $w_1, w_2, \ldots, w_m \in y_3y_2$, with ordering $y_1, v_1, v_2, \ldots, v_n, y_2$ on y_1y_2, ordering $y_1, u_1, u_2, \ldots, u_n, u'_m, \ldots, u'_1, y_3$ on y_1y_3, and ordering $y_3, w_1, \ldots, w_m, y_2$ on y_3y_2, and m and n are even. The segments are $\mathcal{S} = \{v_1u_1, v_2u_2, \ldots, v_nu_n, u'_1w_1, \ldots, u'_mw_m\}$ and $\mathcal{B} = \{v_1u_2, v_2u_1, v_3u_4, v_4u_3, \ldots, v_{n-1}u_n, v_nu_{n-1}\} \cup \{u'_1w_2, u'_2w_1, \ldots, u'_nw_{n-1}, u'_{n-1}w_n\}$.

We say a triangle blocking arrangement $\Delta = (T, \mathcal{S}, \mathcal{B})$ is of type $\mathbf{B_3}$ if it has the following structure. There is an ordering y_1, y_2, y_3 of x_1, x_2, x_3 such that the vertices of $\mathcal{S} \cup \mathcal{B}$ are $v_1, \ldots, v_k, v'_l, \ldots, v'_1 \in y_1y_2$, $w_1, \ldots, w_l, w'_m, \ldots, w'_1 \in y_2y_3$, $u_1, \ldots, u_m, u'_k, \ldots, u'_1 \in y_3y_1$, appearing in orders $y_1, v_1, \ldots, v_k, v'_l, \ldots, v'_1, y_2$ on y_1y_2, y_2, $w_1, \ldots, w_l, w'_m, \ldots, w'_1, y_3$ on y_2y_3, and $y_3, u_1, \ldots, u_m, u'_k, \ldots, u'_1, y_1$ on y_3y_1, and k, l, m are of the same parity.

When k, l, m are even, then the segments are $\mathcal{S} = \{v_1u'_1, \ldots, v_ku'_k\} \cup \{w_1v'_1, \ldots, w_lv'_l\} \cup \{u_1w'_1, \ldots, u_mw'_m\}$ and $\mathcal{B} = \{v_1u'_2, v_2u'_1, \ldots, v_{k-1}u'_k, v_ku'_{k-1}\} \cup \{w_1v'_2, w_2v'_1, \ldots, w_{l-1}v'_l, w_lv'_{l-1}\} \cup \{u_1w'_2, u_2w'_1, \ldots, u_{m-1}w'_m, u_mw'_{m-1}\}$.

When k, l, m are odd, then the segments are $\mathcal{S} = \{v_1u'_1, \ldots, v_ku'_k\} \cup \{w_1v'_1, \ldots, w_lv'_l\} \cup \{u_1w'_1, \ldots, u_mw'_m\}$ and $\mathcal{B} = \{v_1u'_2, v_2u'_1, \ldots, v_{k-1}u'_{k-2}, v_{k-2}u'_{k-1}\} \cup \{w_1v'_2, w_2v'_1, \ldots, w_{l-1}v'_{l-2}, w_{l-2}v'_{l-1}\} \cup \{u_1w'_2, u_2w'_1, \ldots, u_{m-1}w'_{m-2}, u_{m-2}w'_{m-1}\} \cup \{v_kw'_m, v'_lu_m, w_lu'_k\}$.

Intersecting types $\mathbf{I_1}, \mathbf{I_2}$. We say a triangle blocking arrangement $\Delta = (T, \mathcal{S}, \mathcal{B})$ is of type $\mathbf{I_1}$ if it has the following structure. There is an ordering y_1, y_2, y_3 of x_1, x_2, x_3 and there are vertices $z_2 \in y_1y_2, z_3 \in y_1y_3$ such that

(1) The segment z_2z_3 is in $\mathcal{S}$, and the sub-triangle blocking arrangement in the triangle $T' := y_1z_2z_3$ is of type $\mathbf{B_0}$ or $\mathbf{B_1}$, with y_1 as the first vertex and it is even.

(2) There are vertices $v_1, v_2, \ldots, v_{2k}$ on y_2z_2, appearing in that order from y_2 to z_2, and there are vertices $u_1, u_2, \ldots, u_{2k}$ on y_2y_3, appearing in that order from y_2 to y_3, such that

$$\mathcal{S} \setminus \overline{\mathcal{S}_{T'}} = \{u_1v_2, u_2v_1, u_3v_4, u_4v_3, \ldots, u_{2k-1}v_{2k}, u_{2k}v_{2k-1}\},$$
$$\mathcal{B} \setminus \mathcal{B}_{T'} = \{u_1v_1, u_2v_3, u_3v_2, u_4v_5, u_5v_4, \ldots, u_{2k-2}v_{2k-1}, u_{2k-1}v_{2k-2}\}$$
$$\cup \{y_2z_3, y_3v_{2k}, z_2u_{2k}\}.$$

(3) For $i = 1, 2, \ldots, k$, the segments $u_{2i-1}v_{2i}, u_{2i}v_{2i-1}, y_2z_3$ are concurrent. Let p_i be their intersection point. The intersections between $\overline{\mathcal{S}}$ and $\mathcal{B}$, and the intersections of pairs of segments in $\overline{\mathcal{S}}$ are either on ∂T, or the points p_i.

We say a triangle blocking arrangement $\Delta = (T, \mathcal{S}, \mathcal{B})$ is of type $\mathbf{I_2}$ if it has the following structure. There is an ordering y_1, y_2, y_3 of x_1, x_2, x_3 and there are vertices $z_2 \in y_1y_2, z_3 \in y_1y_3$ such that

(1) The segment z_2z_3 is in $\mathcal{S}$, and the sub-triangle blocking arrangement in the triangle $T' := y_1z_2z_3$ is of type $\mathbf{B_0}$ or $\mathbf{B_1}$, with y_1 as the first vertex and it is even.

(2) The vertices from y_2 to z_2 are $u_1, u_2, \ldots, u_{2k}$, in that order, from y_2 to y_3 are $v_1, v_2, \ldots, v_{2k}, w_{2l}, w_{2l-1}, \ldots, w_1$, in that order, and from y_3 to z_3 are

$t_1, t_2, \ldots, t_{2l}$, in that order. The segments of Δ are

$$\mathcal{S} \smallsetminus \overline{\mathcal{S}_{T'}} = \{u_1 v_2, u_2 v_1, \ldots, u_{2k-1} v_{2k}, u_{2k} v_{2k-1}\}$$
$$\cup \{w_1 t_2, w_2 t_1, \ldots, w_{2l-1} t_{2l}, w_{2l} t_{2l-1}\},$$
$$\mathcal{B} \smallsetminus \mathcal{B}_{T'} = \{u_1 v_1, u_2 v_3, u_3 v_2, \ldots, u_{2k-2} v_{2k-1}, u_{2k-1} v_{2k-2}\}$$
$$\cup \{w_1 t_1, w_2 t_3, w_3 t_2, \ldots, w_{2k-2} t_{2k-1}, w_{2k-1} t_{2k-2}\}$$
$$\cup \{u_{2k} y_3, t_{2l} y_2\} \cup \mathcal{B}',$$

where $\mathcal{B}' = \{v_{2k} z_3, w_{2l} z_2\}$ or $\mathcal{B}' = \{v_{2k} z_2, w_{2l} z_3\}$.

(3) For $i = 1, 2, \ldots, k$, the segments $u_{2i-1} v_{2i}, u_{2i} v_{2i-1}, y_2 t_{2l}$ are concurrent, at point p_i, and for $i = 1, 2, \ldots, l$, the segments $w_{2i-1} t_{2i}, w_{2i} t_{2i-1}, y_3 u_{2k}$ are concurrent, at point q_i. The intersections between segments in $\overline{\mathcal{S}}$ and $\mathcal{B}$, and the intersections of pairs of segments in $\overline{\mathcal{S}}$ are either on ∂T, or the points p_i and q_i.

Triangular type T. We say a triangle blocking arrangement $\Delta = (T, \mathcal{S}, \mathcal{B})$ is of type **T** if it has the following structure. There is an integer $k \geqslant 2$ and there are vertices $u_1, u_2, \ldots, u_{2k} \in x_1 x_2$, appearing in that order from x_1 to x_2, vertices $v_1, v_2, \ldots, v_{2k} \in x_2 x_3$, appearing in that order from x_2 to x_3, and vertices $w_1, w_2, \ldots, w_{2k} \in x_3 x_1$, appearing in that order from x_3 to x_1.

(1) Segments $u_1 w_{2k}, v_1 u_{2k}$ and $w_1 v_{2k}$ belong to $\mathcal{S}$, and writing $T_1 = x_1 u_1 w_{2k}$, $T_2 = x_2 v_1 u_{2k}$ and $T_3 = x_3 w_1 v_{2k}$, each sub-triangle blocking arrangement induced by T_i is of type $\mathbf{B_0}$ or $\mathbf{B_1}$ with x_i as the first vertex, and it is even.

(2) The segments are given by

$$\mathcal{S} \smallsetminus \bigcup_{i=1}^{3} \mathcal{S}_{T_i} = \{u_{2i-1} w_{2k+2-2i} : i \in [k]\} \cup \{v_{2i-1} u_{2k+2-2i} : i \in [k]\}$$
$$\cup \{w_{2i-1} v_{2k+2-2i} : i \in [k]\},$$
$$\mathcal{B} \smallsetminus \bigcup_{i=1}^{3} \mathcal{B}_{T_i} = \{u_{2i} w_{2k+1-2i} : i \in [k]\} \cup \{v_{2i} u_{2k+1-2i} : i \in [k]\}$$
$$\cup \{w_{2i} v_{2k+1-2i} : i \in [k]\}.$$

(3) For every triple $(a, b, c) \in [2k]^3$ such that not all of a, b, c are even and $a + b + c = 4k + 2$, the triple of segments $u_a w_{2k+1-a}, v_b u_{2k+1-b}, w_c v_{2k+1-c}$ is concurrent at the point $p_{a,b,c}$. The intersections between segments in $\overline{\mathcal{S}}$ and $\mathcal{B}$, and the intersections of pairs of segments in $\overline{\mathcal{S}}$ are either on ∂T, or the points $p_{a,b,c}$.

We remark that allowing $k = 1$ in the definition of type **T** would actually give type $\mathbf{B_3}$. We keep $\mathbf{B_3}$ as a basic type, as in this case the intersections between initial segments lie on ∂T only, while in the type **T** we insist on having at least one intersection of initial segments that is in the interior of T.

Classifying all triangle blocking arrangements currently seems out of reach, however we are able to prove the following. (The assumption (A) below is exactly the assumption (iii) of Theorem 1.3.)

THEOREM 2.1 (A classification theorem for triangle blocking arrangements.). *Suppose that $\Delta = (T, \mathcal{S}, \mathcal{B})$ is a strong triangle blocking arrangement, i.e., a triangle blocking arrangement such that the following assumption (A) holds.*

(A) *In every minimal $\overline{\mathcal{S}}$-region R, for any consecutive vertices v_1, v_2, v_3, v_4, v_5 appearing in this order on ∂R, we have that, if $l(v_1 v_2)$ and $\beta(v_3)$ intersect in T, and $\beta(v_3)$ meets the interior of R, then $l(v_1 v_2), \beta(v_3), l(v_4 v_5)$ are concurrent.*

Then Δ has one of the types among $\mathbf{B_0}, \mathbf{B_1}, \mathbf{B_2}, \mathbf{B_3}, \mathbf{I_1}, \mathbf{I_2}$ and $\mathbf{T}$.

3. Deducing Theorem 1.2 from the Classification theorem

In this section we prove Theorem 1.2. Immediately, we move to the dual, where the theorem has the following formulation. For $n \geqslant 4$, we say that a pair of disjoint sets $(\mathcal{L}, \mathcal{B})$ of lines in $\mathbb{P}^2 = \mathbb{P}^2(\mathbb{R})$ is an *n-blocking configuration* if $|\mathcal{L}| = |\mathcal{B}| + 1 = n$, no three lines in $\mathcal{L}$ are concurrent, and for every pair l_1, l_2 of lines in $\mathcal{L}$ there is a unique line in $\mathcal{B}$ that passes through $l_1 \cap l_2$. We refer to the lines in $\mathcal{L}$ as the *initial lines*, and to the lines in $\mathcal{B}$ as the *blocking lines*.

THEOREM 3.1. *Let $n \geqslant 4$ and let $(\mathcal{L}, \mathcal{B})$ be an n-blocking configuration. Then $n = 4$.*

We begin the proof by deducing some structural information about the configuration of lines in $\mathcal{L} \cup \mathcal{B}$, which will enable us to apply Theorem 2.1 and deduce Theorem 3.1.

By an *$\mathcal{L}$-region*, we mean the closure of any connected subset of $\mathbb{P}^2 \smallsetminus (\cup L)$, where L is a subset of some lines in $\mathcal{L}$. A *minimal $\mathcal{L}$-region* is the closure of a connected subset of $\mathbb{P}^2 \smallsetminus (\cup \mathcal{L})$. For an $\mathcal{L}$-region R, defined by a subset $L \subset \mathcal{L}$ of initial lines, we define its *edges* to be the segments of lines in L that intersect R, and *vertices* as the intersections of lines in L that lie in R. Finally, a vertex v of $\mathcal{L}$-region is *internally blocked* if the unique blocking line through v meets the interior of R, otherwise, it is *externally blocked*.

LEMMA 3.1. *Let $n \geqslant 4$ and let $(\mathcal{L}, \mathcal{B})$ be an n-blocking configuration. Let R be any $\mathcal{L}$-region, not necessarily minimal. Then the number of internally blocked vertices of R is even.*

PROOF. Draw $l \cap R$ for all the initial lines l that meet R, partitioning R into minimal $\mathcal{L}$-regions $R_1, R_2, \ldots, R_t$. Let V be the set of all intersections of lines in $\mathcal{L}$ that belong to R. We may partition $V = V_1 \cup V_2 \cup V_3$ so that V_1 is the set of extremal points of R, which is the same as the vertices of R, V_2 is the set of all points in V that lie on the boundary of R, but are not vertices of R, and V_3 is the set of all points of V that lie inside the interior of R.

We proceed via a double-counting argument. We count the pairs (R_i, v), where R_i is one of the minimal $\mathcal{L}$-regions above, $v \in V$ is a vertex of R_i and v is internally-blocked in R_i. Let I be the set of all such pairs. Using the partition of V, we may write $I = I_1 \cup I_2 \cup I_3$, where $I_j = \{(R_i, v) \in I : v \in V_j\}$. Writing $N_j = |I_j|$ our goal becomes to show that $|I_1|$ is even. Set $N = |I|$.

Every blocking line b that crosses R intersects ∂R at two points, and these contribute to N by 2, and b also passes through some vertices in int R. But, each such vertex is blocked internally two times by b, for some minimal $\mathcal{L}$-regions. Hence, N is even.

Observe that every initial line l that meets R, but is not one of its edges, intersects ∂R twice at interiors of edges of R. Thus, these two intersections contribute to N_2 by 2, and every such intersection is defined by a unique such initial line l. This shows that N_2 is even.

Finally, every vertex in int R is internally blocked in two minimal $\mathcal{L}$-regions, so N_3 is even, hence $N_1 = N - N_2 - N_3$ is also even, as desired.   □

LEMMA 3.2. *Let $n \geqslant 4$ and let $(\mathcal{L}, \mathcal{B})$ be an n-blocking configuration. Let R be any minimal $\mathcal{L}$-region. Then either all vertices of R are internally blocked or all vertices of R are externally blocked.*

PROOF. Suppose, to the contrary, that R has two consecutive vertices u and v, such that u is internally blocked, but v is externally blocked. Since a blocking line meets int R, R cannot be a triangle. Let u', v' be another two vertices of R, such that u', u, v, v' are consecutive. Let $p = u'u \cap v'v$. Consider $\mathcal{L}$-region S with vertices u, v, p. Inside S, among u and v, exactly one is internally blocked vertex. Therefore, p is an internally blocked vertex of S, with a blocking line $b \in \mathcal{B}$. But b must cross the interior of uv, which is a contradiction with the fact that R is a minimal $\mathcal{L}$-region.   □

LEMMA 3.3. *Let $n \geqslant 4$ and let $(\mathcal{L}, \mathcal{B})$ be an n-blocking configuration. Let R be any minimal $\mathcal{L}$-region, with vertices $v_1, v_2, \ldots, v_k$, sorted in the order as they appear on ∂R. Suppose that the vertices of R are internally blocked. Then k is even, and for every $i, j \in [k/2]$, the line $v_i v_{i+k/2} \in \mathcal{B}$, and the lines $v_{i-j-1} v_{i-j}$, $v_i v_{i+k/2}$, $v_{i+j} v_{i+j+1}$ are concurrent, (indices of vertices are taken modulo k).*

PROOF. Let b_i be the blocking line at the vertex v_i, and let $l_i = v_i v_{i+1} \in \mathcal{L}$. We first prove that l_{i-j-1}, b_i, l_{i+j} are concurrent by induction on $j \in \{0, 1, 2, \ldots, k/2\}$. Observe that when $j = k/2$, we have that $l_{i-k/2-1}, b_i, l_{i+k/2}$ are concurrent. But the lines $l_{i-k/2-1} = v_{i-k/2-1} v_{i-k/2}$ and $l_{i+k/2} = v_{i+k/2} v_{i+k/2+1}$ already meet at $v_{i+k/2}$, which is blocked by $b_{i+k/2}$, so by uniqueness of blocking lines $b_i = b_{i+k/2} = v_i v_{i+k/2}$.

For the base of induction, when $j = 0$, the lines l_{i-1}, b_i, l_i meet at v_i, so the claim holds.

Suppose that the claim holds for some $0 \leqslant j < k/2$, and consider $l_{i-j-2}, b_i, l_{i+j+1}$. Look at the $\mathcal{L}$-region S formed by lines $l_{i-j-2}, l_{i-j-1}, l_{i+j}, l_{i+j+1}$. By induction hypothesis, the triple of lines l_{i-j-1}, b_i, l_{i+j} is concurrent with the common point p_1. Let $p_2 := l_{i-j-2} \cap l_{i+j}, p_3 := l_{i-j-1} \cap l_{i+j+1}$ and $p_4 := l_{i-j-2} \cap l_{i+j+1}$, so the vertices of S are precisely p_1, p_2, p_3 and p_4, and our goal is to show that $p_4 \in b_i$. Look at $\mathcal{L}$-region with vertices v_{i-j-1}, p_1, p_2, formed by initial lines l_{i-j-2}, l_{i-j-1} and l_{i+j}. For this region, v_{i-j-1} and p_1 are externally blocked, so by Lemma 3.1, the vertex p_2 must be externally blocked as well, therefore p_2 is internally blocked in S. Likewise, the vertex p_3 is internally blocked in S, so, since p_1 is also internally blocked, by Lemma 3.1, the remaining vertex p_4 is internally blocked, by

some blocking line b. But b meets R between vertices v_{i-j-1} and v_{i+j}, so it must contain some vertex v_l, $l \in \{i-j, i-j+1, \ldots, i+j-1\}$. By induction hypothesis, b_l meets l_{i-j-2} at a point other than p_4, when $l < i$, and b_l meets l_{i+j+1} at point other than p_4, for $l > i$, proving that $l = i$, as desired.   □

Having acquired enough structural information about blocking configurations, we are ready to prove Theorem 3.1.

PROOF OF THEOREM 3.1. Observe that since no three lines in $\mathcal{L}$ are concurrent, there are $m_1, m_2, m_3 \in \mathcal{L}$ that define a region which is a minimal $\mathcal{L}$-region (this follows from the fact that any line that crosses a triangle splits that triangle into two regions, one of which is also a triangle). Denote the other $\mathcal{L}$-regions formed by m_1, m_2 and m_3 by S_1, S_2 and S_3. Applying Theorem 2.1 to the triangle blocking arrangements induced by S_1, S_2 and S_3, we see that each of them is of type $\mathbf{B_1}$. Let $x_1 := m_2 \cap m_3$, $x_2 := m_3 \cap m_1$, $x_3 := m_1 \cap m_2$ and let $e_1 := m_1 \cap S$, $e_2 := m_2 \cap S$, $e_3 := m_3 \cap S$. Let $a_1, a_2, \ldots, a_r$ be the vertices on $m_1 \setminus e_1$, listed from x_2 to x_3, let $b_1, b_2, \ldots, b_s$ be the vertices on $m_2 \setminus e_2$, listed from x_3 to x_1, and finally let $c_1, c_2, \ldots, c_t$ be the vertices of $m_3 \setminus e_3$, listed from x_1 to x_2. By the definition of type $\mathbf{B_1}$, we have $r = s = t$ and $a_i b_{r+1-i}, b_i c_{r+1-i}, c_i a_{r+1-i} \in \mathcal{L}$, for $i \in [r]$. However, a_1 belongs to the initial lines $a_1 b_r, a_1 c_r$ and m_1, and m_1 is different from $a_1 b_r$ and $a_1 c_r$, therefore $a_1 b_r = a_1 c_r$, making a_1, b_r, c_r collinear. Similarly, b_r, a_1, c_1 are collinear, so c_1, c_r, b_r, a_1 are collinear. If $r > 1$, then $c_1 \neq c_r$, but $c_1, c_r \in m_3$, making $a_1 b_r = m_3$, which is a contradiction. Therefore $r = 1$, so $n = 4$, as desired.   □

4. Proof of the Classification theorem

As the title suggests, this section is devoted to the proof of the Classification theorem. For an integer $n \geqslant 0$, we say that n-Classification theorem holds, if the conclusion of Theorem 2.1 holds for all triangle blocking arrangements $\Delta = (T, \mathcal{S}, \mathcal{B})$ with $|\mathcal{S}| + |\mathcal{B}| \leqslant n$. We denote the quantity $|\mathcal{S}| + |\mathcal{B}|$ by $|\Delta|$ and call it the *size* of Δ. The argument will be based on induction on $|\Delta|$. Note that 0-Classification theorem holds, as $|\Delta| = 0$ implies that $\mathcal{S} = \mathcal{B} = \emptyset$, so Δ is of type $\mathbf{B_0}$.

Before we proceed with the proof, we need a couple of pieces of notation. Firstly, a segment is *minimal* if there are no other vertices in its interior. Also recall that for a segment xy, we write $l(xy)$ for the line that contains the segment, and given a point p, write $\beta(p)$ for the blocking segment through p. Further, also for a segment xy, we write $s(xy)$ for the unique element of $\overline{\mathcal{S}}$ which contains both x and y, if it exists. Given an $\overline{\mathcal{S}}$-region R, and a vertex v of R, we say that v is *internally blocked* in R if $\beta(v)$ passes through the interior of R, and otherwise we say that v is *externally blocked* in R. In particular, if there are no blocking segments through x_i, we say that x_i is externally blocked. (By a vertex of R here, we mean an intersection of initial segments that are edges of R, so a vertex on boundary, but in interior of an edge of R is not counted as a vertex when we talk about internally or externally blocked vertices.)

REMARK. We suggest to the reader to draw sketches while reading the proofs (and modify them when a new point or segment appears in the argument) and to use the figures in the paper as a guide.

We restate Lemma 3.1 here, which will be crucial to our work. As we shall be using this lemma all the time, we will not refer to it explicitly.

LEMMA 4.1. *Suppose that $\Delta = (T, \mathcal{S}, \mathcal{B})$ is a triangle blocking arrangement, and let R be any, not necessarily minimal, $\overline{\mathcal{S}}$-region. Then, the number of internally blocked vertices of R is even.*

PROPOSITION 4.1. *Let $n \geqslant 1$ be an integer and suppose that $(n-1)$-classification theorem holds. Let $\Delta = (T, \mathcal{S}, \mathcal{B})$ be a triangle blocking arrangement of size n, and let x_1, x_2 and x_3 be the vertices of the triangle T. Suppose that no blocking segment passes through x_1. Suppose that $s_1, s_2 \in \mathcal{S}$ are two initial segments, each with the property that one of the endpoints is on the edge $x_1 x_2$ and the other is on the edge $x_1 x_3$. Then s_1 and s_2 are disjoint.*

PROOF. Let $\Delta = (T, \mathcal{S}, \mathcal{B})$ be a triangle blocking arrangement of size n, with vertices of T given by x_1, x_2 and x_3. Assume that no blocking segment passes through x_1 and suppose, to the contrary, that two initial segments $a_2 a_3$ and $b_2 b_3$ intersect at the point c and $a_2, b_2 \in x_1 x_2$, $a_3, b_3 \in x_1 x_3$. Without loss of generality, the vertices appear in order x_1, b_2, a_2, x_2 and x_1, a_3, b_3, x_3 on the segments $x_1 x_2$ and $x_1 x_3$. We consider the following cases on the positions of blocking segments.

Case 1 Inside the region $a_2 b_2 c$, the vertices b_2 and c are both internally blocked.

Case 2 Inside the region $a_2 b_2 c$, the vertex c is internally blocked, but b_2 is externally blocked.

Case 3 Inside the region $a_2 b_2 c$, the vertex c is externally blocked. Thus c is internally blocked in the region $x_1 a_3 c b_2$, and by the parity of the number of internally blocked vertices, exactly one of b_2 and a_3 is internally blocked in that region, without loss of generality, b_2. Thus, b_2 is externally blocked in the region $a_2 b_2 c$.

We treat each case separately and we depict the steps of the proof in Figures 3, 4 and 5.

$\boxed{\textbf{Case 1.}}$ Consider the triangle blocking arrangement Δ_1 induced by the triangle $x_1 a_2 a_3$. Since $|\Delta_1| < |\Delta| = n$, we may apply the classification theorem to Δ_1. All of x_1, a_2, a_3 are externally blocked, so the type of Δ_1 is either $\mathbf{B_i}$, for some i, or $\mathbf{T}$. If the type of Δ_1 is $\mathbf{T}$, then there is an initial segment $s \in \mathcal{S}$, whose restriction to Δ_1 is $y_1 y_2$, with $y_1 \in x_1 a_3$ and $y_2 \in c a_3$. However, that implies that s intersects segment $c b_3$, at some point b_4, so the triangle blocking arrangement Δ_2, induced by the triangle $x_1 b_2 b_3$ has the vertex b_3 externally blocked, and initial segments $y_1 b_4$ and $a_3 c$ intersect. However, this cannot occur in any of the seven types we defined, and by the $(n-1)$-classification theorem, Δ_2 has one of these types, which is a contradiction.

Hence, the type of Δ_1 is one of the four basic types, and analogously the triangle blocking arrangement Δ_2 (induced by the triangle $x_1 b_2 b_3$) has also a basic type. By the definition of the basic types, if we write d', d'', c', c'' for the first
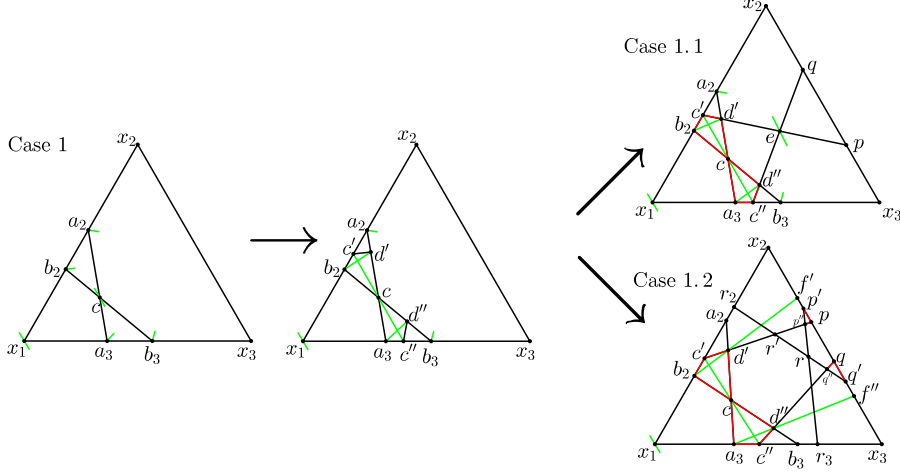
FIGURE 3. Case 1 of proof of Proposition 4.1

vertices next to c on a_2c, next to c on b_3c, next to b_2 on a_2b_2, next to a_3 on b_3a_3, respectively, then $c'd', b_2c', cd', c''d'', a_3c''$ and cd'' are minimal segments, and $c'd' \in \mathcal{S}_{\Delta_1}$, $b_2d', cc' \in \mathcal{B}_{\Delta_1}$, $c''d'' \in \mathcal{S}_{\Delta_2}$, $a_3d'', cc'' \in \mathcal{B}_{\Delta_2}$. Furthermore, we also have that the segments b_2c, ca_3 are minimal. Let p, q be vertices on ∂T such that $c'p := s(c'd')$ and $c''q := s(c''d'')$.

Claim. The vertices p and q lie on x_2x_3.

PROOF OF THE CLAIM. We prove the claim for p, the proof for q is similar. Suppose, to the contrary, that $p \in x_1x_3$. Looking at triangle blocking arrangement Δ_p induced by $x_1c'p$, we see that c' is internally blocked, and $|\Delta_p| < n$, so the $(n-1)$-classification theorem implies that Δ_p has one of types $\mathbf{B_1}, \mathbf{I_1}$ or $\mathbf{I_2}$. We can immediately discard type $\mathbf{B_1}$ as cc' is blocking, but $c \notin x_1p$. On the other hand, intersecting types do not permit $\beta(c')$ crossing segment $d'a_3$ in its interior, as $a_3 \in px_1$, so we get a contradiction.   □

Next, we consider the cases whether the segments $c'p$ and $c''q$ intersect or not.

Case 1.1, $c'p$ and $c''q$ intersect Let $e := c'p \cap c''q$. In the region $x_1c''ec'$, the vertices c' and c'' are internally blocked, while x_1 is not, so e is externally blocked.

Look at the triangle blocking arrangement Δ_3 induced by the triangle $c''qx_3$. Its size is smaller than n, so the $(n-1)$-classification theorem applies. As we have initial segments $d''b_3$ and ep, the type of Δ_3 is not $\mathbf{B_0}$, nor $\mathbf{B_1}$. Suppose for a moment that is one of the two remaining basic types. Then, $\beta(e)$ crosses x_3p, at a point p', say, and $\beta(p)$ crosses ec'' at a point e', say, and we know that $ee', pp', e'p'$ are minimal initial segments. If e' is in the interior of ed'', then $\beta(p)$ crosses $d''c \cup cd'$, which is impossible, as $d''c, d'c$ are minimal and c is externally blocked in $cd''ed'$. But, $c''d''$ is minimal, so we must have $e' = d''$, but then d''

has three initial segments passing through it, $d''p', d''e, d''b_3$, which is impossible as well. Hence, Δ_3 is not one of basic types.

If the type is $\mathbf{I_1}$ or $\mathbf{I_2}$, then q is internally blocked in the region $c''qx_3$. Since e is externally blocked in the region pqe, p is internally blocked in that region. Let the blocking segment through q intersect ep at a point r, say. By the definition of types $\mathbf{I_1}$ and $\mathbf{I_2}$, there is another initial segment r_1r_2 through r, with $r_1 \in px_3$, $r_2 \in eq$. Let $r_3 \in \partial T$ be the vertex such that $r_1r_3 = l(r_1r_2) \cap T$. Since $r_1 \in x_2x_3$ and r_1 and r_3 are on different sides of $l(c'p)$, we have $r_3 \in x_1x_2$. However, p is internally blocked in the region pqe, and thus in the region $x_2c'p$, which thus have one of the types $\mathbf{B_1}, \mathbf{I_1}$ or $\mathbf{I_2}$, and also intersecting segments eq and rr_3, which is a contradiction with the definitions of all these three types. Hence, $c''qx_3$ has type $\mathbf{T}$.

Since the type of Δ_3 is $\mathbf{T}$ and $c''d''$ is a minimal segment, it follows that $c''b_3$ and b_3d'' are also minimal, and that these three segments bound a minimal $\mathcal{S}_{\Delta_3}$-region. Let R be the other minimal $\mathcal{S}_{\Delta_3}$-region that contains the segment b_3d'', which is a hexagon by definition of $\mathbf{T}$, as $\beta(d'')$ crosses qx_3. Let u, v, w be the vertices of R such that u, b_3, d'', v, w are consecutive on ∂R. As $l(a_3d'') \cap T$ is the blocking segment through d'', by the assumption (A) $l(ub_3), l(a_3d'')$ and $l(vw)$ are concurrent. But $l(ub_3) \cap l(a_3d'') = a_3$, and a_2a_3 is another initial segment through a_3, which is a contradiction, as otherwise three initial segments would be concurrent.

Case 1.2, $c'p$ and $c''q$ are disjoint. Look at the triangle blocking arrangement Δ_4 induced by the triangle $x_2c'p$. Let $f' := l(b_2d') \cap x_2x_3$ and $f'' := l(a_3d'') \cap x_2x_3$. Applying the $(n-1)$-classification theorem, Δ_4 has one of the seven defined types. However, it cannot have a basic type, as $d'f'$ gives a blocking segment with endpoints on x_2p and $c'p$, while a_2d' is an initial segment and $a_2 \in c'x_2$.

Suppose for a moment that Δ_4 has type $\mathbf{I_1}$ or $\mathbf{I_2}$. By definition of these types we see that $d'f'$ does not cross any initial segment, except at its endpoints. Thus, the segments $f'd', d'c, cd'', d''c''$ are minimal. Thus, there is no initial segment that crosses qx_3 and qc'' in their interiors. But looking at the triangle blocking arrangement Δ_5 induced by $c''qx_3$, and applying $(n-1)$-classification theorem, we see that Δ_5 has to have one of the seven types defined, and no type satisfies the requirements that $d''b_3$ is an initial segment, $d''f''$ is a blocking segment, and there are no initial segments with a vertex on qx_3 and a vertex on qc'', which is a contradiction. Therefore, Δ_4 must be of type $\mathbf{T}$, and similarly, Δ_5 must be of the same type.

By definition of the type $\mathbf{T}$, there are vertices $p' \in px_2, p'' \in pc'$, such that pp', pp'' are minimal segments and $p'p''$ is an initial segment. Similarly, there are vertices $q' \in qx_3, q'' \in qc''$ such that qq', qq'' are minimal segments and $q'q''$ is an initial segment. Also, recalling that $c'd'$ and $c''d''$ are minimal segments, and using the definition of $\mathbf{T}$, we have that a_2d' and b_3d'' are also minimal segments. Thus $l(p'p'')$ and $l(q'q'')$ cannot cross $a_2c \cup cb_3$, so $l(p'p'')$ crosses x_1x_3, at r_3, say, and $l(q'q'')$ crosses x_1x_2, at r_2, say. In particular, r_2q' and r_3p' intersect, at some point r. Also, r_2q' must intersect $c'p$, at some point r'. However, we then have the following structure. In the triangle blocking arrangement induced by x_2r_2q', the vertex q' is externally blocked (by minimality of the region $qq'q''$), $p, p' \in x_2q', r, r' \in r_2q'$ and
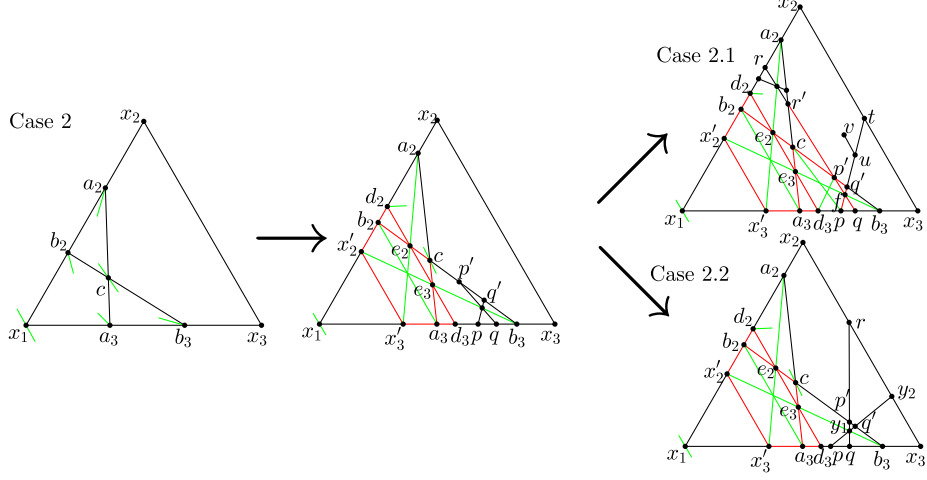
FIGURE 4. Case 2 of proof of Proposition 4.1

$pr', p'r$ are intersecting initial segments. Applying $(n-1)$-classification theorem, we obtain a contradiction, as none of the seven types has this structure. This concludes the proof in the Case 1.

Case 2. Consider the triangle blocking arrangement Δ_1 induced by the triangle $x_1 a_2 a_3$. By the classification theorem Δ_1 has type $\mathbf{B_1}, \mathbf{I_1}$ or $\mathbf{I_2}$. Immediately, we see that the basic type is not possible here, as initial segment $b_2 c$ and $\beta(a_2)$ intersect. Further, the point $e_2 := b_2 c \cap \beta(a_2)$, has the property that $b_2 e_2$ and $e_2 c$ are both minimal segments. Let $d_2 e_3$ be the other initial segment through e_2, with $d_2 \in x_1 a_2$, $e_3 \in a_2 a_3$. By the definition of types $\mathbf{I_1}$ and $\mathbf{I_2}$, we have $d_2 \in a_2 b_2$, $e_3 \in c a_3$. Let $d_3 := l(d_2 e_2) \cap x_1 x_3$. Arguing similarly for the triangle blocking arrangement Δ_2 induced by the triangle $x_1 b_2 b_3$, we obtain that the segments $b_2 d_2, d_2 e_2, e_2 b_2, e_2 c, c e_3, e_2 e_3, e_3 d_3, d_3 a_3, a_3 e_3$ are all minimal. From this, we also see that Δ_1 and Δ_2 must both have type $\mathbf{I_1}$. Writing $x_2' := \beta(b_3) \cap x_1 x_2$, $x_3' := \beta(a_2) \cap x_1 x_3$, we also obtain that $x_2' x_3'$ is a minimal initial segment, that $x_2' b_2$ and $x_3' a_3$ are also minimal and $b_2 a_3$ is a blocking segment.

Observe $d_2 c$ and $d_3 c$ cannot both be blocking segments, as otherwise $d_2 d_3$ would simultaneously be blocking and initial. Without loss of generality, $d_3 c$ is not a blocking segment. By the definition of type $\mathbf{I_1}$, there are vertices $p, q \in d_3 b_3$ and $p', q' \in c b_3$, such that $d_3 p, p q, c p', p' q'$ are minimal and $p' q, p q'$ are initial segments and $p' q, p q', b_3 e_3$ are concurrent. We now consider cases on the position of intersection r of $l(p'q)$ and ∂T, other than q.

Case 2.1, $r \in x_1 x_2$**.** The type of triangle blocking arrangement Δ_3 induced by $x_1 r q$ must be $\mathbf{T}$, as x_1 and q are externally blocked, and some of the initial segments intersect. As $d_2 e_2, e_2 c, c p'$ are minimal segments, we must have $r \in d_2 x_2$. If $r \in a_2 x_2$, then we obtain a contradiction from the type $\mathbf{T}$ for Δ_3 and the fact that $a_2 a_3$ and $d_2 d_3$ intersect, while $a_2, d_2 \in x_1 r$, $a_3, d_3 \in x_1 q$. Therefore $r \in d_2 a_2$,

FIGURE 5. Case 3 of proof of Proposition 4.1

and rq intersects a_2c, at point r', say. Recall that the segments $q'p'$ and cp' are minimal. The minimality of the segment cp' and the fact that the type of triangle blocking arrangement Δ_3 is $\mathbf{T}$ imply that segments cr' and $p'r'$ are minimal as well. Returning to the triangle blocking arrangement Δ_1 induced by $x_1a_2a_3$, which is of type $\mathbf{I_1}$, as r is internally blocked in a_2rr', we have vertices $s \in d_2r, s' \in a_2r'$, such that $rs, r's'$ are minimal and rr', ss' are initial segments that are concurrent with a_2e_2.

Extend pq' to the other intersection t of $l(pq')$ and ∂T, other than p. If $t \in x_1x_2$, applying the $(n-1)$-classification theorem for the triangle blocking arrangement Δ_4 induced by x_1tp, we have that Δ_4 has type $\mathbf{B_1}, \mathbf{I_1}$ or $\mathbf{I_2}$, as p is internally blocked. But, the fact that hexagonal region $x_2'x_3'a_3e_3e_2b_2$ is minimal $\overline{\mathcal{S}_{\Delta_4}}$-region with edges on x_1t and x_1p is in contradiction with the definitions of these two types. Therefore, $t \in x_2x_3$.

Let $f := pq' \cap p'q$. Applying $(n-1)$-classification theorem to the triangle blocking arrangement induced by x_3pt, and observing that $q'f, qf$ are minimal segments, $q'b_3$ is initial and fb_3 is blocking, we must have that qq' is also a blocking segment. Consider the minimal $\overline{\mathcal{S}}$-region R, that contains vertices s', r', p' and q'. Let u, v be the two vertices of R, such that p', q', u, v are consecutive. As $p'r'$ and $\beta(q') = q'q$ intersect at q, by the assumption (A), the line $l(uv)$ must also pass through q. However, the only initial segment other than rq through q is x_1x_3, which is disjoint from R, and we reach a contradiction.

Case 2.2, $r \in x_2x_3$. Applying the classification theorem to the triangle blocking arrangement induced by qrx_3, we obtain a contradiction as q is internally blocked in this region, while $y_1y_2 := l(pq') \cap qrx_3$ and $p'b_3$ intersect, while $p', y_1 \in qr, y_2 \in rx_3, b_3 \in qx_3$, and none of the defined types has this substructure.

$\boxed{\textbf{Case 3.}}$ Let Δ_1 be the triangle blocking arrangement induced by the triangle $x_1b_2b_3$. Applying the classification theorem we see that Δ_1 has type $\mathbf{B_1}, \mathbf{I_1}$ or $\mathbf{I_2}$. But, $\beta(b_3)$ crosses ca_3, discarding $\mathbf{B_1}$ as an option. Therefore, there are collinear vertices $d_3 \in x_1a_3, e_3 \in a_3c, f_3 \in cb_3$ such that $d_3a_3, e_3d_3, e_3f_3, e_3a_3, e_3c$ and cf_3 are minimal, d_3f_3 is initial and e_3b_3 is blocking. Let $p \neq d_3$ be the other intersection

point of $l(d_3 f_3)$ and ∂T. If $p \in x_1 x_2$, then segments $b_2 b_3$ and $d_3 p$ satisfy conditions of the Case 2, which we have proved to be impossible. Hence, $p \in x_2 x_3$.

Apply the $(n-1)$-classification theorem to the triangle blocking arrangement induced by $p d_3 x_3$. Since $e_3 f_3$ and $e_3 a_3$ are minimal, it follows that $f_3 b_3$ and $a_3 b_3$ are also minimal and $a_3 f_3$ is blocking. But, look at the minimal $\overline{\mathcal{S}}$-region R, that has f_3 and c as two vertices, but not e_3. Since $l(a_2 c)$ and $\beta(f_3)$ meet at a_3, by the assumption (A), we have $a_3 \in l(s)$, where s is another segment of R. But, the only other initial segment through a_3 is $x_1 x_3$, which is disjoint from R, thus we have a contradiction. $\square$

PROPOSITION 4.2. *Let $n \geqslant 1$ be an integer and suppose that $(n-1)$-classification theorem holds. Let $\Delta = (T, \mathcal{S}, \mathcal{B})$ be a triangle blocking arrangement of size n, and let x_1, x_2 and x_3 be the vertices of the triangle T. Suppose there are blocking segments through x_1 and x_2. Suppose also that $a_1 a_2, b_1 b_2 \in \mathcal{S}$ are two initial segments, such that $a_1 \in x_1 x_2$, $b_2 \in x_2 x_3$, $a_2, b_1 \in x_1 x_3$. Then $a_1 a_2$ and $b_1 b_2$ are disjoint.*

PROOF. Let c be the intersection $a_1 a_2 \cap b_1 b_2$. Depending on the blocking segment through c and a_1, we have the following four cases.

Case 1 In the region $x_2 a_1 c b_2$, both a_1 and c are internally blocked.
Case 2 In the region $x_2 a_1 c b_2$, a_1 is internally blocked, while c is externally blocked.
Case 3 In the region $x_2 a_1 c b_2$, a_1 is externally blocked, while c is internally blocked.
Case 4 In the region $x_2 a_1 c b_2$, both a_1 and c are externally blocked.

We treat each case separately and we depict the steps of the proof in Figures 6, 7, 8 and 9.

$\boxed{\textbf{Case 1.}}$ We have b_2 externally blocked in the region $b_1 b_2 x_3$, and in the region $b_1 c a_2$, the vertex b_1 is externally blocked, while a_2 is internally blocked. Applying the $(n-1)$-classification theorem to the triangle blocking arrangement induced by $b_1 b_2 x_3$, it has a basic type or $\textbf{T}$. In either of these cases, from the definition of the types, there are vertices $d \in b_1 a_2$, $e \in b_1 c$ such that ce, ed, da_2 are minimal initial segments, and $cd, a_2 e$ are blocking. Applying the $(n-1)$-classification theorem to the triangle blocking arrangement induced by the triangle $x_1 a_1 a_2$, we see that it has type $\textbf{I}_1$ or $\textbf{I}_2$, so $l(de)$ must cross the segment $a_1 c$, at some point f, and also fc is minimal. Let R be the minimal $\overline{\mathcal{S}}$-region with vertices c, f, but not e. Let f', u, v be the vertices of R, such that f', f, c, u, v are consecutive, appearing in this order of ∂R. As $l(ff') = l(de)$ and $\beta(c)$ intersect at d, by the assumption (A), we must have $d \in l(uv)$. However, the other initial segment through d, apart from de, is $x_1 x_3$, which is disjoint from R, and we have a contradiction in this case.

$\boxed{\textbf{Case 2.}}$ We have b_2 internally blocked in the region $b_1 b_2 x_3$, and in the region $b_1 c a_2$, the vertices b_1 and a_2 are internally blocked. Let $e := \beta(a_2) \cap b_1 c$. Applying the $(n-1)$-classification theorem to the triangle blocking arrangement induced by $x_1 a_1 a_2$, we see that its type is either $\textbf{I}_1$ or $\textbf{I}_2$. In either case, there are vertices $d \in x_1 b_1$, $f \in a_2 c$ such that $b_1 d, de, ef, fc$ are minimal initial segments. Note that
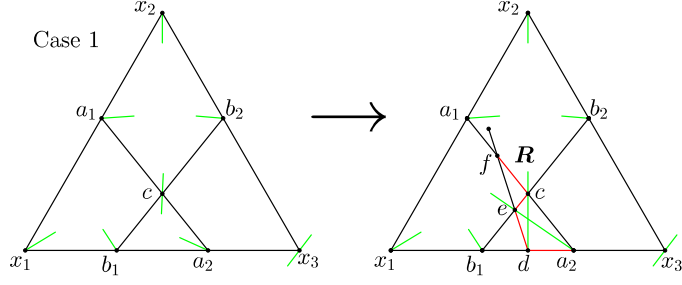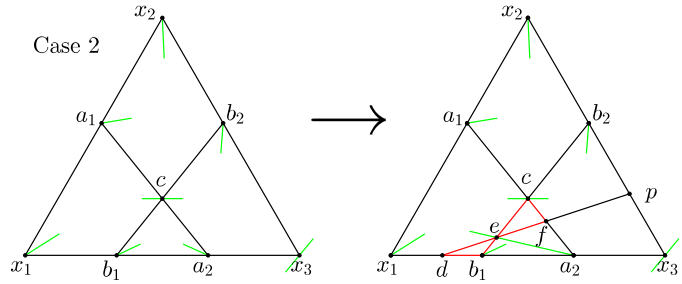
FIGURE 6. Case 1 of proof of Proposition 4.2

FIGURE 7. Case 2 of proof of Proposition 4.2

$l(ef)$ intersects the segment b_2x_3, at some point p, say. However, applying $(n-1)$-classification theorem to the triangle blocking arrangement induced by $b_1b_2x_3$, we obtain a contradiction, as no type permits a configuration where b_1, b_2 are internally blocked, and ca_2, ep cross.

Case 3. We have b_2 internally blocked in the region $b_1b_2x_3$, and in the region b_1ca_2, the vertex b_1 is internally blocked, while a_2 is externally blocked. Apply the $(n-1)$-classification theorem to the triangle blocking arrangement induced by $x_1a_1a_2$. Its type is one of $\mathbf{B_1}, \mathbf{I_1}$ and $\mathbf{I_2}$, but in any case, b_1c is minimal, and there are vertices $d \in b_1a_2$, $e \in ca_2$ such that ce, b_1d and de are minimal initial segments, and cd and b_1e are blocking. Apply the $(n-1)$-classification theorem to the triangle blocking arrangement induced by $b_1b_2x_3$, which must then have type $\mathbf{I_1}$ or $\mathbf{I_2}$, and in particular $l(de)$ crosses cb_2, at a point f, say.

Let $p \neq d$ be the other point of intersection of $l(de)$ with ∂T. If $p \in x_2x_3$, then the segments pd, b_1b_2 and the vertex x_3 form a configuration that is impossible by Proposition 4.1, hence $p \in x_1x_2$. As pd crosses a_1a_2 at e, we actually have $p \in a_1x_2$. However, applying $(n-1)$-classification theorem to the triangle blocking arrangement induced by px_1d, we obtain a contradiction, as no type allows a subconfiguration, where b_1f and a_1e cross, x_1 and d are internally blocked, and $e, f \in pd$, $b_1 \in x_1d$, $a_1 \in x_1p$.
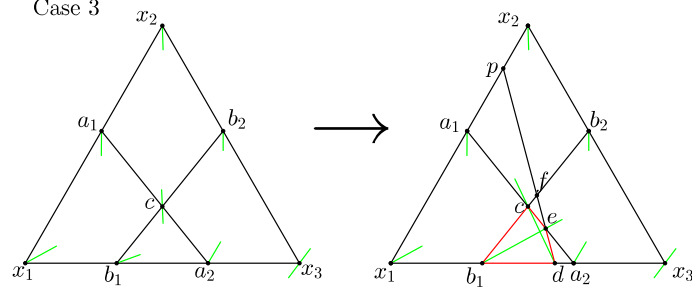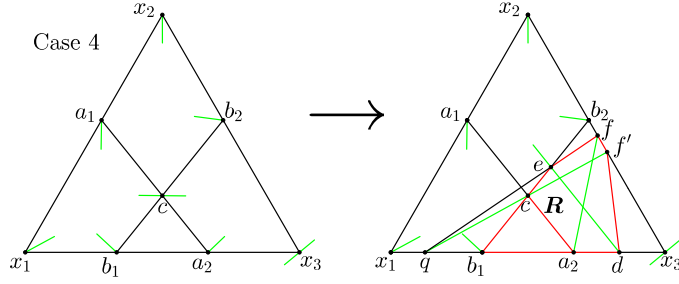
FIGURE 8. Case 3 of proof of Proposition 4.2

FIGURE 9. Case 4 of proof of Proposition 4.2

Case 4. We have b_2 externally blocked in the region $b_1b_2x_3$, and in the region b_1ca_2, the vertices b_1 and a_2 are externally blocked. Applying the $(n-1)$-classification theorem to the triangle blocking arrangement Δ_1 induced by $x_1a_1a_2$, we see that its type is $\mathbf{B_1}, \mathbf{I_1}$ or $\mathbf{I_2}$. In particular, the triangle blocking arrangement induced by b_1a_2c is of type $\mathbf{B_0}$ or $\mathbf{B_1}$ and b_1c is minimal. On the other hand, applying the $(n-1)$-classification theorem to the triangle blocking arrangement Δ_2 induced by $b_1b_2x_3$, which must have a basic type or $\mathbf{T}$, we also see that as b_1c is minimal, so are b_1a_2 and a_2c.

Assume now that there is no initial segment with one vertex on cb_2 and the other vertex on a_2x_3. Thus Δ_2 has type $\mathbf{T}$, and there are vertices $d \in a_2x_3$, $f, f' \in x_3b_2$, $e \in b_2c$ such that $a_2d, df', f'f, fe, ec$ are minimal initial segments, that bound a minimal $\overline{\mathcal{S}}$-region R, and a_2f, de, cf' are blocking. Observe that by the assumption (A) for R, as $l(a_2c)$ and $l(ff')$ are disjoint in T, we must have $l(de)$ disjoint from these two lines in T, as well. Hence $l(de)$ crosses segment a_1x_2. Thus, $\beta(x_2)$ does not pass through e and c, and fe, ec, cb_1 are minimal, so $\beta(x_2)$ has to cross a_1c and either crosses x_1b_1 or contains b_1.

Next, we show that $l(cf')$ crosses x_1x_3. If Δ_1 has type $\mathbf{B_1}$, then this is true. If $l(cf')$ does not cross x_1x_3, then, Δ_1 has type $\mathbf{I_1}$ or $\mathbf{I_2}$, but in that case, the only blocking segments that could cross $l(cf')$ in Δ_1 are $\beta(x_1), \beta(a_1)$ and $\beta(b_1)$. Thus $\beta(x_2)$ passes through b_1. But $\beta(b_1)$ crosses the interior of a_1c, but types $\mathbf{I_1}$ and $\mathbf{I_2}$ imply that $\beta(b_1)$ crosses a_1x_1, which is a contradiction.

Hence, $l(cf')$ crosses x_1x_3. Then, by the assumption (A) for region R, $l(ef)$, $l(cf')$, $l(a_2d)$ are concurrent at a point $q \neq x_2$. However, Proposition 4.1 gives a contradiction, when applied to the vertex x_3 and the segments fq and b_1b_2.

Finally, we assume that there is an initial segment de with $e \in cb_2$ and $d \in a_2x_3$. To obtain a contradiction, we consider the following three cases on the position of $p \neq d$, the other intersection of $l(de)$ with ∂T.

Case 4.1. Suppose that $p \in x_1a_1$. Applying $(n-1)$-classification theorem to the triangle blocking arrangement induced by x_1pd, we see that d must be internally blocked in this region. It follows that e is internally blocked in the region ca_2de. However, pd and b_1b_2 satisfy the conditions of the Case 1, which is impossible.

Case 4.2. Suppose that $p \in a_1x_2$. Applying $(n-1)$-classification theorem to the triangle blocking arrangement induced by x_1pd results in a contradiction as a_1a_2 and b_1e cross, but x_1 is internally blocked.

Case 4.3. Suppose that $p \in x_2x_3$. We actually have $p \in x_2b_2$ and Proposition 4.1 says that configuration where x_3 is externally blocked, and b_1b_2 and dp cross is impossible. $\square$

PROPOSITION 4.3. *Let $n \geqslant 1$ be an integer and suppose that $(n-1)$-classification theorem holds. Let $\Delta = (T, \mathcal{S}, \mathcal{B})$ be a triangle blocking arrangement of size n, and let x_1, x_2 and x_3 be the vertices of the triangle T. Suppose there are blocking segments through x_1 and x_3. Suppose also that $a_1a_2, b_1b_2 \in \mathcal{S}$ are two initial segments, such that $a_1 \in x_1x_2$, $b_2 \in x_2x_3$, $a_2, b_1 \in x_1x_3$. Then a_1a_2 and b_1b_2 are disjoint.*

PROOF. Suppose, to the contrary, that a_1a_2 and b_1b_2 intersect at point c. Depending on the blocking segments through c and b_1, up to symmetry, we have the following three cases.

Case 1 In the region cb_1a_2, c is externally blocked, while b_1 is internally blocked.
Case 2 In the region cb_1a_2, both b_1 and c are externally blocked.
Case 3 In the region cb_1a_2, both b_1 and c are internally blocked.

As before, we treat each case separately and we depict the steps of the proof in Figures 10, 11 and 12.

Case 1. Note that the vertex a_2 is internally blocked in the region cb_1a_2, and that a_1 and b_2 are internally blocked in the region $x_2a_1cb_2$. Let $d := \beta(a_2) \cap cb_1$, $e := \beta(b_1) \cap ca_2$. Applying the $(n-1)$-classification theorem to the triangle blocking arrangement Δ_1 induced by $a_1a_2x_1$, we see that Δ_1 has type among $\mathbf{I_1}$ and $\mathbf{I_2}$. In either case, dc and db_1 are minimal segments, and no initial segment may cross the interior of the segment b_1e. Similarly, by looking at the triangle blocking arrangement Δ_2 induced by $b_1b_2x_3$, we have that ce, ea_2 are minimal, and no initial segment crosses the interior of a_2d. It follows that the initial segment through d, which is different from cb_1, must be de, so de is an initial segment.

However, the type of Δ_1 implies that $l(de)$ crosses x_1b_1, while the type of Δ_2 implies that $l(de)$ crosses a_2x_3, so $l(de)$ crosses segment x_1x_3 twice, which is impossible.
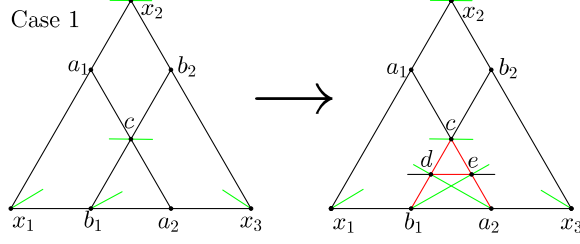
FIGURE 10. Case 1 of Proposition 4.3

Case 2. Note that the vertex a_2 is externally blocked in the region cb_1a_2, and that a_1 and b_2 are externally blocked in the region $x_2a_1cb_2$. Applying the $(n-1)$-classification theorem to the triangle blocking arrangement Δ_1 induced by $a_1a_2x_1$, the type of Δ_1 is among $\mathbf{B_1}, \mathbf{I_1}$ or $\mathbf{I_2}$. In either case, the segment b_1c is minimal. Similarly, looking at the triangle blocking arrangement Δ_2 induced by $b_1b_2x_3$, the segment ca_2 is also minimal.

Suppose for a moment that $\beta(c)$ crosses a_2x_3, at a point e. Let R be the minimal $\overline{\mathcal{S}}$-region with vertices b_1 and c, but not a_2. Let u, v, w be the vertices of R, such that u, b_1, c, v, w are consecutive and appear in that order on ∂R, thus wv, vc are minimal. As $l(ub_1)$ and $\beta(c)$ intersect at e, by the assumption (A), $e \in l(vw)$. Also $l(vw)$ crosses cb_2, let d be their intersection point. From the $(n-1)$-classification theorem applied to Δ_2, we see that cd, de, a_2e are minimal and a_2d is blocking. As cd, cv are minimal, so is vd. But, if we look at the minimal $\overline{\mathcal{S}}$-region R', with vertices v, d, but not c, since $l(vc) = l(s_1)$ and $\beta(d)$ intersect at a_2 (where s_1 is the segment of $\partial R'$ through v, different from vd), it follows by the assumption (A), that $a_2 \in l(s_2)$ for another segment s_2 of $\partial R'$. However, $a_2 \in x_1x_3$, and x_1x_3 is disjoint from R', which is a contradiction. Therefore, $\beta(c)$ is disjoint from a_2x_3, and by symmetry $\beta(c)$ is also disjoint from x_1b_1.

Thus $\beta(c)$ crosses a_1x_1, at some point d, and crosses b_2x_3, at some point e. Hence, the triangle blocking arrangements Δ_1 and Δ_2 are of the interesecting types. Further, $\beta(x_1)$ crosses a_1c, at a point f, and $\beta(x_3)$ crosses b_2c, at a point f'. From the types of Δ_1 and Δ_2, we also have that $fc, f'c$ are minimal, and there are points $p \in a_1d$, and $p' \in b_2e$, such that pf and $p'f'$ are initial segments. Further, also from the types of Δ_1 and Δ_2, we have that if an initial segment s crosses fx_1, then s must have one vertex on a_1x_1 and the other on x_1b_1, and we have that if an initial segment s crosses x_3f', then s has one vertex on b_2x_3, and the other on a_2x_3. It follows that $l(pf) \cap \text{int } f'x_3 = l(p'f') \cap \text{int } fx_1 = \emptyset$.

Using the facts that $l(pf) \cap \text{int } f'x_3 = \emptyset$ and that cf' is a minimal segment, we infer that $l(pf)$ intersects the segment x_2x_3. However, Proposition 4.2 applies to segments $s(pf)$ (the segment in $\overline{\mathcal{S}}$ that contains the points p and f) and a_1a_2 inside $x_1x_2x_3$ to give a contradiction.

Case 3. Note that the vertex a_2 is externally blocked in the region cb_1a_2, and that a_1 is externally and b_2 is internally blocked in the region $x_2a_1cb_2$. Applying the $(n-1)$-classification theorem to the triangle blocking arrangement Δ_1 induced by
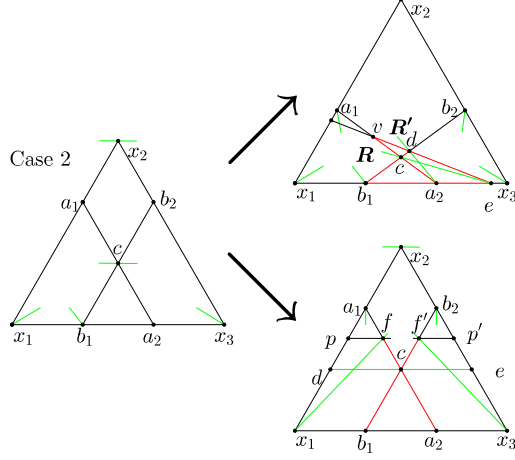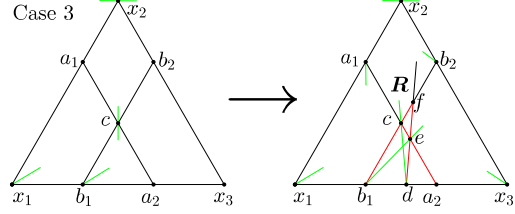
FIGURE 11. Case 2 of Proposition 4.3

FIGURE 12. Case 3 of Proposition 4.3

$a_1 a_2 x_1$, we have that the type of Δ_1 is among $\mathbf{B_1}, \mathbf{I_1}$ and $\mathbf{I_2}$. In either case, $b_1 c$ is a minimal segment. Next, applying the $(n-1)$-classification theorem to the triangle blocking arrangement Δ_2 induced by $b_1 b_2 x_3$, the type of Δ_2 is either $\mathbf{I_1}$ or $\mathbf{I_2}$. Since $b_1 c$ is minimal, setting $d := \beta(c) \cap b_1 a_2$, $e := \beta(b_1) \cap c a_2$, we must have that de is an initial segment, $l(de)$ crosses cb_2, at some point f, and $cf, ce, fe, de, ea_2, da_2$ are minimal. Let R be a minimal $\overline{\mathcal{S}}$-region such that $c, f \in \partial R$, but $e \notin \partial R$. Let u, v, w be the vertices of R, such that u, f, c, v, w are consecutive and appear in this order on ∂R. Since $\beta(c)$ and $l(uf) = l(fe)$ meet at d, by the assumption (A), it follows that $d \in l(vw)$. However, the other initial segment through d, apart from de, is $x_1 x_3$, which is disjoint from R, which is a contradiction. $\qquad \square$

LEMMA 4.2. *Let $n \geqslant 1$ be an integer and suppose that $(n-1)$-classification theorem holds. Let $\Delta = (T, \mathcal{S}, \mathcal{B})$ be a triangle blocking arrangement of size n, and let x_1, x_2 and x_3 be the vertices of the triangle T. Suppose that there are vertices $a_1 \in x_1 x_2$, $a_2, b_1 \in x_1 x_3$, $b_2 \in x_2 x_3$ such that $a_1 a_2$ and $b_1 b_2$ are initial segments, intersecting at a point c. Then, in the region $a_1 c b_2 x_2$, the vertices c, x_2 are externally blocked and a_1, b_2 are internally blocked. Also, x_1, x_3 are externally blocked, and in the region $cb_1 a_2$, the vertices a_2, b_1 are externally blocked.*
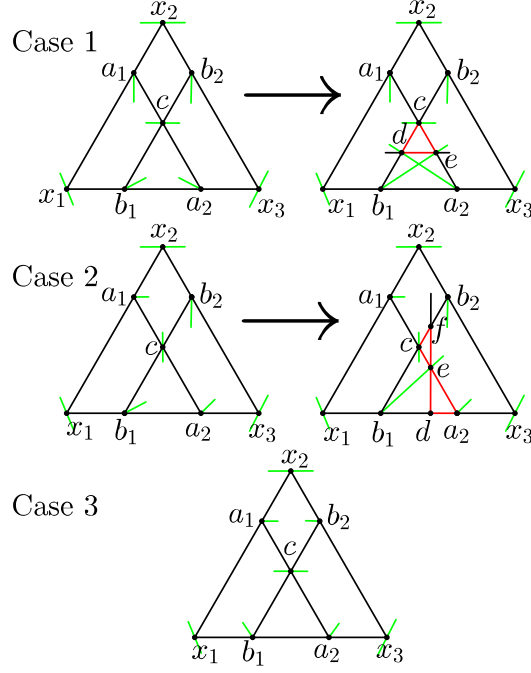
FIGURE 13. Cases in the proof of Lemma 4.2

PROOF. By Propositions 4.2 and 4.3, we have x_1, x_2, x_3 all externally blocked. Looking at the blocking segments $\beta(c)$ and $\beta(a_2)$, we have the following three cases, up to symmetry.

Case 1 In the region $a_1cb_2x_2$, the vertices a_1, c, b_2, x_2 are externally blocked. In the region cb_1a_2, the vertices a_2, b_1 are internally blocked.

Case 2 In the region $a_1cb_2x_2$, the vertices a_1, c are internally blocked and b_2, x_2 are externally blocked. In the region cb_1a_2, the vertex b_1 is internally blocked and a_2 is externally blocked.

Case 3 In the region $a_1cb_2x_2$, the vertices a_1, b_2 are internally blocked and c, x_2 are externally blocked. In the region cb_1a_2, the vertices a_2, b_1 are externally blocked.

Observe that the Case 3 is exactly the conclusion of the lemma, so we just need to discard the first two cases. As before, we depict the steps in the proof in Figure 13.

$\boxed{\textbf{Case 1.}}$ Let $d := \beta(a_2) \cap b_1c$, $e := \beta(b_1) \cap a_2c$. By the $(n-1)$-classification theorem applied to the triangle blocking arrangement Δ_1 induced by $a_1a_2x_1$, the segment cd is minimal, and there is another initial segment through d that crosses ce, possibly through e. Similarly, by applying the $(n-1)$-classification theorem to the triangle blocking arrangement Δ_2 induced by $b_1b_2x_3$, ce is also minimal segment, so it follows that de is itself a minimal initial segment. But, as Δ_1 has type $\mathbf{I_1}$ or
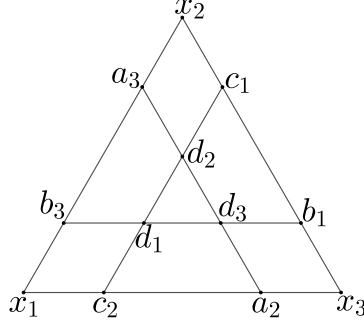
FIGURE 14. Assumptions in Proposition 4.4

$\mathbf{I_2}$, it follows that $l(de)$ crosses x_1b_1. Similarly, from the type of Δ_2, $l(de)$ crosses a_2x_3, hence $l(de)$ meets x_1x_3 twice, which is impossible.

$\boxed{\text{Case 2.}}$ Let $e := \beta(b_1) \cap ca_2$. By the $(n-1)$-classification theorem applied to the triangle blocking arrangement Δ_1 induced by $b_1b_2x_3$, Δ_1 has type $\mathbf{I_1}$ or $\mathbf{I_2}$, and in particular, there are vertices $d \in b_1a_2$, $f \in cb_2$, such that d, e, f lie on the same initial segment, and $ce, ef, fc, ed, da_2, a_2e$ are minimal. Let $p \neq d$, be the other intersection of the line $l(de)$ with ∂T. If $p \in x_2x_3$, then we have a substructure where b_1b_2, dp intersect, and x_3 is externally blocked, which is forbidden by Proposition 4.1, yielding a contradiction. Therefore, $p \in x_1x_2$. Once again, Proposition 4.1 gives contradiction, as a_1a_2, dp intersect and x_1 is externally blocked. $\square$

PROPOSITION 4.4. *Let $n \geqslant 1$ be an integer and suppose that $(n-1)$-classification theorem holds. Let $\Delta = (T, \mathcal{S}, \mathcal{B})$ be a triangle blocking arrangement of size n, and let x_1, x_2 and x_3 be the vertices of the triangle T. Suppose that there are vertices $a_3, b_3 \in x_1x_2$, appearing in order x_1, b_3, a_3, x_2, $c_2, a_2 \in x_1x_3$, appearing in order x_1, c_2, a_2, x_3, and $b_1, c_1 \in x_3x_2$, appearing in order x_3, b_1, c_1, x_2, such that $a_2a_3, b_1b_3, c_1c_2 \in \mathcal{S}$. Let $d_1 = b_1b_3 \cap c_1c_2$, $d_2 = a_2a_3 \cap c_1c_2$, $d_3 = a_2a_3 \cap b_1b_3$. Suppose additionally that d_1 is in the interior of $x_1a_2a_3$. Then Δ has type $\mathbf{T}$.*

The assumptions of Proposition 4.4 are depicted in Figure 14.

PROOF. Apply the $(n-1)$-classification theorem to the triangle blocking arrangement Δ_a induced by $x_1a_3a_2$. As $b_1b_3 \cap \Delta_a$ and $c_1c_2 \cap \Delta_a$ intersect, it must have type $\mathbf{T}$. By definition, there are an integer $k \geqslant 2$ and vertices $u_1^{(a)}, u_2^{(a)}, \ldots, u_{2k}^{(a)}$ appearing on x_1a_3, in that order from x_1 to a_3, there are vertices $v_1^{(a)}, v_2^{(a)}, \ldots, v_{2k}^{(a)}$ appearing on a_3a_2, in that order from a_3 to a_2, and there are vertices $w_1^{(a)}, w_2^{(a)}, \ldots, w_{2k}^{(a)}$ appearing on a_2x_1, in that order from a_2 to x_1, such that segments $u_i^{(a)}u_{i+1}^{(a)}$, $v_i^{(a)}v_{i+1}^{(a)}$, $w_i^{(a)}w_{i+1}^{(a)}$ are minimal, and the triangle blocking arrangements induced by $x_1u_1^{(a)}w_{2k}^{(a)}$, $a_3v_1^{(a)}u_{2k}^{(a)}$ and $a_2w_1^{(a)}v_{2k}^{(a)}$, are of type $\mathbf{B_0}$ or $\mathbf{B_1}$, with minimal initial segments $u_1^{(a)}w_{2k}^{(a)}, v_1^{(a)}u_{2k}^{(a)}$ and $w_1^{(a)}v_{2k}^{(a)}$. The initial segments, outside the three small regions $x_1u_1^{(a)}w_{2k}^{(a)}$, $a_3v_1^{(a)}u_{2k}^{(a)}$, $a_2w_1^{(a)}v_{2k}^{(a)}$ are given by $u_{2i-1}^{(a)}w_{2k+2-2i}^{(a)}$,

$v_{2i-1}^{(a)} u_{2k+2-2i}^{(a)}$, $w_{2i-1}^{(a)} v_{2k+2-2i}^{(a)}$, for $i = 1, 2, \ldots, k$, and the blocking segments outside the three small regions are given by $w_{2i-1}^{(a)} u_{2k+2-2i}^{(a)}$, $u_{2i-1}^{(a)} v_{2k+2-2i}^{(a)}$, $v_{2i-1}^{(a)} w_{2k+2-2i}^{(a)}$, for $i = 1, 2, \ldots, k$.

Similarly, apply the $(n-1)$-classification theorem to the triangle blocking arrangement Δ_b induced by $x_2 b_1 b_3$. It must have type $\mathbf{T}$. By definition, there are an integer $l \geqslant 2$ and vertices $u_1^{(b)}, u_2^{(b)}, \ldots, u_{2l}^{(b)}$ appearing on $b_3 x_2$, in that order from b_3 to x_2, there are vertices $v_1^{(b)}, v_2^{(b)}, \ldots, v_{2l}^{(b)}$ appearing on $x_2 b_1$, in that order from x_2 to b_1, and there are vertices $w_1^{(b)}, w_2^{(b)}, \ldots, w_{2l}^{(b)}$ appearing on $b_1 b_3$, in that order from b_1 to b_3, such that segments $u_i^{(b)} u_{i+1}^{(b)}$, $v_i^{(b)} v_{i+1}^{(b)}$, $w_i^{(b)} w_{i+1}^{(b)}$ are minimal, and the triangle blocking arrangements induced by $b_3 u_1^{(b)} w_{2l}^{(b)}$, $x_2 v_1^{(b)} u_{2l}^{(b)}$ and $b_1 w_1^{(b)} v_{2l}^{(b)}$, are of type $\mathbf{B_0}$ or $\mathbf{B_1}$, with minimal initial segments $u_1^{(b)} w_{2l}^{(b)}$, $v_1^{(b)} u_{2l}^{(b)}$, $w_1^{(b)} v_{2l}^{(b)}$. The initial segments, outside the three small regions $b_3 u_1^{(b)} w_{2l}^{(b)}$, $x_2 v_1^{(b)} u_{2l}^{(b)}$, $b_1 w_1^{(b)} v_{2l}^{(b)}$ are given by $u_{2i-1}^{(b)} w_{2l+2-2i}^{(b)}$, $v_{2i-1}^{(b)} u_{2l+2-2i}^{(b)}$, $w_{2i-1}^{(b)} v_{2l+2-2i}^{(b)}$, for $i = 1, 2, \ldots, l$, and the blocking segments outside the three small regions are given by $w_{2i-1}^{(b)} u_{2l+2-2i}^{(b)}$, $u_{2i-1}^{(b)} v_{2l+2-2i}^{(b)}$, $v_{2i-1}^{(b)} w_{2l+2-2i}^{(b)}$, for $i = 1, 2, \ldots, l$.

Apply the $(n-1)$-classification theorem to the triangle blocking arrangement Δ_c induced by $x_3 c_1 c_2$. It must have type $\mathbf{T}$, so by definition, there are and integer $m \geqslant 2$ and vertices $u_1^{(c)}, u_2^{(c)}, \ldots, u_{2m}^{(c)}$ appearing on $c_2 c_1$, in that order from c_2 to c_1, there are vertices $v_1^{(c)}, v_2^{(c)}, \ldots, v_{2m}^{(c)}$ appearing on $c_1 x_3$, in that order from c_1 to x_3, and there are vertices $w_1^{(c)}, w_2^{(c)}, \ldots, w_{2m}^{(c)}$ appearing on $x_3 c_2$, in that order from x_3 to c_2, such that segments $u_i^{(c)} u_{i+1}^{(c)}$, $v_i^{(c)} v_{i+1}^{(c)}$, $w_i^{(c)} w_{i+1}^{(c)}$ are minimal, and the triangle blocking arrangements induced by $c_2 u_1^{(c)} w_{2m}^{(c)}$, $c_1 v_1^{(c)} u_{2m}^{(c)}$ and $x_3 w_1^{(c)} v_{2m}^{(c)}$, are of type $\mathbf{B_0}$ or $\mathbf{B_1}$, with minimal initial segments $u_1^{(c)} w_{2m}^{(c)}$, $v_1^{(c)} u_{2m}^{(c)}$, $w_1^{(c)} v_{2m}^{(c)}$. The initial segments, outside the three small regions $c_2 u_1^{(c)} w_{2m}^{(c)}$, $c_1 v_1^{(c)} u_{2m}^{(c)}$, $x_3 w_1^{(c)} v_{2m}^{(c)}$ are given by $u_{2i-1}^{(c)} w_{2m+2-2i}^{(c)}$, $v_{2i-1}^{(c)} u_{2m+2-2i}^{(c)}$, $w_{2i-1}^{(c)} v_{2m+2-2i}^{(c)}$, for $i = 1, 2, \ldots, m$, and the blocking segments outside the three small regions are given by $w_{2i-1}^{(c)} u_{2l+2-2i}^{(c)}$, $u_{2i-1}^{(c)} v_{2m+2-2i}^{(c)}$, $v_{2i-1}^{(c)} w_{2m+2-2i}^{(c)}$, for $i = 1, 2, \ldots, m$.

If we look at the triangle blocking arrangement induced by $a_3 u_{2k}^{(a)} v_1^{(a)}$, which we know has type $\mathbf{B_0}$ or $\mathbf{B_1}$ with $u_{2k}^{(a)} v_1^{(a)}$ minimal, since Δ_b has type $\mathbf{T}$, it follows that $a_3 u_{2k}^{(a)} v_1^{(a)}$ actually is a minimal $\overline{\mathcal{S}}$-region, so $a_3 u_{2k}^{(a)}$ is also minimal. Similarly, it follows that $c_1 v_1^{(c)}, b_1 v_{2l}^{(b)}, a_2 w_1^{(a)}, c_2 w_{2m}^{(c)}$ and $b_3 u_1^{(b)}$ are minimal segments. Let i_a, i_b, i_c be such that $a_3 = u_{i_a}^{(b)}, b_1 = v_{i_b}^{(c)}, c_2 = w_{i_c}^{(a)}$. Define

$$u_i = u_i^{(a)}, \text{ for } 1 \leqslant i \leqslant 2k, \text{ and } u_{2k+i} = u_{i_a+i-1}^{(b)}, \text{ for } 1 \leqslant i \leqslant 2l+1-i_a,$$
$$v_i = v_i^{(b)}, \text{ for } 1 \leqslant i \leqslant 2l, \text{ and } v_{2l+i} = v_{i_b+i-1}^{(c)}, \text{ for } 1 \leqslant i \leqslant 2m+1-i_b,$$
$$w_i = w_i^{(c)}, \text{ for } 1 \leqslant i \leqslant 2m, \text{ and } w_{2m+i} = w_{i_c+i-1}^{(a)}, \text{ for } 1 \leqslant i \leqslant 2k+1-i_c.$$

Let $p_u = 2k + 2l + 1 - i_a$, $p_v = 2l + 2m + 1 - i_b$, $p_w = 2m + 2k + 1 - i_c$ (which are the lengths of these sequences). We now show that $p_u = p_v = p_w$. Let $p_{u,1}$ be the number of vertices u_i, whose initial segment $\neq x_1 x_2$ crosses $x_2 x_3$, let $p_{v,1}$

be the number of vertices v_i, whose initial segment $\neq x_2 x_3$ crosses $x_3 x_1$, let $p_{w,1}$ be the number of vertices w_i, whose initial segment $\neq x_3 x_1$ crosses $x_1 x_2$, and let $p_{u,2} = p_u - p_{u,1}$, $p_{v,2} = p_v - p_{v,1}$, $p_{w,2} = p_w - p_{w,1}$.

Firstly, we show that $p_{u,1} = p_{u,2}$ (so by symmetry $p_{v,1} = p_{v,2}$ and $p_{w,1} = p_{w,2}$). Note immediately that the initial segment $u_1 w_{p_w}$ crosses $x_1 x_2$ and $x_1 x_3$, and that the initial segment $u_{p_u} v_1$ crosses $x_1 x_2$ and $x_2 x_3$. Also, for any $i < p_u$, observe that, by the definition of type **T**, the initial segments $\neq x_1 x_2$ through u_i and u_{i+1} cannot both intersect $x_1 x_3$, nor can both intersect $x_2 x_3$. To spell out details, looking at Δ_a, if $i < 2k$, then one of u_i, u_{i+1} has the other initial segment with a vertex on $x_1 a_2$, and the second has the other initial segment with a vertex on $a_3 a_2$. Write temporarily q for this second vertex among u_i, u_{i+1} and $r \in a_3 a_2$ for the vertex such that qr is initial. But $l(qr)$ crosses $a_3 a_2$, and we must have $l(qr)$ cross $x_2 x_3$, as otherwise we obtain a contradiction by Proposition 4.1. We argue similarly for $i \geqslant 2k$, by considering Δ_b. The claim follows.

Secondly, we show $p_{u,1} = p_{v,2}$ (and by symmetry $p_{v,1} = p_{w,2}$ and $p_{w,1} = p_{u,2}$). But, if a segment crosses $x_1 x_2$ at some u_i, and crosses $x_2 x_3$ at a point q, then by minimality of $u_{p_u} v_1$ and $v_{p_v} w_1$, $q = v_j$ for some j. However, this is injective map $i \mapsto j$, so $p_{u,1} \geqslant p_{v,2}$, and by symmetry $p_{u,1} = p_{v,2}$. From these observations, it follows that $p_u = p_v = p_w$, and we may write p for this common value.

Next, we show that initial and blocking segments at u_i, v_i, w_i satisfy the conditions of the type **T**. As we have seen already, initial segments through u_i are $u_1 w_{i_1}, u_3 w_{i_3}, \ldots, u_{p-1} w_{i_{p-1}}$ and $u_2 v_{i_2}, u_4 v_{i_4}, \ldots, u_p v_{i_p}$, where $i_1, i_3, \ldots, i_{p-1} \in [p]$ are distinct and even, and $i_2, i_4, \ldots, i_p \in [p]$ are distinct and odd. However, if $i_j > i_{j'}$ holds for some $j > j'$ of the same parity, then we obtain a contradiction by Proposition 4.1. Hence, $i_j = p + 1 - j$, for all j, as desired, and a similar argument shows that all initial segments have desired structure. For blocking segments, observe that all initial and blocking segments between $x_1 x_2$ and $x_2 x_3$ through some u_i are disjoint, so the blocking segments have the desired structure. The intersections structure follows from the structure of $\Delta_a, \Delta_b, \Delta_c$.

Finally, we know from before that the triangle blocking arrangements induced by $x_1 u_1 w_p$, $x_2 v_1 u_p$, $x_3 w_1 v_p$ are of type **B$_0$** or **B$_1$**, with minimal segments $u_1 w_p$, $v_1 u_p$, $w_1 v_p$. This completes the proof. $\square$

COROLLARY 4.1. *Let $n \geqslant 1$ be an integer and suppose that $(n-1)$-classification theorem holds. Let $\Delta = (T, \mathcal{S}, \mathcal{B})$ be a triangle blocking arrangement of size n, and let x_1, x_2 and x_3 be the vertices of the triangle T. Suppose that there are vertices $a_3, b_3 \in x_1 x_2$, appearing in order x_1, b_3, a_3, x_2, $c_2, a_2 \in x_1 x_3$, appearing in order x_1, c_2, a_2, x_3, and $b_1, c_1 \in x_3 x_2$, appearing in order x_3, b_1, c_1, x_2, such that $a_2 a_3, b_1 b_3, c_1 c_2 \in \mathcal{S}$. Then Δ has type* **T**.

The two possibilities for the structure of the given points and segments are depicted in Figure 15. Notice that the first possibility already appears in Figure 14.

PROOF. Let $d_1 = b_1 b_3 \cap c_1 c_2$, $d_2 = a_2 a_3 \cap c_1 c_2$, $d_3 = a_2 a_3 \cap b_1 b_3$. By previous proposition, we may suppose that d_1 is not in the interior of $x_1 a_2 a_3$, and moreover
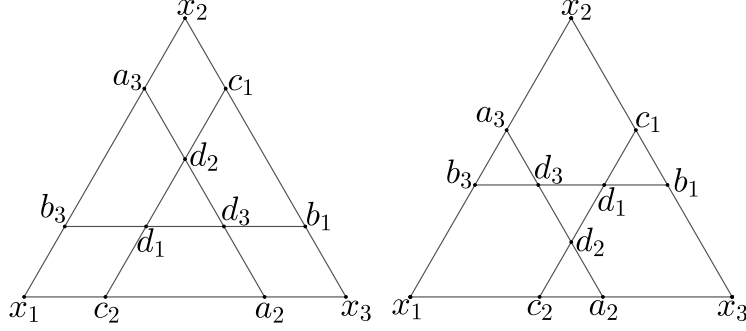
FIGURE 15. Two possibilities for assumptions in Corollary 4.1

that there are no triples of segments where each pair intersects and form a small triangle that satisfies the conditions of Proposition 4.4.

By Lemma 4.2, we have that x_1, x_2, x_3 are externally blocked, and in regions $a_3 b_3 d_3, a_2 c_2 d_2, b_1 c_1 d_1$, the vertices $a_3, b_3, d_3, a_2, c_2, d_2, b_1, c_1, d_1$ are externally blocked as well. Suppose for a moment that $a_3 d_3$ is not a minimal segment. Let $q \in a_3 d_3$ be another vertex. Let pr be another initial segment through q, with $p, r \in \partial T$, with $p \in a_3 x_1 \cup x_1 a_2$ and $r \in a_3 x_2 \cup x_2 x_3 \cup x_3 a_2$. If pr crosses $b_3 b_1$, then either Proposition 4.1 gives a contradiction, for pr and $b_3 b_1$ or pr and $a_3 a_2$, or the segments $pr, b_3 b_1, a_3 a_2$ form a triple we forbade at the beginning of the proof. Hence, $pr \cap b_3 b_1 = \emptyset$, so $p \in a_3 b_3, r \in x_2 b_1$. However, if we apply the $(n-1)$-classification theorem to the triangle blocking arrangement induced by $x_2 b_3 b_1$, it must have type **T**, and it follows that there is an initial segment s which crosses both $d_3 a_3$ and $d_3 b_3$. By Proposition 4.1, it follows that s crosses $x_1 x_3$ and $x_2 x_3$, thus $s, b_3 b_1, a_3 a_2$ forms a structure that we forbade at the beginning of the proof. This is a contradiction, and it follows similarly that $a_3 d_3, b_3 d_3, d_2 a_2, d_2 c_2, d_1 b_1, d_1 c_1$ are minimal, and further, $a_3 b_3, a_2 c_2, b_1 c_1$ are minimal as well.

Therefore, we have actually shown that any configuration of segments like $a_2 a_3, b_1 b_3, c_1 c_2$ either implies that the corollary is true, or it implies the minimality of the segments $a_3 d_3, b_3 d_3$, etc. Without loss of generality, we may assume the latter. Using this observation and Proposition 4.1, it follows also that $d_1 d_2, d_2 d_3$ and $d_3 d_1$ can be assumed to be minimal. Applying the $(n-1)$-classification theorem to the triangle blocking arrangements induced by $x_1 a_3 a_2, x_2 b_3 b_1$ and $x_3 c_2 c_1$, all three have the type **B₃**, and there are vertices $a'_2 \in x_1 c_2$, $a'_3 \in b_3 x_1$, $b'_3 \in a_3 x_2$, $b'_1 \in x_2 c_1$, $c'_1 \in b_1 x_3$, $c'_2 \in a_2 x_3$, such that $c_2 a'_2, a'_2 a'_3, a'_3 b_3, a_3 b'_3, b'_3 b'_1, b'_1 c_1, b_1 c'_1, c'_1 c'_2, c'_2 a_2$ are minimal initial segments, and $\beta(d_1) = b'_3 c'_2$, $\beta(d_2) = c'_1 a'_3$, $\beta(d_3) = a'_2 b'_1$. From the types of same triangle blocking arrangements, it follows that triangle blocking arrangements induced by $x_1 a'_2 a'_3, x_2 b'_3 b'_1, x_3 c'_1 c'_2$ are of type **B₀** or **B₁**, with $a'_2 a'_3, b'_1 b'_3, c'_1 c'_2$ minimal. Thus, Δ has the type **T**. $\square$

PROPOSITION 4.5. *Let $n \geqslant 1$ be an integer and suppose that $(n-1)$-classification theorem holds. Let $\Delta = (T, \mathcal{S}, \mathcal{B})$ be a triangle blocking arrangement of size n, and let x_1, x_2 and x_3 be the vertices of the triangle T. Suppose that there are vertices*
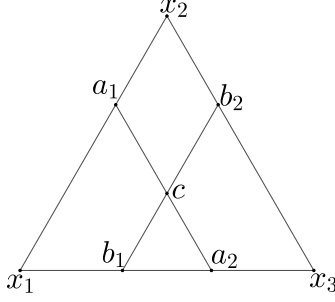
FIGURE 16. Assumptions in Proposition 4.5

$a_1 \in x_1 x_2$, $a_2, b_1 \in x_1 x_3$, $b_2 \in x_2 x_3$ such that $a_1 a_2$ and $b_1 b_2$ are initial segments, intersecting at a point c. Then Δ has type $\mathbf{T}$.

The assumptions of Proposition 4.5 are depicted in Figure 16.

PROOF. By Lemma 4.2, the vertices x_1, x_2, x_3 are externally blocked. Let $c_1, c_2, \ldots, c_k$ be the vertices that lie on $a_1 a_2$, in that order from a_1 to a_2. Thus, $k \geqslant 1$. Let the initial segment $\neq a_1 a_2$ through c_i be $p_i q_i$. Note that at least one of p_i, q_i must be on $x_2 x_3$, otherwise we obtain a contradiction using Proposition 4.1, without loss of generality, $q_i \in x_2 x_3$. Also, if $p_i \in x_1 x_3$, $p_{i+1} \in x_1 x_2$, then by Corollary 4.1, Δ has type $\mathbf{T}$. Thus, assume that there is i_0 such that $p_i \in x_1 x_2$ for $i \leqslant i_0$ and $p_i \in x_1 x_3$ for $i > i_0$. Moreover, by Proposition 4.1, on $a_1 x_1$, the vertices $a_1, p_1, p_2, \ldots, p_{i_0}, x_1$ appear in this order, and on $x_1 a_2$, the vertices $x_1 p_{i_0+1} p_{i_0+2} \ldots p_k a_2$ appear in this order. By Lemma 4.2, we also have a_1, p_i and c_i externally blocked in the region $a_1 p_i c_i$, for $i \leqslant i_0$, and a_2, p_i and c_i externally blocked in the region $a_2 p_i c_i$ for $i > i_0$. However, applying the $(n-1)$-classification theorem to the triangle blocking arrangement Δ_a induced by $a_1 a_2 x_1$, the only type that can be satisfied by Δ_a is $\mathbf{B_3}$.

From the definition of type $\mathbf{B_3}$, it follows that $k = 2$, $i_0 = 1$ and that $p_1 c_1, c_1 c_2, c_2 p_2$ are minimal, that there are vertices $r_1 \in x_1 p_1, r_2 \in x_1 p_2$ such that $r_1 r_2$ is a minimal initial segment, and also that $r_1 p_1, r_2 p_2$ are minimal. Furthermore, $r_2 c_1, r_1 c_2, p_1 p_2$ are all blocking segments.

Observe that, by the assumption (A), if $l(c_1 r_2)$ crosses $x_1 x_2$, then $l(p_2 c_2)$ must pass through the same point, however, $l(p_2 c_2)$ cuts $x_2 x_3$, which is a contradiction. Hence, $\beta(c_1)$ crosses the segment $x_2 q_1$, and similarly, $\beta(c_2)$ cuts $q_2 x_3$. Therefore, $(n-1)$-classification theorem applied to the triangle blocking arrangements Δ_1, Δ_2 induced respectively by regions $p_1 x_2 q_1$ and $p_2 x_3 q_2$, both have type $\mathbf{T}$ or $\mathbf{B_3}$. In particular, looking at Δ_1, there are vertices $u \in x_2 q_1, v \in c_1 q_1$, such that uv is an initial segment. As $c_1 c_2$ and $c_2 a_2$ are minimal, $l(uv)$ is disjoint from $c_1 a_2$, so $l(uv)$ must cross $a_2 x_3$, at some point w. But, uw and $p_2 q_2$ must cross as well, and this is a final contradiction granted by Proposition 4.1. □

LEMMA 4.3. Let $n \geqslant 1$ be an integer and suppose that $(n-1)$-classification theorem holds. Let $\Delta = (T, \mathcal{S}, \mathcal{B})$ be a triangle blocking arrangement of size n, and
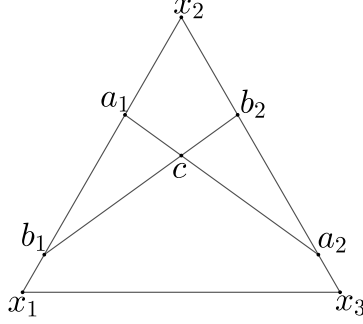
FIGURE 17. Assumptions in Lemma 4.3

let x_1, x_2 and x_3 be the vertices of the triangle T. Suppose that there are vertices $a_1, b_1 \in x_1x_2$, $a_2, b_2 \in x_2x_3$ such that x_1, b_1, a_1, x_2 and x_3, a_2, b_2, x_2 appear in these orders, and a_1a_2, b_1b_2 are initial segments, with an intersection point c. Then, in the region $a_1x_2b_2c$, the vertices x_2 and c are internally blocked.

The assumptions of Lemma 4.3 (which are also the assumptions of the next few results) are depicted in Figure 17.

PROOF. By Proposition 4.1, we have x_2 internally blocked. Thus, Δ cannot have type **T**, so by Proposition 4.5 it follows that any two initial segments that intersect have to have their endpoints on the same edges of T.

Suppose, to the contrary, that vertex c is externally blocked in $x_2a_1cb_2$. Thus, exactly one of a_1, b_2 is internally blocked in this region, by symmetry, we may assume that a_1 is internally blocked. So b_1 is internally blocked in a_1b_1c. Let $q = \beta(b_1) \cap a_1c$. Applying the $(n-1)$-classification theorem to the triangle blocking arrangement induced by $x_2b_1b_2$, implies that it has type **I₁** or **I₂**. Thus, there are vertices $p \in a_1b_1$, $r \in cb_2$, such that p, q, r are collinear, and pq, qr are minimal initial segments. Let s be the intersection $l(pr) \cap b_2a_2$. However, cb_2, qs intersect, while a_2 is externally blocked in $a_1a_2x_2$, so application of Proposition 4.1 results in contradiction.   □

LEMMA 4.4. *Let Δ and vertices $x_1, x_2, x_3, a_1, a_2, b_1, b_2, c$ satisfy the assumptions of Lemma* 4.3. *Then, in the regions a_1b_1c and a_2b_2c, the vertices a_1, a_2, b_1, b_2, c are externally blocked.*

PROOF. As in the proof of Lemma 4.3, intersecting initial segments must have endpoints on the same edges of T. Also, by that lemma, x_2 is internally blocked and c is internally blocked in the region $a_1cb_2x_2$.

Consider the vertex b_1. If we prove that b_1 is externally blocked in the a_1b_1c, then it follows that so is a_1, and looking at regions $a_1a_2x_2$ and $b_1b_2x_2$, the conclusion follows. Therefore, assume, to the contrary, that b_1 is internally blocked in a_1b_1c.

Set $q = \beta(b_1) \cap a_1c$. By $(n-1)$-classification theorem applied to the triangle blocking arrangement Δ_b induced by $b_1b_2x_2$, the type of Δ_b is either **I₁** or **I₂**, but in either case we have vertices $p \in a_1x_2$, $r \in b_1c$, such that p, q, r are collinear, and

pq, qr are minimal initial segments. Looking at $l(pr)$ and b_1b_2, as these intersect, the line $l(pr)$ must cross a_2x_3, with intersection point s, say. Recalling that the type of Δ_b is either $\mathbf{I_1}$ or $\mathbf{I_2}$, we have that p is externally blocked in pqa_1, hence in x_2ps, the vertices x_2, p are internally blocked, and therefore s is externally blocked. But qa_2 and rb_2 cross at c, which is a contradiction by Proposition 4.1 applied to x_2ps. $\qquad\square$

LEMMA 4.5. *Let Δ and vertices $x_1, x_2, x_3, a_1, a_2, b_1, b_2, c$ satisfy the assumptions of Lemma* 4.3 *. Then, a_1c and b_2c are minimal and x_2c is blocking.*

PROOF. By previous lemma, we have that a_1, b_1, a_2, b_2, c are externally blocked in regions a_1b_1c and a_2b_2c. Apply the $(n-1)$-classification theorem to the triangle blocking arrangement Δ_b induced by $b_1b_2x_2$, thus Δ_b has type $\mathbf{B_1}$, $\mathbf{I_1}$ or $\mathbf{I_2}$. But, in either case, a_1c is minimal. Similarly, b_2c is minimal, and $\beta(x_2)$ must pass through c. $\qquad\square$

LEMMA 4.6. *Let Δ and vertices $x_1, x_2, x_3, a_1, a_2, b_1, b_2, c$ satisfy the assumptions of Lemma* 4.3. *Then, a_2c and b_1c are minimal.*

PROOF. Suppose, to the contrary, that b_1c is not minimal. Thus, there is a vertex $q \in b_1c$. Let $pr \neq b_1b_2$ be the initial segment through q, with $p, r \in \partial T$. As a_1c is minimal, without loss of generality, $p \in a_1b_1$. As in the proof of Lemma 4.3, since pr and b_1b_2 intersect, $r \in x_2x_3$. However, we may apply the previous lemma to b_1b_2 and pr, to obtain that b_2q is minimal, which is a contradiction as $c \in b_2q$. $\quad\square$

COROLLARY 4.2. *Let $n \geqslant 1$ be an integer and suppose that $(n-1)$-classification theorem holds. Let $\Delta = (T, \mathcal{S}, \mathcal{B})$ be a triangle blocking arrangement of size n, and let x_1, x_2 and x_3 be the vertices of the triangle T. If Δ is not of type $\mathbf{T}$, then for every $\{j_1, j_2, j_3\} = [3]$ there are vertices $p_1, p_2, \ldots, p_k \in x_{j_1}x_{j_2}$ and $q_1, q_2, \ldots, q_k \in x_{j_1}x_{j_3}$ such that $x_{j_1}p_1, p_ip_{i+1}, x_{j_1}q_1, q_iq_{i+1}$ are minimal, for all $i < k$, and one of the following alternatives holds.*

(1) *Vertex x_{j_1} is externally blocked. Each p_iq_i is a minimal initial segment, and every initial segment with one vertex on $x_{j_1}x_{j_2}$ and the other vertex on $x_{j_1}x_{j_3}$ is among p_iq_i. Furthermore, k is odd, and $p_{2i-1}q_{2i}, p_{2i}q_{2i-1}$ are blocking segments, for $i \leqslant \frac{k-1}{2}$, and p_k, q_k are externally blocked in $x_{j_1}p_kq_k$.*

(2) *Vertex x_{j_1} is externally blocked. Each p_iq_i is a minimal initial segment, and every initial segment with one vertex on $x_{j_1}x_{j_2}$ and the other vertex on $x_{j_1}x_{j_3}$ is among p_iq_i. Furthermore, k is even, and $p_{2i-1}q_{2i}, p_{2i}q_{2i-1}$ are blocking segments, for $i \leqslant \frac{k}{2}$.*

(3) *Vertex x_{j_1} is internally blocked and k is even. For all $i \leqslant k/2$, the segments $p_{2i-1}q_{2i}$ and $p_{2i}q_{2i-1}$ are initial segments and intersect at point r_i. Every initial segment with one vertex on $x_{j_1}x_{j_2}$ and the other vertex on $x_{j_1}x_{j_3}$ is among these. The initial segments $r_ip_{2i-1}, r_ip_{2i}, r_iq_{2i-1}r_iq_{2i}$ are minimal. The vertices $r_1, \ldots, r_{k/2}$ all lie on $\beta(x_{j_1})$. Also, p_1q_1 is blocking, and $p_{2i}q_{2i+1}, p_{2i+1}q_{2i}$ are blocking for $i < k/2$, and p_k is externally blocked in $x_{j_1}p_kq_{k-1}$, and q_k is externally blocked in $x_{j_1}q_kp_{k-1}$.*
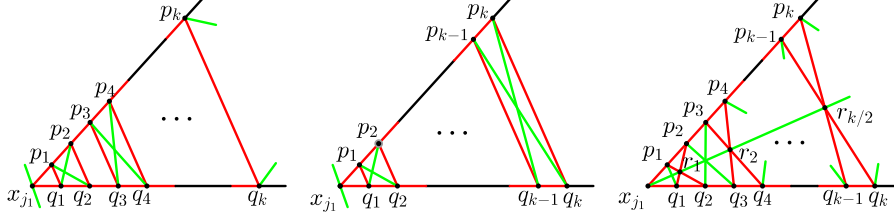
FIGURE 18. Possibilities in Corollary 4.2

PROOF. Without loss of generality, $j_1 = 1$, $j_2 = 2$, $j_3 = 3$. We split into two cases, depending on whether some initial segments between x_1x_2 and x_1x_3 intersect or not. The possible outcomes are shown in Figure 18.

Case 1: there is an intersecting pair. By Lemma 4.6, x_1 is internally blocked. Let pq be any initial segment with $p \in x_1x_2$, $q \in x_1x_3$. Then $\beta(x_1)$ crosses pq, at a point r, say, and let $p'q'$ be another initial segment through r, with $p', q' \in \partial T$. However, Δ is not of type **T**, and by Proposition 4.5, without loss of generality, $p' \in x_1x_2$, $q' \in x_1x_3$. Applying Lemma 4.6 to pq and $p'q'$, it follows that pr, $p'r$, qr, $q'r$, pp', qq' are minimal. Observe further that if an initial segment s has an endpoint on x_1p, unless the second endpoint is on x_1q, s crosses pq, and thus $s = p'q'$. Combining these observations, we conclude that there are points p_1, $p_2, \ldots, p_k \in x_1x_2$, $q_1, q_2, \ldots, q_k \in x_1x_3$, such that x_1p_1, $p_1p_2, \ldots, p_{k-1}p_k$, x_1q_1, $q_1q_2, \ldots, q_{k-1}q_k$ are minimal, k is even, $p_{2i-1}q_{2i}$, $p_{2i}q_{2i-1}$ are initial segments, and every initial segment with a vertex on x_1x_2 and another vertex on x_1x_3 is one of $p_{2i-1}q_{2i}$, $p_{2i}q_{2i-1}$. Furthermore, $p_{2i-1}q_{2i}$, $p_{2i}q_{2i-1}$ intersect at a point r_i, and r_ip_{2i-1}, r_ip_{2i}, r_iq_{2i-1}, r_iq_{2i} are minimal for all $i \leqslant k/2$.

From the information about minimal segments, we are forced to have $r_1, r_2, \ldots, r_{k/2} \in \beta(x_1)$, p_1q_1 blocking, in the minimal $\overline{\mathcal{S}}$-region $p_{2i}r_iq_{2i}q_{2i+1}\,r_{i+1}p_{2i+1}$, all six vertices are internally blocked, for every $1 \leqslant i < k/2$, and finally p_k is externally blocked in $x_1p_kq_{k-1}$ and q_k is externally blocked in $x_1q_kp_{k-1}$. It remains to prove that for $i < k/2$, $p_{2i}q_{2i+1}$ and $p_{2i+1}q_{2i}$ are blocking. Suppose, to the contrary, that there is some $i < k/2$, such that, without loss of generality, $p_{2i}q_{2i+1}$ is not blocking. Looking at minimal $\overline{\mathcal{S}}$-region $p_{2i}r_iq_{2i}q_{2i+1}r_{i+1}p_{2i+1}$, it follows that $p_{2i}q_{2i}$ is blocking. However, looking at $\beta(p_{2i})$ and $l(r_iq_{2i})$, which meet at q_{2i}, by assumption (A), it follows that $l(p_{2i+1}r_{i+1})$ also contains q_{2i}, which is a contradiction.

Case 2: there are no intersecting pairs. Observe that if $p \in x_1x_2$, $q \in x_1x_3$ and pq is an initial segment, then, if any other segment s crosses pq, then, by the assumption of this case, s must have at least one vertex on x_2x_3, which is impossible by Proposition 4.5. Thus, pq is minimal, and as in the previous case, if a segment s has an endpoint in x_1p, its other endpoint is bound to be in x_1q. From this, we conclude that there are points $p_1, p_2, \ldots, p_k \in x_1x_2$, $q_1, q_2, \ldots, q_k \in x_1x_3$, such that $x_1p_1, p_1p_2, \ldots, p_{k-1}p_k, x_1q_1, q_1q_2, \ldots, q_{k-1}q_k$ are minimal, and p_iq_i is a minimal initial segment for every $i \leqslant k$, and if s is an initial segment with endpoints on x_1x_2 and x_1x_3, then $s = p_iq_i$ for some i. As x_1p_1, x_1q_1 are minimal, p_1, q_1 are externally

blocked in $x_1 p_1 q_1$. Thus, $p_1 q_2$ and $p_2 q_1$ are blocking segments. Hence, in the region $p_2 q_2 q_3 p_3$, p_2 and q_2 externally blocked and so are p_3, q_3. Proceeding in this fashion, the conclusion of the corollary follows. □

PROPOSITION 4.6. *Let $n \geqslant 1$ be an integer and suppose that $(n-1)$-classification theorem holds. Let $\Delta = (T, \mathcal{S}, \mathcal{B})$ be a triangle blocking arrangement of size n, and let x_1, x_2 and x_3 be the vertices of the triangle T. Suppose that x_1, x_2 are internally blocked. Then, Δ has one the types $\mathbf{B_1}, \mathbf{I_1}$ or $\mathbf{I_2}$.*

PROOF. We consider three cases, depending on the outcomes of Corollary 4.2. We say that x_i *has no segments* if there are no segments with one vertex on $x_i x_{i'}$ and the other on $x_i x_{i''}$, where $i \in \{1, 2\}$ and $\{i, i', i''\} = \{1, 2, 3\}$. Otherwise, we say that x_i *has segments*.

Case 1: both x_1, x_2 have no segments. Let $u = \beta(x_1) \cap x_2 x_3$, $v = \beta(x_2) \cap x_1 x_3$. By Corollary 4.2, any initial segment in $\mathcal{S}$ is of the form pq, where $p \in x_1 x_3$, $q \in x_2 x_3$ and all these are minimal (and hence disjoint). In particular, no initial segment can cross $x_1 u$, $x_2 v$, and also, $x_1 v, x_2 u$ are minimal segments. Thus, the other initial segment through u, must cross $x_3 v$, and the other initial segment through v must cross $x_3 u$. However, all initial segments are disjoint, so actually uv is an initial segment, and it is minimal. It follows from Corollary 4.2 at vertex x_3 that the type of Δ is $\mathbf{B_1}$.

Case 2: x_1 has, but x_2 has no segments. By Corollary 4.2, we have vertices $a_1, a_2, \ldots, a_k \in x_1 x_3$, $b_1, b_2, \ldots, b_k \in x_1 x_2$ such that k is even and $x_1 a_1, a_1 a_2, \ldots,$ $a_{k-1} a_k, x_1 b_1, b_1 b_2, \ldots, b_{k-1} b_k, b_k x_2$ are minimal, $a_{2i-1} b_{2i}, a_{2i} b_{2i-1}$ are initial segments that intersect at a point c_i. Let $u = \beta(x_1) \cap x_2 x_3$. By Corollary 4.2 applied to vertex x_3, we see that every initial segment is either among $a_i b_j$, or has vertices on $x_1 x_3$ and $x_2 x_3$ and is minimal. Hence, $x_2 u$ is minimal. Let $v \in x_1 x_3$ be such that uv is an initial segment, and thus minimal. Hence, $v a_k$ is also minimal, as otherwise, an initial segment with a vertex on $v a_k$ would have the second endpoint on $x_2 x_3$, so it would have to be minimal, but would cross $\beta(x_1) = x_1 u$ or vu, which is impossible. Finally, $a_k c_{k/2} b_k x_2 uv$ is a minimal $\overline{\mathcal{S}}$-region. Using assumption (A) as before, we see that $x_2 a_k, v b_k$ are blocking, and it follows that the type of Δ is $\mathbf{I_1}$.

Case 3: both x_1, x_2 have segments. By Corollary 4.2, we have vertices $a_1, a_2, \ldots, a_k \in x_1 x_3$, $b_1, b_2, \ldots, b_k, d_1, d_2, \ldots, d_l \in x_1 x_2$, $e_1, e_2, \ldots, e_l \in x_2 x_3$ such that $x_1 a_1, a_1 a_2, \ldots, a_{k-1} a_k, x_1 b_1, b_1 b_2, \ldots, b_{k-1} b_k, x_2 d_1, d_1 d_2, \ldots, d_{l-1} d_l, x_2 e_1, e_1 e_2, \ldots,$ $e_{l-1} e_l$ are minimal, and $a_1 b_2, a_2 b_1, \ldots, a_{k-1} b_k, a_k b_{k-1}, d_1 e_2, d_2 e_1, \ldots, d_{l-1} e_l, d_l e_{l-1}$ are initials segments, k, l are even and these are all initial segments that have at least one vertex on $x_1 x_2$. Furthermore, $a_{2i-1} b_{2i}, a_{2i} b_{2i-1}$ intersect at point c_i, $d_{2i-1} e_{2i}, d_{2i} e_{2i-1}$ intersect at a point f_i, such that $c_1, c_2, \ldots, c_{k/2} \in \beta(x_1)$, $f_1, f_2, \ldots, f_{l/2} \in \beta(x_2)$. We also have that $b_k d_l$ is minimal.

Suppose for a moment that $a_k x_3$ is minimal. Then, as $e_l f_{l/2}$, $f_{l/2} d_l$, $d_l b_k$, $b_k c_{k/2}$, $c_{k/2} a_k$ are minimal, it follows that $e_l x_3$ is also minimal, and hence $a_k c_{k/2} b_k d_l f_{l/2} e_l x_3$ is a minimal $\mathcal{L}$-region. But inside this minimal region, $\beta(f_{l/2})$ can only pass through a_k, and $\beta(c_{k/2})$ can only pass through e_l. However, then we have $\beta(b_k) = \beta(d_l)$, which is a contradiction. Therefore, $a_k x_3$ is not minimal.

Let v be the vertex in $a_k x_3$ such that $a_k v$ is minimal. As in the previous case, any initial segment with vertices on $x_1 x_3$ and $x_2 x_3$ is minimal. It follows that the initial segment through v, not equal to $x_1 x_3$, is vu with $u \in x_3 e_l$ and it is minimal. Since $uv, va_k, a_k c_{k/2}, c_{k/2} b_k, b_k d_l, d_l f_{l/2}, f_{l/2} e_l$ are minimal, it follows that ue_l is minimal as well. Therefore, $R = a_k c_{k/2} b_k d_l f_{l/2} e_l uv$ is a minimal $\overline{\mathcal{S}}$-region. Using Corollary 4.2, to prove that Δ has type $\mathbf{I_2}$, it suffices to show that $a_k \in \beta(x_2)$, $e_l \in \beta(x_1)$.

Suppose, to the contrary, that $\beta(x_1) \cap x_2 x_3 \neq e_l$. By minimality of R, we must have $\beta(x_1) \cap x_2 x_3 = u$. But, we would then have $\beta(u) \cap l(va_k) = x_1$. By the assumption (A), it follows that $l(e_l f_{l/2})$ also passes through x_1, which is impossible. Thus $e_l \in \beta(x_1)$, and similarly we obtain $a_k \in \beta(x_2)$. Thus, Δ has type $\mathbf{I_2}$. $\square$

Finally, it remains to classify the triangle blocking arrangements without intersecting initial segments.

PROPOSITION 4.7. *Let $n \geqslant 1$ be an integer and suppose that $(n-1)$-classification theorem holds. Let $\Delta = (T, \mathcal{S}, \mathcal{B})$ be a triangle blocking arrangement of size n, and let x_1, x_2 and x_3 be the vertices of the triangle T. Suppose that no two initial segments in $\mathcal{S}$ intersect. Then Δ has a basic type or $\mathbf{T}$ (with $k = 1$ in the definition of $\mathbf{T}$).*

PROOF. We say that a vertex x_i is empty, if for $\{i, j, j'\} = \{1, 2, 3\}$, there are no initial segments between $x_i x_j$ and $x_i x_{j'}$. We distinguish between four cases, depending on the number of empty vertices.

Case 0: All four vertices are empty. Then $\mathcal{L} = \emptyset$, and Δ has type $\mathbf{B_0}$.

Case 1: Only x_1 is nonempty. Applying Corollary 4.2, Δ has type $\mathbf{B_1}$.

Case 2: Vertices x_1, x_2 are nonempty. By Corollary 4.2, there are vertices $a \in x_1 x_3$, $b, c \in x_1 x_2$, $d \in x_2 x_3$ such that ab, cd are initial segments and all initial segments are either in $x_1 ab$ or in $x_2 cd$, and are disjoint. Moreover, we obtain the desired structure of blocking lines in regions $x_1 ab, x_2 cd$. Moreover, bc, dx_3, ax_3 are minimal, so $R = x_3 abcd$ is a minimal $\overline{\mathcal{S}}$-region, inside which a is internally blocked iff b is, and c is internally blocked iff d is. If none of these four vertices are internally blocked in R, Δ is of type $\mathbf{B_2}$. Assume for contradiction that some vertex among them is internally blocked in R. Without loss of generality, one of a, b is internally blocked, so both of them must be internally blocked. However, x_3 is externally blocked in R, so $\beta(b)$ must pass through d. However, we have $l(cd) \cap \beta(b) = d$, so by the assumption (A) applied to R and vertex b, we have that $l(ax_3)$ passes through d, which is a contradiction, as desired.

Case 3: All three vertices are nonempty. Similarly to the previous case, there are vertices $a, f \in x_1 x_3$, $b, c \in x_1 x_2$, $d, e \in x_2 x_3$ such that ab, cd, ef are minimal initial segments, and all initial segments are in regions $x_1 ab$, $x_2 cd$, $x_3 ef$, and are minimal. From this, fa, bc, de are also minimal. Moreover, we know the structure of blocking lines in $x_1 ab, x_2 cd, x_3 ef$, and it remains to determine the structure of blocking lines in the minimal $\overline{\mathcal{S}}$-region $R = abcdef$.

We have that in R, a is internally blocked iff b is, c is internally blocked iff d is, and e is internally blocked iff f. If all these are externally blocked, then Δ has

$\mathbf{B_3}$, as desired. Now, assume that, without loss of generality, one of vertices a, b is internally blocked in R. But then both a and b must be internally blocked. Suppose for a moment that there is a blocking segment in R, which is a small diagonal of hexagon $abcdef$. By symmetry, we may suppose it contains a, so it is ac or ae. If ac is blocking, however, $\beta(a) \cap l(bc) = c$, so by the assumption (A), $l(ef)$ has to contain c, which is impossible. Similarly, if ae is blocking, $\beta(a) \cap l(fe) = e$, so by the assumption (A), $l(bc)$ has to contain e, which is also impossible. Hence, the only possible blocking segments in R are the main diagonals ad, be, cf. As a, b are internally blocked, we have that ad, be are blocking segments. But, as d is internally blocked in R, so is c, so cf is also blocking, showing that Δ has type $\mathbf{B_3}$. $\qquad\square$

Combining all ingredients, we are ready to prove the classification theorem.

PROOF OF THE CLASSIFICATION THEOREM. We prove the theorem by induction on the size n of triangle blocking arrangement. The base of induction is $n = 0$, when the triangle blocking arrangement has type $\mathbf{B_0}$.

Now, assume that $n \geqslant 1$ and $(n-1)$-classification theorem holds and let Δ be a triangle blocking arrangement of size n with vertices x_1, x_2, x_3. If Δ has intersecting initial segments that satisfy the conditions of Proposition 4.5, we are done. Otherwise, if there are any intersecting initial segments in Δ at all, by Proposition 4.1, we must have some of x_1, x_2, x_3 internally blocked. Then, we are done by Proposition 4.6. Finally, if there are no intersecting initial segments in Δ, we may apply Proposition 4.7 to finish the proof. $\qquad\square$

5. Concluding remarks

Our first remark is that it would also be very interesting to classify all triangle blocking arrangements, without the assumption (A). However, this is probably much harder, as the following discussion suggests.

A comment about the assumption (A). As we have seen before, the assumption (A) is necessary in the classification theorem. However, there could be hope that we are using this assumption only locally, and that the arrangement types are rigid enough so that after some point, large arrangements are forced to combine as in the classification theorem. However, Figure 2 shows that we cannot localize the assumption (A). Namely, a natural weaker assumption would be that for some fixed K, and for any minimal $\bar{\mathcal{S}}$-region R, for any consecutive vertices v_1, v_2, v_3, v_4, v_5 appearing in this order on ∂R, we have that, if $l(v_1 v_2)$ and $\beta(v_3)$ intersect in T as some point p, $\beta(v_3)$ meets the interior of R, and $v_2 p$ or $v_3 p$ has at most K points, then $l(v_1 v_2), \beta(v_3), l(v_4 v_5)$ are concurrent. But this figure shows that we may have as many points between as we want; the only region where (A) fails is $abcdef$, namely $l(ab), \beta(c)$ meet at x_1, but $x_1 \notin l(de)$, and this region satisfies the weaker assumption.

Relationship with Green Tao theorem on ordinary lines. We discuss very briefly the proof of the result about ordinary lines of Green and Tao [5]. It can be summarized as follows.
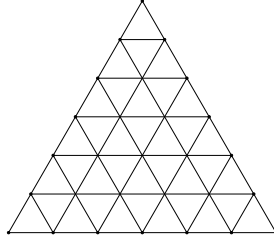
Figure 19. Hexagonal grid

Step 1. Move to the dual.

Step 2. Apply Melchior's inequality (which is a consequence of Euler's formula) to get some control over point-line incidences.

Step 3. Use the incidence information to find large pieces with 'triangular structure'.

Step 4. Study 'triangular structure' to show that it looks like a hexagonal lattice.

Step 5. Apply the dual version of Chasles' theorem to place the points on a cubic.

Step 4 corresponds to our classification theorem, and to emphasize the similarity, we phrase it as the following Classification Lemma. The conclusion is written slightly informally.

LEMMA 5.1 (Classification of triangular arrangements, Green and Tao [**5**].). *Let $T = x_1 x_2 x_3$ be a triangle in the plane, and let $\mathcal{S}$ be a collection of segments with endpoints on ∂T with the property that whenever two segments in $\bar{\mathcal{S}} = \mathcal{S} \cup \{x_1 x_2, x_2 x_3, x_3 x_1\}$ intersect, there is a unique third segment in $\bar{\mathcal{S}}$ that contains the intersection point, except possibly if the intersection is one of x_1, x_2, x_3, in which case there might not be the third segment. Then, $\bar{\mathcal{S}}$ forms a hexagonal grid shown in Figure* 19.

PROOF. We prove the claim by induction on $|\mathcal{S}|$. If $\mathcal{S}$ is empty, we are done. Assume now that we are given $\mathcal{S}$ and the claim holds for all smaller arrangements.

We begin the proof by showing that there are no segments in $\mathcal{S}$ that pass through vertices x_1, x_2 and x_3. Suppose, to the contrary, that we have $u \in \operatorname{int} x_2 x_3$ such that $x_1 u$ is one of the given segments. Then we may apply the induction hypothesis to arrangements induced by triangles $x_1 x_2 u$ and $x_1 x_3 u$, from which we deduce that the only segments through u are actually $x_1 u$ and $x_2 x_3$, which is a contradiction.

Next, observe that if v is an intersection point on some edge of T, but not among the vertices x_1, x_2, x_3, then, we have $u, w \in \partial T$ such that $uv, wv \in \mathcal{S}$. Without loss of generality, $v \in x_1 x_2$, $u \in x_1 x_3$. If $w \in x_1 x_3$ also, then, without loss of generality, u is between x_1 and w, so applying the induction hypothesis to $v x_1 w$ gives a contradiction, as hexagonal structure does not allow three segments at v. Hence, we must always have the two segments that meet on ∂T between different pairs of edges of T.

Similarly, we show that if two segment in $\mathcal{S}$ intersect internally, then they are between different pairs of edges of T. Suppose for the sake of contradiction that $a, c \in x_1 x_2$ and $b, d \in x_1 x_3$ are such that ab and cd intersect internally at a point e. Note in particular that a, b, c, d are distinct points. Moreover, since segments in $\mathcal{S}$ do not pass through x_1, x_2 or x_3, we also see that in fact $a, b, c, d, x_1, x_2, x_3$ are distinct. Without loss of generality, c is between x_1 and a. Applying the induction hypothesis to $x_1 ab$, we obtain a segment ef with $f \in x_1 b$. But, applying the induction hypothesis to $x_1 cd$, we obtain a contradiction.

Without loss of generality, we have a segment between $x_1 x_2$ and $x_1 x_3$. Pick an endpoint $v \in x_1 x_2$ of such a segment with the property that v is closest to x_2 among all such points. Let $u \in x_1 x_3$ be such that $uv \in \mathcal{S}$. By observations before, there are no other points in $v x_2$ and all the segments between $x_1 x_2$ and $x_1 x_3$ are in $u x_1 v$. In particular, $u x_3$ also has no points in its interior. We may apply the induction hypothesis to $x_1 vu$, to obtain hexagonal structure there, with points $w_1, w_2, \ldots, w_k$ appearing from v to u. Consider segment $w_1 b_1$ with $b_1 \in x_1 u$. Then $l(w_1 b_1)$ must cross $x_2 x_3$, at some point t_1. But, then at t_1 we also have a segment with other endpoint on $x_1 x_2$. However, by the choice of w_1, this may only be v. Next, consider w_2, and apply the same argument. We obtain a point $t_2 \in x_2 x_3$ such that $t_2 w_2$ is a part of a segment with other endpoint on $x_1 x_3$, so similarly we obtain $t_2 w_1$ is a subset of a segment in $\mathcal{S}$. Proceeding further in this fashion, we eventually obtain the hexagonal grid. □

It is therefore plausible that an extremal result could be proved with a similar general strategy, but given the significant differences in the difficulty of the relevant Classification Theorem, we expect that the new interesting difficulties will arise, in particular because not all types we defined come from duals of points on cubic curves. Nevertheless, we will investigate this further.

Classification Theorem for curves in the plane. Going back to the proof of classification theorem, we made a heavy use of topological properties of the real plane. However, we mainly focused on order of points on a line, and did not rely too much on the fact that the lines are straight (except that at intersection points the lines change sides with respect to one another). Instead of asking what happens over a different field, it could be possible that a similar, if not the same theorem holds for curves instead lines. Here we need some conditions on the curves, e.g. that we have some family of curves $\mathcal{C}$ with the property that through any two distinct points, there is a unique line in $\mathcal{C}$ containing them. Then, we could consider configurations where segments are intersections of curves in $\mathcal{C}$ with T. Or, we might not need to go that far and maybe we could consider curves with endpoints on ∂T which are not self-intersecting and any two intersect in at most one point. This is something we shall also study further.

Returning to the possibility of using a different field, this is of course another interesting question. However, over $\mathbb{C}$ we have, for example, the Hesse configuration (which can be realized as inflection points of a cubic curve), which gives 9 points, without ordinary lines. In this setting the interesting phenomenon is actually of a different nature.

THEOREM 5.1 (Kelly, [**6**]). *Any finite set of points in a complex space without ordinary lines is coplanar.*

We also expect that a classification theorem over finite fields would be very different from the one proved here.

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