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SOME PROPERTIES OF ANTI-KÄHLER MANIFOLDS EQUIPPED WITH QUARTER-SYMMETRIC F-CONNECTIONS

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Abstract. We construct quarter-symmetric metric and nonmetric F-connections on anti-Kähler manifolds and analyze some properties of torsion and curvature tensors of these connections.

1. Introduction

The idea of metric connection with torsion tensor on a Riemannian manifold was introduced by Hayden [**3**]. Later, Yano [**13**] considered a semi-symmetric metric connection and studied some of its properties. Golab [**2**] defined and studied quarter-symmetric linear connections on differentiable manifolds, which generalizes the idea of semi-symmetric connection. After that, Rastogi [**7, 8, 9**] continued the systematic study of quarter-symmetric metric connection. In 1980, Mishra and Pandey [**6**] defined a quarter-symmetric F-connection and studied the conditions for Einstein manifolds, Sasakian manifolds and Kähler manifolds equipped with this connection to be flat, projectively flat or conharmonically flat. In 1982, Yano and Imai [**12**] obtained the most general expression for quarter-symmetric metric connections on Riemannian, Hermitian and Kählerian manifolds and gave some special examples of them. Recently, Chaubey and Ojha [**1**] proved that an Einstein manifold admitting a quarter-symmetric F-connection whose Ricci tensor vanishes is conformally flat.

Our aim is to systematically study quarter-symmetric metric and nonmetric F-connections on anti-Kähler manifolds by focusing on properties of their torsion and curvature tensors.

2. Preliminaries

Let M_n be an $n = 2m$-dimensional differentiable manifold of class C^∞ covered by any system of coordinate neighbourhoods (x^h), where here and in the sequel the

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indices $h, i, j, k, \dots$ run over the range $1, 2, \dots, n$. Also note that summation over repeated indices is always implied.

An almost complex structure $F = (F_i{}^k)$ on M_n is a $(1,1)$-tensor field on M_n such that $F_i{}^k F_k{}^j = -\delta_i^j$. The pair (M_n, F) is called an almost complex manifold. When the almost complex structure F is integrable, it is called a complex structure, and (M_n, F) is a complex manifold. A pseudo-Riemannian manifold (M_n, g), endowed with an almost complex structure F, satisfying the relations

$$F_i{}^k g_{kj} = F_j{}^k g_{ki} \quad \text{and} \quad \nabla_k F_i{}^j = 0 \tag{2.1}$$

is called an anti-Kähler manifold (or a Kähler-Norden manifold), where ∇_k denotes the operator of covariant derivation w.r.t. the Levi-Civita connection of g. It is well known that the condition $\nabla_k F_i^j = 0$ is equivalent to holomorphicity (analyticity) of the pseudo-Riemannian metric g [**4, 10**], i.e., $\phi_F g = 0$, where ϕ_F is the Tachibana operator. Also note that the first condition in (2.1) is the purity condition of the pseudo-Riemannian metric g w.r.t. the almost complex structure F.

In general, any (p,q)-tensor K with components $K_{i_1 i_2 \dots i_q}^{j_1 j_2 \dots j_p}$ on an almost complex manifold (M_n, F) satisfying the relations

$$\begin{aligned} K_{m i_2 \dots i_q}^{j_1 \dots j_p} F_{i_1}^m &= K_{i_1 m \dots i_q}^{j_1 \dots j_p} F_{i_2}{}^m = \dots = K_{i_1 i_2 \dots m}^{j_1 \dots jp} F_{i_q}{}^m \\ &= K_{i_1 \dots i_q}^{m j_2 \dots j_p} F_m{}^{j_1} = K_{i_1 \dots i_q}^{j_1 m \dots j_p} F_m{}^{j_2} = \dots = K_{i_1 \dots i_q}^{j_1 j_2 \dots m} F_m{}^{j_p} \end{aligned}$$

(purity condition) and

$$\begin{aligned} (\phi_F K)_{k i_1 \dots i_q}^{j_1 \dots j_p} &= F_k{}^m \partial_m t_{i_1 \dots i_q}^{j_1 \dots j_p} \\ -\partial_k (K \circ F)_{i_1 \dots i_q}^{j_1 \dots j_p} + \sum_{\lambda=1}^{q} (\partial_{i_\lambda} F_k{}^m) K_{i_1 \dots m \dots i_q}^{j_1 \dots j_p} &+ \sum_{\mu=1}^{p} (\partial_k F_m{}^{j_\mu} - \partial_m F_k{}^{j_\mu}) K_{j_1 \dots j_s}^{i_1 \dots m \dots i_r} = 0 \end{aligned}$$

(Tachibana operator) is called as a holomorphic (analytic) tensor w.r.t. the almost complex structure F [**11**] (see also [**10**] and [**14**]). We recall that the Riemannian curvature tensor R of an anti-Kähler manifold is a holomorphic tensor [**4, 10**].

3. Quarter-symmetric metric F-connection

Let (M_n, g, F) be an anti-Kähler manifold. A linear connection $\widetilde{\nabla}$ with components $\widetilde{\Gamma}_{ij}^k$ in the anti-Kähler manifold (M_n, g, F) satisfying the relations

$$\widetilde{\nabla}_h g_{ij} = 0, \quad \text{and} \quad \widetilde{\nabla}_h F_i{}^j = 0$$

is called a metric F-connection. When we consider a metric F-connection whose torsion tensor is in the form

$$\widetilde{S}_{ij}{}^k = p_j F_i^k - p_i F_j^k + p_t F_j{}^t \delta_i{}^k - p_t F_i{}^t \delta_j{}^k, \tag{3.1}$$

standard calculations give the components $\widetilde{\Gamma}_{ij}^k$ of the metric F-connection as follows

$$\widetilde{\Gamma}_{ij}^k = \Gamma_{ij}^k + p_j F_i^k - p^k F_{ij} + p_t F_j{}^t \delta_i{}^k - p_t F^{kt} g_{ij},$$

where p_i and p^k are covariant and contravariant components of any covector field p and $F^{kt} = F_i{}^t g^{ik}$, $F_{ij} = F_j{}^k g_{ik}$. This covector field p is called the generator of the metric F-connection. We shall call such a connection *a quarter-symmetric metric F-connection.*

Let f be a function on M_n. It is locally holomorphic if [**10**]

$$\begin{aligned}(\phi_F(df))_{kj} &= F_k^m\partial_m\partial_j f - \partial_k(F_j^m\partial_m f) + (\partial_j F_k^m)\partial_m f\\ &= F_k{}^m(\partial_m\nabla_j f) - \partial_k(F_j{}^m\nabla_m f)\\ &= F_k{}^m(\nabla_m\nabla_j f + \Gamma^s_{mj}\nabla_s f) - F_j{}^m(\nabla_k\nabla_m f + \Gamma^s_{km}\nabla_s f)\\ &= F_k{}^m(\nabla_m\nabla_j f) - F_j{}^m(\nabla_k\nabla_m f) = 0.\end{aligned}$$

For the sake of simplicity, we specialize the quarter-symmetric metric F-connection in such a way that its generator p_i is gradient of a locally holomorphic function f on M_n. In what follows, we call such a quarter-symmetric metric F-connection simply *a quarter-symmetric metric F-connection.*

3.1. Properties of the torsion tensor. The section deals with some properties concerning the torsion tensor of the quarter-symmetric metric F-connection.

THEOREM 3.1. *The torsion tensor $\widetilde{S}$ of the quarter-symmetric metric F-connection is a pure tensor w.r.t. the complex structure F.*

PROOF. The result follows by a straightforward calculation. □

In [**10**], it was proved that an F-connection is pure if and only if its torsion tensor is pure. Thus we can say that the quarter-symmetric metric F-connection is pure w.r.t. the complex structure F.

THEOREM 3.2. *The torsion tensor $\widetilde{S}$ of the quarter-symmetric metric F-connection is a holomorphic tensor.*

PROOF. In the anti-Kähler manifold (M_n, g, F), denote by ∇ the Levi-Civita connection of g. It is well-known that a torsion-free F-connection is always pure. Hence the Levi-Civita connection ∇ on M_n is pure w.r.t. the complex structure F.

The Tachibana operator ϕ_F applied to the torsion tensor $\widetilde{S}$ can be expressed in the form

$$\begin{aligned}(\phi_F\widetilde{S})_{kij}{}^l &= F_k{}^m(\partial_m\widetilde{S}^l_{ij}) - \partial_k(\widetilde{S}^m_{ij}F^l_m)\\ &= F_k{}^m(\nabla_m\widetilde{S}^l_{ij} + \Gamma^s_{mi}\widetilde{S}^l_{sj} + \Gamma^s_{mj}\widetilde{S}^l_{is} - \Gamma^l_{ms}\widetilde{S}^s_{ij})\\ &\quad - F_m{}^l(\nabla_k\widetilde{S}^m_{ij} + \Gamma^s_{ki}\widetilde{S}^m_{sj} + \Gamma^s_{kj}\widetilde{S}^m_{is} - \Gamma^m_{ks}\widetilde{S}^s_{ij})\\ &= F_k{}^m(\nabla_m\widetilde{S}^l_{ij}) - F_m{}^l(\nabla_k\widetilde{S}^m_{ij}).\end{aligned}$$

Substituting (3.1) into the above relation, it follows that

$$\begin{aligned}(\phi_F\widetilde{S})_{kij}{}^l = {}&[F_k{}^m(\nabla_m p_j) - F_j{}^m(\nabla_k p_m)]F^l_i - [F_k{}^m(\nabla_m p_i) - F_i{}^m(\nabla_k p_m)]F^l_j\\ &- [F_k{}^m F_i{}^s(\nabla_m p_s) + \nabla_k p_i]\delta^l_j + [F_k{}^m F_j{}^s(\nabla_m p_s) + \nabla_k p_j]\delta^l_i.\end{aligned}$$

Using the fact that the generator p_i is a gradient, we get $(\phi_F\widetilde{S})_{kij}{}^l = 0$ i.e., the torsion tensor $\widetilde{S}$ is holomorphic. □

THEOREM 3.3. *The torsion tensor $\widetilde{S}$ w.r.t. the quarter-symmetric metric F-connection is recurrent, i.e., $\widetilde{\nabla}_k S^l_{ij} = \omega_k S^l_{ij}$ if and only if the generator p is recurrent w.r.t. the quarter-symmetric metric F-connection, where ω_k is the recurrence covector field.*

PROOF. Let the torsion tensor $\widetilde{S}$ be recurrent w.r.t. the quarter-symmetric metric F-connection, i.e., $\widetilde{\nabla}_k \widetilde{S}^l_{ij} = \omega_k \widetilde{S}^l_{ij}$. By contracting this w.r.t. i and l, we obtain

$$\widetilde{\nabla}_k \widetilde{S}_{lj}^{\ l} = \omega_k \widetilde{S}_{lj}^{\ l}, \quad \widetilde{\nabla}_k[(n-2)p_l F_j^{\ l}] = \omega_k[(n-2)p_l F_j^{\ l}],$$
$$(\widetilde{\nabla}_k p_l) F_j^{\ l} = \omega_k p_l F_j^{\ l}, \quad \widetilde{\nabla}_k p_j = \omega_k p_j$$

which means that the generator p_i is recurrent.

Conversely, let the generator p_i is recurrent. On taking covariant derivative of (3.1) w.r.t. the quarter-symmetric metric F-connection, we get

$$\begin{aligned}\widetilde{\nabla}_k \widetilde{S}_{ij}^{\ l} &= (\widetilde{\nabla}_k p_j) F^l_i - (\widetilde{\nabla}_k p_i) F^l_j - (\widetilde{\nabla}_k p_t) F_j^{\ t} \delta^l_i + (\widetilde{\nabla}_k p_t) F_i^{\ t} \delta^l_j \\ &= \omega_k p_j F^l_i - \omega_k p_i F^l_j - \omega_k p_t F_j^{\ t} \delta^l_i + \omega_k p_t F_i^{\ t} \delta^l_j = \omega_k \widetilde{S}_{ij}^{\ l}. \qquad \square\end{aligned}$$

3.2. Properties of the curvature tensor. If (M_n, g, F) is the anti-Kähler manifold, then denote by $\widetilde{R}_{ijk}^{\ \ \ l}$ and $R_{ijk}^{\ \ \ l}$ its curvature tensors of the quarter-symmetric metric F-connection and the Levi-Civita connection, respectively. The curvature $(0,4)$-tensor $\widetilde{R}_{ijkl} = \widetilde{R}_{ijk}[]^h g_{hl}$ can be expressed in the form

$$\begin{aligned}\widetilde{R}_{ijkl} &= R_{ijkl} + F_{jl}\sigma_{ik} - F_{il}\sigma_{jk} + F_{ik}\sigma_{jl} - F_{jk}\sigma_{il} \\ &\quad + g_{jl} F_k^{\ t} \sigma_{it} - g_{il} F_k^{\ t} \sigma_{jt} + g_{ik} F_l^{\ t} \sigma_{jt} - g_{jk} F_l^{\ t} \sigma_{it},\end{aligned} \tag{3.2}$$

where we use the following abbreviation

$$\sigma_{jk} = \nabla_j p_k - p_j p_m F_k^{\ m} + \tfrac{1}{2} p^m p_m F_{kj} - p_k p_m F_j^{\ m} + \tfrac{1}{2} p^m p_t F_m^{\ t} g_{jk}. \tag{3.3}$$

It is easy to see that $\sigma_{jk} - \sigma_{kj} = \nabla_j \nabla_k f - \nabla_k \nabla_j f = 0$. Also, it is not too hard to see that the curvature $(0,4)$-tensor $\widetilde{R}$ satisfies $\widetilde{R}_{ijkl} = -\widetilde{R}_{jikl}$, $\widetilde{R}_{ijkl} = -\widetilde{R}_{ijlk}$, $\widetilde{R}_{ijk}^{\ \ \ k} = 0$, $\widetilde{R}_{ijkl} = \widetilde{R}_{klij}$, $\widetilde{R}_{ijkl} + \widetilde{R}_{kijl} + \widetilde{R}_{jkil} = 0$.

PROPOSITION 3.1. *The $(0,2)$-tensor σ given by (3.3) is a holomorphic tensor.*

PROOF. Taking into account (3.3), we infer

$$F_k^{\ t}\sigma_{it} - F_i^{\ t}\sigma_{tk} = (\nabla_i p_t) F_k^{\ t} - (\nabla_t p_k) F_i^{\ t} = (\nabla_i \nabla_t f) F_k^{\ t} - (\nabla_t \nabla_k f) F_i^{\ t} = 0,$$

i.e., the $(0,2)$-tensor σ is pure w.r.t. the complex structure F.

The Tachibana operator applied to the $(0,2)$-tensor σ in the anti-Kähler manifold (M_n, g, F) is given by

$$(\phi_F \sigma)_{kij} = (\nabla_m \sigma_{ij}) F_k^{\ m} - (\nabla_k \sigma_{im}) F_j^{\ m}.$$

Substituting (3.3) into the last relation, we find

$$(\phi_F \sigma)_{kij} = (\nabla_m \nabla_i p_j) F_k^{\ m} - (\nabla_k \nabla_m p_j) F_i^{\ m}. \tag{3.4}$$

On the other hand, we write down the Ricci identity for the generator p_i

$$\big(\nabla_m\nabla_i p_j\big)F_k{}^m = \big(\nabla_i\nabla_m p_j\big)F_k{}^m - \tfrac{1}{2}p_s R_{mij}{}^s F_k{}^m,$$
$$\big(\nabla_k\nabla_i p_m\big)F_j{}^m = \big(\nabla_i\nabla_k p_m\big)F_j{}^m - \tfrac{1}{2}p_s R_{kim}{}^s F_j{}^m.$$

Finally, in view of the above two relations, (3.4) becomes

$$(\phi_F\sigma)_{kij} = -\tfrac{1}{2}p_s(R_{mij}{}^s F_k{}^m - R_{kim}{}^s F_j{}^m) = 0. \qquad \square$$

THEOREM 3.4. *The curvature* $(0,4)$*-tensor* $\widetilde{R}$ *of the quarter-symmetric metric* F*-connection is a holomorphic tensor.*

PROOF. With the help of the purity of the $(0,2)$-tensor σ, it follows immediately that

$$\widetilde{R}_{mjlt}F_i{}^m = \widetilde{R}_{imlt}F_j{}^m = \widetilde{R}_{ijmt}F_l{}^m = \widetilde{R}_{ijlm}F_t{}^m.$$

In the anti-Kähler manifold (M_n, g, F), the Tachibana operator ϕ_F applied to the curvature $(0,4)$-tensor $\widetilde{R}$ is

$$\begin{aligned}(\phi_F\widetilde{R})_{kijlt} &= F_k{}^m(\partial_m\widetilde{R}_{ijlt}) - \partial_k(\widetilde{R}_{ijlm}F_t{}^m)\\ &= F_k{}^m(\nabla_m\widetilde{R}_{ijlt} + \Gamma^s_{mi}\widetilde{R}_{sjlt} + \Gamma^s_{mj}\widetilde{R}_{islt} + \Gamma^s_{ml}\widetilde{R}_{ijst} + \Gamma^s_{mt}\widetilde{R}_{ijls})\\ &\quad - F_t{}^m(\nabla_k\widetilde{R}_{ijlm} + \Gamma^s_{ki}\widetilde{R}_{sjlm} + \Gamma^s_{kj}\widetilde{R}_{islm} + \Gamma^s_{kl}\widetilde{R}_{ijsm} + \Gamma^s_{km}\widetilde{R}_{ijls})\\ &= (\nabla_m\widetilde{R}_{ijlt})F_k{}^m - (\nabla_k\widetilde{R}_{ijlm})F_m{}^t.\end{aligned}$$

By (3.2), we find

$$\begin{aligned}(\phi_F\widetilde{R})_{kijlt} &= (\phi_F R)_{kijlt}\\ &\quad + [(\nabla_m\sigma_{il})F_k{}^m - (\nabla_k\sigma_{im})F_l{}^m]F_{jt} - [(\nabla_m\sigma_{jl})F_k{}^m - (\nabla_k\sigma_{jm})F_l{}^m]F_{it}\\ &\quad + [(\nabla_m\sigma_{jt})F_k{}^m - (\nabla_k\sigma_{jm})F_t{}^m]F_{il} - [(\nabla_m\sigma_{it})F_k{}^m - (\nabla_k\sigma_{im})F_t{}^m]F_{jl}\\ &\quad + [(\nabla_m\sigma_{is})F_k{}^m F_l{}^s + \nabla_k\sigma_{il}]g_{jt} - [(\nabla_m\sigma_{js})F_k{}^m F_l{}^s + \nabla_k\sigma_{jl}]g_{it}\\ &\quad + [(\nabla_m\sigma_{js})F_k{}^m F_t{}^s + \nabla_k\sigma_{jt}]g_{il} - [(\nabla_m\sigma_{is})F_k{}^m F_t{}^s + \nabla_k\sigma_{it}]g_{jl}.\end{aligned}$$

Since the $(0,2)$-tensor σ is holomorphic, it satisfies

$$(\nabla_m\sigma_{ij})F^m_k = (\nabla_k\sigma_{mj})F^m_i = (\nabla_k\sigma_{im})F^m_j$$

(see the proof of Proposition 3.1). Hence, the last relation becomes $(\phi_F\widetilde{R})_{kijlt} = 0$, i.e., the curvature $(0,4)$-tensor $\widetilde{R}$ is a holomorphic tensor. $\square$

EXAMPLE 3.1. The pseudo-Euclidean space $\mathbb{R}^{2n}$ is given by pseudo-Euclidean metric $(g_{\alpha\beta}) = \left(\begin{smallmatrix}\delta_{ij} & 0\\ 0 & -\delta_{ij}\end{smallmatrix}\right)$ Let $\mathbb{C}^n$ be the complex space. The usual identification r of $\mathbb{C}^n$ with $\mathbb{R}^{2n}$ is given by

$$r : z = (z^1, z^2, \dots, z^n) \in \mathbb{C}^n \to r(z) = Z = (x^1, x^2, \dots, x^n; y^1, y^2, \dots, y^n) \in \mathbb{R}^{2n}$$

where $z^k = x^k + iy^k$, $k = 1, \dots, n$. The canonical complex structure F on $\mathbb{R}^{2n}$ is determined by the matrix

$$(F^\beta_\alpha) = \begin{pmatrix}0 & \delta^j_i\\ -\delta^j_i & 0\end{pmatrix} \quad \text{or} \quad (F_{\alpha\beta}) = \begin{pmatrix}0 & \delta_{ij}\\ \delta_{ij} & 0\end{pmatrix}$$

w.r.t. the natural basis of $\mathbb{R}^{2n}$. In the example, Greek indices take on values 1 to $2n$. For all Z, W on $\mathbb{R}^{2n}$ the metric g and the complex structure F on $\mathbb{R}^{2n}$ are related by the equality $g(FZ, FW) = -g(Z, W)$, that is, g is pure w.r.t. F. Hence $(\mathbb{R}^{2n}, g, F)$ is an anti-Kähler Euclidean space.

The components of the quarter semisymmetric metric F-connection in $(\mathbb{R}^{2n}, g, F)$ are

$$\begin{aligned}\widetilde{\Gamma}_{ij}^{k} = \widetilde{\Gamma}_{\bar{i}j}^{\bar{k}} = \widetilde{\Gamma}_{i\bar{j}}^{\bar{k}} = -\widetilde{\Gamma}_{\bar{i}\bar{j}}^{k} &= -(\partial_{\bar{j}} f)\delta_i^k - \left(\partial_{\bar{h}} f\right)\delta^{\overline{hk}}\delta_{ij},\\ \widetilde{\Gamma}_{\bar{i}j}^{k} = \widetilde{\Gamma}_{i\bar{j}}^{k} = \widetilde{\Gamma}_{\bar{i}\bar{j}}^{\bar{k}} = -\widetilde{\Gamma}_{ij}^{\bar{k}} &= (\partial_j f)\,\delta_i^k - (\partial_h f)\,\delta^{hk}\delta_{ij}.\end{aligned}$$

The torsion tensor of the complex semisymmetric metric F-connection has the components

$$\begin{aligned}\widetilde{S}_{ij}^{\;k} = \widetilde{S}_{\bar{i}j}^{\;\bar{k}} = \widetilde{S}_{i\bar{j}}^{\;\bar{k}} = -\widetilde{S}_{\bar{i}\bar{j}}^{\;k} &= (\partial_{\bar{i}} f)\delta_j^k - (\partial_{\bar{j}} f)\delta_i^k,\\ \widetilde{S}_{\bar{i}j}^{\;k} = \widetilde{S}_{i\bar{j}}^{\;k} = \widetilde{S}_{\bar{i}\bar{j}}^{\;\bar{k}} = -\widetilde{S}_{ij}^{\;\bar{k}} &= (\partial_j f)\delta_i^k - (\partial_i f)\delta_j^k.\end{aligned}$$

One verifies that the torsion tensor $\widetilde{S}$ is pure w.r.t. F and moreover $(\phi_F \widetilde{S})_{\sigma\alpha\beta}^{\gamma} = 0$, i.e., $\widetilde{S}$ is holomorphic.

The components of the curvature $(0,4)$-tensor $\widetilde{R}$ of the quarter semi-symmetric metric F-connection are

$$\begin{aligned}\widetilde{R}_{\bar{i}jkl} &= \widetilde{R}_{i\bar{j}kl} = \widetilde{R}_{ij\bar{k}l} = -\widetilde{R}_{\bar{i}\bar{j}k\bar{l}} = -\widetilde{R}_{\bar{i}j\bar{k}\bar{l}} = -\widetilde{R}_{i\bar{j}\bar{k}\bar{l}} = -\widetilde{R}_{\bar{i}\bar{j}\bar{k}l} = \widetilde{R}_{ijk\bar{l}}\\ &= \delta_{jl}\sigma_{ki} - \delta_{il}\sigma_{kj} + \delta_{ki}\sigma_{jl} - \delta_{kj}\sigma_{il},\\ \widetilde{R}_{\bar{i}j\bar{k}l} &= -\widetilde{R}_{ijkl} = \widetilde{R}_{ij\bar{k}\bar{l}} = \widetilde{R}_{\bar{i}jk\bar{l}} = \widetilde{R}_{\bar{i}\bar{j}kl} = -\widetilde{R}_{\bar{i}\bar{j}\bar{k}\bar{l}} = \widetilde{R}_{i\bar{j}\bar{k}l} = \widetilde{R}_{i\bar{j}k\bar{l}}\\ &= \delta_{jl}\sigma_{\bar{k}i} - \delta_{il}\sigma_{\bar{k}j} + \delta_{ki}\sigma_{\bar{j}l} - \delta_{kj}\sigma_{\bar{i}l}\end{aligned}$$

where

$$\begin{aligned}\sigma_{\bar{k}j} = \sigma_{k\bar{j}} &= \partial_{\bar{k}}\partial_j f + (\partial_{\bar{k}} f)(\partial_{\bar{j}} f) - (\partial_k f)(\partial_j f)\\ &\quad + \tfrac{1}{2}\delta^{hm}\delta_{kj}[(\partial_{\bar{h}} f)(\partial_{\bar{m}} f) + (\partial_h f)(\partial_m f)],\\ \sigma_{kj} = -\sigma_{\bar{k}\bar{j}} &= \partial_k\partial_j f + (\partial_{\bar{k}} f)(\partial_j f) + (\partial_k f)(\partial_{\bar{j}} f)\\ &\quad + \tfrac{1}{2}\delta_{kj}[\delta^{\bar{h}\bar{m}}(\partial_{\bar{h}} f)(\partial_m f) - \delta^{hm}(\partial_h f)(\partial_{\bar{m}} f)].\end{aligned}$$

Simple calculations show that $(\phi_F \sigma)_{\sigma\alpha\beta} = 0$. Using this, one checks that the curvature tensor $\widetilde{R}$ is pure w.r.t. F and furthermore $(\phi_F \widetilde{R})_{\sigma\alpha\beta\gamma\eta} = 0$, i.e., $\widetilde{R}$ is holomorphic.

Denote by $\widetilde{R}_{jk} = \widetilde{R}_{ijkl} g^{il}$ the Ricci tensor of the quarter-symmetric metric F-connection and R_{jk} the Ricci tensor of the Levi-Civita connection. Then the Ricci tensor has the components

$$\widetilde{R}_{jk} = R_{jk} + (4-n)F_k^{\;t}\sigma_{jt} - g_{jk}F_t^{\;m}\sigma_m^{\;t} - F_{jk}(\operatorname{trace}\sigma). \tag{3.5}$$

It is easy to see that $\widetilde{R}_{jk} = \widetilde{R}_{kj}$. In fact,

$$\widetilde{R}_{jk} - \widetilde{R}_{kj} = (4-n)\,F_k^{\;t}(\sigma_{jt} - \sigma_{tj}) = 0.$$

The scalar curvature $\widetilde{\tau} = \widetilde{R}_{jk} g^{jk}$ of the quarter-symmetric metric F-connection is given by $\widetilde{\tau} = \tau + 2(2-n) F_t{}^m \sigma_m{}^t$, where τ is the Riemann scalar curvature of g.

The conharmonic curvature tensor w.r.t. the quarter-symmetric metric F-connection is

$$\widetilde{V}_{ijkl} = \widetilde{R}_{ijkl} - \frac{1}{n-2}\big[\widetilde{R}_{jk} g_{il} - \widetilde{R}_{ik} g_{jl} - \widetilde{R}_{jl} g_{ik} + \widetilde{R}_{il} g_{jk}\big].$$

It follows from (3.2) and (3.5) that

$$\begin{aligned}\widetilde{V}_{ijkl} = {} & V_{ijkl} + F_{jl}\sigma_{ik} - F_{il}\sigma_{jk} + F_{ik}\sigma_{jl} - F_{jk}\sigma_{il} \\ & - \frac{1}{n-2}\big[(2F_k{}^t \sigma_{jt} - g_{jk} F_s{}^t \sigma_t^s - F_{jk}(\operatorname{trace}\sigma)) g_{il} \\ & \quad - (2F_k{}^t \sigma_{it} - g_{ik} F_s{}^t \sigma_t^s - F_{ik}(\operatorname{trace}\sigma)) g_{jl} \\ & \quad - (2F_l{}^t \sigma_{jt} - g_{jl} F_s{}^t \sigma_t^s - F_{jl}(\operatorname{trace}\sigma)) g_{ik} \\ & \quad + (2F_l{}^t \sigma_{it} - g_{il} F_s{}^t \sigma_t^s - F_{il}(\operatorname{trace}\sigma)) g_{jk}\big],\end{aligned} \tag{3.6}$$

where V_{ijkl} is the conharmonic curvature tensor w.r.t. the Levi-Civita connection. It is easy to see that it satisfies the relations

$$\widetilde{V}_{ijkl} = -\widetilde{V}_{jikl}, \quad \widetilde{V}_{ijkl} = -\widetilde{V}_{ijlk}, \quad \widetilde{V}_{ijkl} = \widetilde{V}_{klij}, \quad \widetilde{V}_{ijkl} + \widetilde{V}_{kijl} + \widetilde{V}_{jkil} = 0,$$

THEOREM 3.5. *If the conharmonic curvature tensor w.r.t. the quarter-symmetric metric F-connection vanishes, then*

$$(\nabla_l \nabla^t f) F_t^l + \frac{\tau}{2(2-n)} + \frac{4-n}{2} (\nabla_l f)(\nabla^l f) = 0$$

where τ is the Riemann scalar curvature of g.

PROOF. Suppose that $\widetilde{V}_{ijkl} = 0$; from (3.6) we get

$$\begin{aligned}0 = {} & V_{ijkl} + F_{jl}\sigma_{ik} - F_{il}\sigma_{jk} + F_{ik}\sigma_{jl} - F_{jk}\sigma_{il} \\ & - \frac{1}{n-2}\big[(2F_k{}^t \sigma_{jt} - g_{jk} F_s{}^t \sigma_t^s - F_{jk}(\operatorname{trace}\sigma)) g_{il} \\ & \quad - (2F_k{}^t \sigma_{it} - g_{ik} F_s{}^t \sigma_t^s - F_{ik}(\operatorname{trace}\sigma)) g_{jl} \\ & \quad - (2F_l{}^t \sigma_{jt} - g_{jl} F_s{}^t \sigma_t^s - F_{jl}(\operatorname{trace}\sigma)) g_{ik} \\ & \quad + (2F_l{}^t \sigma_{it} - g_{il} F_s{}^t \sigma_t^s - F_{il}(\operatorname{trace}\sigma)) g_{jk}\big],\end{aligned}$$

When we transvect the last relation by g^{il}, using $V_{ijkl} g^{il} = V_{ljk}{}^l = -\frac{\tau}{n-2} g_{jk}$, we obtain

$$(\nabla_l \nabla^t f) F_t^l + \frac{\tau}{2(2-n)} + \frac{4-n}{2} (\nabla_l f)(\nabla^l f) = 0. \qquad \square$$

4. Quarter-symmetric nonmetric F-connections

We define a linear connection $\overline{\nabla}$ in the anti-Kähler manifold (M_n, g, F) whose torsion is in the form (3.1), as follows:

$$\overline{\Gamma}_{ij}^k = \Gamma_{ij}^k + (1-\lambda)(p_j F_i{}^k + p_t F_j{}^t \delta_i{}^k) - \lambda(p_i F_j{}^k + p_t F_i{}^t \delta_j{}^k), \tag{4.1}$$

where the generator p_i is gradient of a locally holomorphic function f on M_n and $\lambda \neq 1 \in \mathbb{R}$. Calculating the covariant derivative of g and F w.r.t. the connection, we obtain

$$\overline{\nabla}_k g_{ij} = (\lambda-1)(p_i F_{jk} + p_j F_{ik} + p_t F_i{}^t g_{kj} + p_t F_j{}^t g_{ki}) - 2\lambda(p_k F_{ij} + p_t F_k{}^t g_{ij}) \neq 0$$

and $\overline{\nabla}_k F_i{}^j = 0$. We shall call the connection *a quarter-symmetric nonmetric F-connection*.

The curvature $(0,4)$-tensor of the quarter-symmetric nonmetric F- connection can be written

$$\overline{R}_{ijkl} = R_{ijkl} + F_{jl}\theta_{ik} - F_{il}\theta_{jk} + g_{jl}F_k{}^t\theta_{it} - g_{il}F_k{}^t\theta_{jt}, \tag{4.2}$$

where

$$\theta_{jk} = (1-\lambda)\nabla_j p_k - (1-\lambda)^2 p_k p_t F_j{}^t - (1-\lambda)^2 p_j p_t F_k{}^t. \tag{4.3}$$

The $(0,2)$-tensor θ is a symmetric tensor and also pure w.r.t. the complex structure F.

PROPOSITION 4.1. *The $(0,2)$-tensor θ given by (4.3) is a holomorphic tensor.*

PROOF. In the anti-Kähler manifold (M_n, g, F), in view of (4.3) we calculate

$$\begin{aligned}(\phi_F\theta)_{kij} &= (\nabla_m\theta_{ij})F_k{}^m - (\nabla_k\theta_{im})F_j{}^m \\ &= (\lambda-1)\big[(\nabla_m\nabla_i\nabla_j f)F_k{}^m - (\nabla_k\nabla_m\nabla_j f)F_i{}^m\big].\end{aligned}$$

Using the Ricci identity for the generator p_i, the above relation reduces to

$$(\phi_F\theta)_{kij} = \frac{1-\lambda}{2} p_s(R_{mij}{}^s F_k{}^m - R_{kim}{}^s F_j{}^m)$$

from which we get $(\phi_F\theta)_{kij} = 0$. □

It is clear that the curvature $(0,4)$-tensor has the properties $\overline{R}_{ijkl} = -\overline{R}_{jikl}$ and $\overline{R}_{ijkl} + \overline{R}_{kijl} + \overline{R}_{jkil} = 0$.

THEOREM 4.1. *The curvature $(0,4)$-tensor $\overline{R}$ of the quarter-symmetric nonmetric F-connection is a holomorphic tensor.*

PROOF. By the purity of the $(0,2)$-tensor θ, it follows from (4.2) that

$$\overline{R}_{mjkl}F_i{}^m = \overline{R}_{imkl}F_j{}^m = \overline{R}_{ijml}F_k{}^m = \overline{R}_{ijkm}F_l{}^m.$$

The Tachibana operator ϕ_F applied to the curvature tensor $\overline{R}$ is in the form

$$(\phi_F\overline{R})_{kijlt} = (\nabla_m\overline{R}_{ijlt})F_k{}^m - (\nabla_k\overline{R}_{ijlm})F_m{}^t.$$

Substitution (4.2) into the above relation gives

$$\begin{aligned}(\phi_F\overline{R})_{kijlt} &= (\phi_F R)_{kijlt} \\ &\quad + \big[(\nabla_m\theta_{il})F_k{}^m - (\nabla_k\theta_{im})F_l{}^m\big]F_j t - \big[(\nabla_m\theta_{jl})F_k{}^m - (\nabla_k\theta_{jm})F_l{}^m\big]F_{it} \\ &\quad + \big[(\nabla_m\theta_{is})F_k{}^m F_l{}^s + \nabla_k\theta_{jl}\big]g_{jt} - \big[(\nabla_m\theta_{is})F_k{}^m F_l{}^s + \nabla_k\theta_{il}\big]g_{it}.\end{aligned}$$

Thus, by Proposition 4.1, the last relation reduces to $(\phi_F\overline{R})_{kijlt} = 0$. □

The Ricci tensor of the quarter-symmetric nonmetric F-connection has the form $\overline{R}_{jk} = R_{jk} + (2-n)\, F_k{}^t\theta_{jt}$, wherefrom $\overline{R}_{jk} - \overline{R}_{kj} = (2-n)F_k{}^t(\theta_{jt} - \theta_{tj}) = 0$ i.e., the Ricci tensor is symmetric. The scalar curvature is $\overline{\tau} = \tau + (2-n)F_l{}^t\theta_t^l$.

The dual connection ${}^*\overline{\nabla}$ of any linear connection $\overline{\nabla}$ on a differentiable manifold M_n is given in [**5**] by

$$Xg(Y,Z) = g(\overline{\nabla}_X Y, Z) + g(Y, {}^*\overline{\nabla}_X Z)$$

for all vector fields X, Y and Z on M_n. In local coordinates, this equation can be written $\partial_k g_{ij} = \overline{\Gamma}_{ki}^m g_{mj} + {}^*\overline{\Gamma}_{kj}^m g_{im}$, where $\overline{\Gamma}_{ki}^m$ and ${}^*\overline{\Gamma}_{kj}^m$ are respectively the components of $\overline{\nabla}$ and ${}^*\overline{\nabla}$. The dual connection ${}^*\overline{\nabla}$ of the quarter-symmetric nonmetric F-connection has the components

$$(4.4) \qquad {}^*\overline{\Gamma}_{ij}^k = \Gamma_{ij}^k + (\lambda - 1)(p^k F_{ij} + p^t F_t^k g_{ij}) + \lambda(p_i F_j{}^k + p_t F_i{}^t \delta_j^k),$$

where $p^k = p_t g^{tk}$. The torsion tensor ${}^*\overline{S}_{ij}^k$ of the dual connection (4.4) is ${}^*\overline{S}_{ij}^k = -\lambda \widetilde{S}_{ij}^k$, where $\widetilde{S}_{ij}^k$ is the torsion tensor of the quarter-symmetric metric F- connection. Taking the covariant derivatives of the pseudo-Riemannian metric g and the complex structure F w.r.t. the dual connection (4.4), we find ${}^*\overline{\nabla}_k g_{ij} \neq 0$ and ${}^*\overline{\nabla}_k F_i{}^j = 0$, i.e., the dual connection (4.4) is another quarter-symmetric nonmetric F-connection.

Taking account of the relation between the curvature tensors of any linear connection and its dual connection (see, [**5**]), the curvature $(0,4)$-tensor of the dual connection (4.4) has the components

$${}^*\overline{R}_{ijlk} = -\overline{R}_{ijkl}, \quad {}^*\overline{R}_{ijkl} = R_{ijkl} + F_{ik}\theta_{jl} - F_{jk}\theta_{il} + g_{ik}F_l{}^t\theta_{jt} - g_{jk}F_l{}^t\theta_{it},$$

where $\overline{R}_{ijkl}$ is the curvature $(0,4)$-tensor of the quarter-symmetric nonmetric F-connection.

In view of the relation between ${}^*\overline{R}$ and $\overline{R}$, we directly state the following theorem.

THEOREM 4.2. *Let* (M_n, g, F) *be an anti-Kähler manifold endowed with the dual connection* (4.4)*. The curvature* $(0,4)$*-tensor* ${}^*\overline{R}$ *of the dual connection* (4.4) *satisfies:*

$${}^*\overline{R}_{ijkl} = -{}^*\overline{R}_{jikl}, \quad {}^*\overline{R}_{ijkl} + {}^*\overline{R}_{kijl} + {}^*\overline{R}_{jkil} = 0, \quad (\phi_F^* \overline{R})_{kijlt} = 0,$$

i.e., the curvature $(0,4)$*-tensor* ${}^*\overline{R}$ *is a holomorphic tensor.*

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