

## SOME INTEGRAL INEQUALITIES FOR THE POLAR DERIVATIVE OF POLYNOMIALS

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ABSTRACT. As a generalization of well-known result due to Turán [24] for polynomials having all their zeros in  $|z| \leq 1$ , Malik [17] proved that, if  $P(z)$  is a polynomial of degree  $n$ , having all its zeros in  $|z| \leq 1$ , then for any  $\delta > 0$ ,

$$n \left\{ \int_0^{2\pi} |P(e^{i\theta})|^\delta d\theta \right\}^{1/\delta} \leq \left\{ \int_0^{2\pi} |1 + e^{i\theta}|^\delta d\theta \right\}^{1/\delta} \max_{|z|=1} |P'(z)|.$$

We generalize the above inequality to polar derivatives, which as special cases include several known results in this area. Besides the paper contains some more results that generalize and sharpen several results known in this direction.

### 1. Introduction

Let  $P(z)$  be a polynomial of degree  $n$ . Then according to a classical result due to Bernstein [4],

$$(1.1) \quad \max_{|z|=1} |P'(z)| \leq n \max_{|z|=1} |P(z)|.$$

Inequality (1.1) is sharp and equality holds, if  $P(z)$  has all its zeros at the origin. If  $P(z)$  is a polynomial of degree  $n$ , having no zeros in  $|z| < 1$ , then Erdős [6] conjectured and later Lax [14] proved that

$$(1.2) \quad \max_{|z|=1} |P'(z)| \leq \frac{n}{2} \max_{|z|=1} |P(z)|.$$

Inequality (1.2) is best possible and equality holds for  $P(z) = a + bz^n$ , where  $|a| = |b|$ . For polynomials having no zeros in  $|z| > 1$  the corresponding inequality was proved by Turán [24]. Since the equality in Bernstein's inequality (1.1) holds for polynomials which have all their zeros at the origin, improvement in (1.1) is not possible if we consider polynomials having all their zeros inside the unit circle. For this reason, in this case, it may be interesting to obtain inequality in the reverse

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direction, and in this connection, Turán [24] proved that, if a polynomial  $P(z)$  has all its zeros in  $|z| \leq 1$ , then

$$(1.3) \quad \max_{|z|=1} |P'(z)| \geq \left(\frac{n}{2}\right) \max_{|z|=1} |P(z)|.$$

The result is best possible and equality holds in (1.3) for any polynomial which has all its zeros on  $|z| = 1$ .

The concept of ‘derivative of a polynomial’ has been generalized to ‘polar derivative of a polynomial’ as follows. If  $P(z)$  is a polynomial of degree  $n$ , then the *polar derivative of  $P(z)$  with respect to a complex number  $\alpha$*  is defined as

$$D_\alpha P(z) = nP(z) + (\alpha - z)P'(z).$$

Note that  $D_\alpha\{P(z)\}$  is a polynomial of degree at most  $n - 1$  and one could get the sense of ‘generalization’ from the fact that  $\lim_{\alpha \rightarrow \infty} \frac{D_\alpha\{P(z)\}}{\alpha} = P'(z)$ , uniformly with respect to  $z$  for  $|z| \leq R$ ;  $R > 0$ . More information on this topic can be obtained from the monographs [18–20].

Zygmund [25] extended Bernstein’s inequality (1.1) to  $L^p$  norm as

$$(1.4) \quad \left\{ \int_0^{2\pi} |P'(e^{i\theta})|^\delta d\theta \right\}^{1/\delta} \leq n \left\{ \int_0^{2\pi} |P(e^{i\theta})|^\delta d\theta \right\}^{1/\delta},$$

for any polynomial  $P(z)$  of degree  $n$  and for any  $\delta \geq 1$ . Inequality (1.4) is sharp and equality holds if  $P(z)$  has all its zeros at the origin. The above inequality of Zygmund was extended by Arestov [1] for  $0 < p < 1$ . Extensions of (1.4) to polar derivatives of complex polynomials can be seen in the book [10] (see also [11]).

Malik [17] obtained the  $L^p$  extension of (1.3) due to Turán [24] by proving that, if  $P(z)$  has all its zeros in  $|z| \leq 1$  then for any  $\delta > 0$ ,

$$(1.5) \quad n \left\{ \int_0^{2\pi} |P(e^{i\theta})|^\delta d\theta \right\}^{1/\delta} \leq \left\{ \int_0^{2\pi} |1 + e^{i\theta}|^\delta d\theta \right\}^{1/\delta} \max_{|z|=1} |P'(z)|.$$

In this paper we will extend and generalize the above inequality (1.5) to the class of polar derivatives of polynomials. In addition to this main result, we will prove a few more results vis-a-vis polar derivatives of polynomials.

## 2. Lemmas

LEMMA 2.1. *If  $P(z)$  is a polynomial of degree  $n \geq 1$  then for all  $R > 1$ ,*

$$\max_{|z|=R} |P(z)| \leq R^n \max_{|z|=1} |P(z)| - (R^n - R^{n-2})|P(0)|,$$

taking the value for  $R^{-1} = 1$ .

The above result is due to Frappier et al. [7]. The following lemma is due to Rahman and Schmeisser [21] (see also [5]).

LEMMA 2.2. *If  $P(z)$  is a polynomial of degree  $n$  which does not vanish in  $|z| < 1$ , then for every  $R \geq 1$  and  $\delta > 0$ , we have*

$$\int_0^{2\pi} |P(Re^{i\theta})|^\delta d\theta \leq \frac{\left\{ \int_0^{2\pi} |1 + R^n e^{i\theta}|^\delta d\theta \right\}}{\left\{ \int_0^{2\pi} |1 + e^{i\theta}|^\delta d\theta \right\}} \int_0^{2\pi} |P(e^{i\theta})|^\delta d\theta.$$

LEMMA 2.3. *If  $P(z)$  is a polynomial of degree at most  $n$  having all its zeros in  $|z| \geq 1$  then on  $|z| = 1$ ,  $|Q'(z)| \geq |P'(z)| + n(\min_{|z|=1} |P(z)|)$ , where  $Q(z) = z^n \overline{P(1/\bar{z})}$ .*

A proof of this Lemma is contained in the proof of Theorem 1 in [9].

LEMMA 2.4. *Let  $a \geq b > 0$ , and  $\delta > 0$ . Then*

$$\int_0^{2\pi} |a + be^{i\theta} f(e^{i\theta})|^\delta d\theta \leq \int_0^{2\pi} |a + be^{i\theta}|^\delta d\theta$$

for any analytic function  $f(z)$  in  $|z| \leq 1$  with  $|f(e^{i\theta})| \leq 1$ .

PROOF. The proof follows from subordination inequality [13] (see also [16]). If  $u$  is subharmonic in the unit disk  $|z| \leq 1$ ,  $v$  is the least harmonic majorant on the unit circle  $|z| = 1$ , and  $\phi$  is the analytic mapping from the unit disk into itself with  $\phi(0) = 0$  then  $I(u \circ \phi) \leq I(v \circ \phi) = v \circ \phi(0) = v(0) = I(v) = I(u)$ , implying

$$(2.1) \quad I(u \circ \phi) \leq I(u),$$

where  $I$  is the integral average over the unit circle  $|z| = 1$ . Now let  $F(z) = 1 + \frac{b}{a}z$  and  $\phi(z) = zf(z)$ . Then  $\phi$  is an analytic function mapping the unit disk into itself, and  $\phi(0) = 0$ . Taking  $u = |F|^\delta$  in (2.1) we get  $\int_0^{2\pi} |F \circ \phi(e^{i\theta})|^\delta d\theta \leq \int_0^{2\pi} |F(e^{i\theta})|^\delta d\theta$ , which is nothing but

$$\int_0^{2\pi} \left| 1 + \frac{b}{a} e^{i\theta} f(e^{i\theta}) \right|^\delta d\theta \leq \int_0^{2\pi} \left| 1 + \frac{b}{a} e^{i\theta} \right|^\delta d\theta.$$

Simple manipulations will yield the required inequality. □

LEMMA 2.5. *For any real numbers  $a, p, q > 0$ ,*

$$(2.2) \quad (p - q) \log a \leq \left[ (a)^{\frac{p-q}{2}} - (a)^{\frac{q-p}{2}} \right].$$

PROOF. One can observe that the result is straightforward when  $p = q$  and the equality holds in this case. Hence let us assume that  $p \neq q$ . We denote  $f(t) = a^t$ . Then  $f'(t) = a^t \log a$  is strictly convex on  $\mathbf{R}$ , and therefore

$$a^{\frac{p+q}{2}} \log a = f'\left(\frac{p+q}{2}\right) < \frac{f(p) - f(q)}{p - q} \implies \log a < \frac{a^p - a^q}{p - q} \cdot \frac{1}{a^{\frac{p+q}{2}}}.$$

With a minor simplification, we get inequality (2.2). □

We prove another simple lemma.

LEMMA 2.6. *If  $P(z) = \sum_0^n a_k z^k$  is a polynomial of degree  $n$  having no zeros in  $|z| < 1$  and  $\alpha$  is any complex number with  $|\alpha| < 1$ , then*

$$(2.3) \quad \int_0^{2\pi} \log \left| P(e^{i\theta}) + \frac{(\alpha - e^{i\theta})P'(e^{i\theta})}{n} \right| d\theta \leq \int_0^{2\pi} \log |P(e^{i\theta})| d\theta, \quad \text{if } \left| a_0 + \frac{\alpha a_1}{n} \right| \leq |a_0|,$$

$$(2.4) \quad \int_0^{2\pi} \log \left| P(e^{i\theta}) + \frac{(\alpha - e^{i\theta})P'(e^{i\theta})}{n} \right| d\theta \geq \int_0^{2\pi} \log |P(e^{i\theta})| d\theta, \quad \text{if } \left| a_0 + \frac{\alpha a_1}{n} \right| \geq |a_0|.$$

PROOF. Let  $P_1(z) = (P(z) + \frac{(\alpha-z)P'(z)}{n})$ . Since  $P(z)$  has no zeros in the disc  $|z| < 1$ , by Laguerre's theorem [15, 18] it follows that  $P_1(z)$  also has no zeros in  $|z| < 1$  for any  $\alpha$  with  $|\alpha| < 1$ . By applying Jensen's formula for the polynomials  $P(z)$  and  $P_1(z)$  separately, we get

$$(2.5) \quad \log |P(0)| = \frac{1}{2\pi} \int_0^{2\pi} \log |P(e^{i\theta})| d\theta, \quad \log |P_1(0)| = \frac{1}{2\pi} \int_0^{2\pi} \log |P_1(e^{i\theta})| d\theta.$$

Note that

$$\begin{aligned} \left| a_0 + \frac{\alpha a_1}{n} \right| \leq |a_0| &\Rightarrow |P_1(0)| \leq |P(0)| \Rightarrow \log |P_1(0)| \leq \log |P(0)|, \\ \left| a_0 + \frac{\alpha a_1}{n} \right| \geq |a_0| &\Rightarrow |P_1(0)| \geq |P(0)| \Rightarrow \log |P_1(0)| \geq \log |P(0)|. \end{aligned}$$

Using these two facts in (2.5), we get inequalities (2.3) and (2.4), respectively.  $\square$

LEMMA 2.7. *If  $P(z)$  is a polynomial of degree  $n$  having no zeros in  $|z| < 1$ , such that  $P'(0) = 0$  and  $\alpha$  is any complex number with  $|\alpha| < 1$  then*

$$(2.6) \quad \int_0^{2\pi} \log \left| P(e^{i\theta}) + \frac{(\alpha - e^{i\theta})P'(e^{i\theta})}{n} \right| d\theta = \int_0^{2\pi} \log |P(e^{i\theta})| d\theta.$$

PROOF. Since  $P(z)$  is a polynomial of degree  $n$  having no zeros in  $|z| < 1$ , such that  $P'(0) = 0$  and  $\alpha$  is any complex number with  $|\alpha| < 1$ , (2.5) is true here as well with the additional property  $P(0) = P_1(0)$ . But then  $\log |P(0)| = \log |P_1(0)|$ . Using this fact in (2.5), we get identity (2.6).  $\square$

### 3. Main Results and Proofs

One can see in the literature (refer [10] and [11]) that Turán-type inequalities have been extended and presented in more general  $L^P$  settings. Here we extend and generalize inequality (1.5) to the polar derivatives of complex polynomials. Theorem 3.1 given below might be a good companion for the recently published similar results due to Rather and Bhat [22]. In fact now we prove,

THEOREM 3.1. *If  $P(z)$  is a polynomial of degree  $n \geq 1$  having all its zeros in  $|z| \leq K$ ,  $K \geq 1$ , then for each  $\delta > 0$  and for any real or complex  $\alpha$  with  $|\alpha| \geq K$ ,*

$$(3.1) \quad n(|\alpha| - K) \left\{ \int_0^{2\pi} |P(e^{i\theta})|^\delta d\theta \right\}^{\frac{1}{\delta}} \\ \leq \left\{ \int_0^{2\pi} |1 + K^n e^{i\theta}|^\delta d\theta \right\}^{\frac{1}{\delta}} \left( \max_{|z|=1} |D_\alpha P(z)| - \frac{(K^n - K^{n-2})}{K^n} |D_\alpha P(0)| - \frac{nm}{K^{n-1}} \right),$$

where  $m = \min_{|z|=K} |P(z)|$  and take the value for  $K^{-1} = 1$

PROOF. The polynomial  $G(z) = P(Kz)$  has all its zeros in  $|z| \leq 1$  and hence its conjugate reciprocal polynomial  $H(z)$  has all its zeros in the disc  $|z| \geq 1$ . But then by Lemma 2.3, it follows that  $|H'(z)| \leq |G'(z)| - n \min_{|z|=1} |G(z)|$ , which is on  $|z| = 1$  equivalent to

$$(3.2) \quad |H'(z)| \leq |G'(z)| - nm.$$

Now on  $|z| = 1$ ,

$$|D_{\alpha/K}G(z)| \geq \frac{|\alpha|}{K}|G'(z)| - |zG(z) - zG'(z)| \geq \frac{|\alpha|}{K}|G'(z)| - |H'(z)|.$$

Using (3.2) we get  $|D_{\alpha/K}G(z)| \geq \left(\frac{|\alpha|}{K} - 1\right)|G'(z)| + nm$  on  $|z| = 1$ . Hence

$$\begin{aligned} (3.3) \quad & \max_{|z|=1} |D_{\alpha/K}G(z)| \geq \left(\frac{|\alpha|}{K} - 1\right) \max_{|z|=1} |G'(z)| + nm \\ \implies & \max_{|z|=K} |D_{\alpha}P(z)| \geq \left(\frac{|\alpha| - K}{K}\right) \max_{|z|=1} |G'(z)| + nm \\ \implies & K \max_{|z|=K} |D_{\alpha}P(z)| \geq (|\alpha| - K) \max_{|z|=1} |G'(z)| + Knm, \end{aligned}$$

which can be expressed after applying Lemma 2.1 on  $|D_{\alpha}P(z)|$  for any polynomial  $P(z)$  of degree  $n \geq 2$  with the understanding of taking  $K^{-1} = 1$ , as

$$K^n \max_{|z|=1} |D_{\alpha}P(z)| - (K^n - K^{n-2})|D_{\alpha}P(0)| \geq (|\alpha| - K) \max_{|z|=1} |G'(z)| + Knm,$$

and hence for any  $\delta > 0$ ,

$$(3.4) \quad \left\{ K^n \max_{|z|=1} |D_{\alpha}P(z)| - (K^n - K^{n-2})|D_{\alpha}P(0)| - Knm \right\}^{\delta} \geq (|\alpha| - K)^{\delta} \left\{ \max_{|z|=1} |G'(z)| \right\}^{\delta}.$$

Now let  $f(z) = \frac{H'(z)}{nH(z) - zH'(z)}$ . One can observe that  $nH(z) - zH'(z) \equiv z^{n-1} \overline{G'(\frac{1}{z})} \neq 0$  in  $|z| < 1$  and hence the function  $f(z)$  is analytic for  $|z| \leq 1$  with  $|f(z)| \leq 1$  on  $|z| = 1$ . Hence by Lemma 2.4 we have for any  $\delta > 0$ ,

$$(3.5) \quad \int_0^{2\pi} |1 + e^{i\theta} f(e^{i\theta})|^{\delta} d\theta \leq \int_0^{2\pi} |1 + e^{i\theta}|^{\delta} d\theta.$$

Now

$$n|H(z)| = |nH(z) - zH'(z) + zH'(z)| = \left| 1 + \frac{zH'(z)}{nH(z) - zH'(z)} \right| |nH(z) - zH'(z)|.$$

But then  $n|H(z)| = |1 + zf(z)||G'(z)|$  on  $|z| = 1$  and hence

$$(3.6) \quad n|H(z)| \leq |1 + zf(z)| \max_{|z|=1} |G'(z)|$$

on  $|z| = 1$ . Therefore from (3.6) and (3.5) we have

$$(3.7) \quad n^{\delta} \int_0^{2\pi} |H(e^{i\theta})|^{\delta} d\theta \leq \int_0^{2\pi} |1 + e^{i\theta}|^{\delta} d\theta \left\{ \max_{|z|=1} |G'(z)| \right\}^{\delta}.$$

Using (3.7) in (3.4) we get

$$(3.8) \quad \left\{ K^n \max_{|z|=1} |D_{\alpha}P(z)| - (K^n - K^{n-2})|D_{\alpha}P(0)| - Knm \right\}^{\delta} \geq n^{\delta} (|\alpha| - K)^{\delta} \frac{\int_0^{2\pi} |H(e^{i\theta})|^{\delta} d\theta}{\int_0^{2\pi} |1 + e^{i\theta}|^{\delta} d\theta}$$

Since  $H(z)$  does not vanish in  $|z| < 1$ , by Lemma 2.2, for  $K \geq 1$  and any  $\delta > 0$ , we have

$$\int_0^{2\pi} |H(Ke^{i\theta})|^\delta d\theta \leq \frac{\left\{ \int_0^{2\pi} |1 + K^n e^{i\theta}|^\delta d\theta \right\}}{\left\{ \int_0^{2\pi} |1 + e^{i\theta}|^\delta d\theta \right\}} \int_0^{2\pi} |H(e^{i\theta})|^\delta d\theta.$$

From the fact that  $|H(Ke^{i\theta})| = K^n |P(e^{i\theta})|$  it follows that

$$(3.9) \quad K^{n\delta} \int_0^{2\pi} |P(e^{i\theta})|^\delta d\theta \leq \frac{\left\{ \int_0^{2\pi} |1 + K^n e^{i\theta}|^\delta d\theta \right\}}{\left\{ \int_0^{2\pi} |1 + e^{i\theta}|^\delta d\theta \right\}} \int_0^{2\pi} |H(e^{i\theta})|^\delta d\theta, \quad K \geq 1.$$

The required inequality follows from inequalities (3.9), and (3.8).  $\square$

Dividing (3.1) by  $|\alpha|$  and taking  $|\alpha| \rightarrow \infty$  we get the following result.

**COROLLARY 3.1.** *If  $P(z)$  has all its zeros in  $|z| \leq K$ ,  $K \geq 1$ , then for any  $p > 0$ ,*

$$(3.10) \quad n \left\{ \int_0^{2\pi} |P(e^{i\theta})|^\delta d\theta \right\}^{1/\delta} \leq \left\{ \int_0^{2\pi} |1 + K^n e^{i\theta}|^\delta d\theta \right\}^{1/\delta} \max_{|z|=1} |P'(z)|.$$

*This is independently proved by Aziz [2].*

Taking  $\delta \rightarrow \infty$  in (3.1) we get the analogous result in supremum norm as follows.

**COROLLARY 3.2.** *If  $P(z)$  is a polynomial of degree  $n$  having all its zeros in  $|z| \leq K$ ,  $K \geq 1$ , then for each  $\delta > 0$  and for any real or complex  $\alpha$  with  $|\alpha| \geq K$ ,*

$$(3.11) \quad n(|\alpha| - K) \max_{|z|=1} |P(z)| \leq (1 + K^n) \left( \max_{|z|=1} |D_\alpha P(z)| - \frac{(K^n - K^{n-2})}{K^n} |D_\alpha P(0)| - \frac{nm}{K^{n-1}} \right),$$

*where  $m = \min_{|z|=K} |P(z)|$  and take the value for  $K^{-1} = 1$ . This is an improvement to the inequality with the same hypothesis given by*

$$(3.12) \quad n(|\alpha| - K) \max_{|z|=1} |P(z)| \leq (1 + K^n) \max_{|z|=1} |D_\alpha P(z)|,$$

*which is independently proved by Aziz and Rather [3].*

Dividing (3.11) by  $|\alpha|$ , and taking  $|\alpha| \rightarrow \infty$ , we get the following result, which is independently proved by Govil [8].

**COROLLARY 3.3.** *If  $P(z)$  is a polynomial of degree  $n$  having all its zeros in  $|z| \leq K$ , where  $K \geq 1$ , then*

$$(3.13) \quad \max_{|z|=1} |P'(z)| \geq \left( \frac{n}{1 + K^n} \right) \max_{|z|=1} |P(z)|.$$

*The above result is also best possible and equality holds in for  $P(z) = z^n + K^n$ .*

The below result is quite significant in the context of the inequalities involving polar derivative  $D_\alpha P(z)$  of a polynomial  $P(z)$ . We did not find any results on inequalities involving the polar derivative  $D_\alpha P(z)$  of  $P(z)$  having no zeros in  $|z| < 1$  and  $|\alpha| < 1$ . The result we prove below might be a beginning in this direction.

**THEOREM 3.2.** *If  $P(z)$  is a polynomial of degree  $n$  having no zeros in  $|z| < 1$ , such that  $P'(0) = 0$  and  $\alpha$  is any complex number with  $|\alpha| < 1$  then for any  $p, q > 0$ ,*

$$(3.14) \quad (p - q) \int_0^{2\pi} \log |P(e^{i\theta})| d\theta \leq \int_0^{2\pi} \left\{ \left| \frac{D_\alpha P(e^{i\theta})}{n} \right|^{\frac{p-q}{2}} - \left| \frac{D_\alpha P(e^{i\theta})}{n} \right|^{\frac{q-p}{2}} \right\} d\theta.$$

**PROOF.** Since  $P(z)$  has no zeros in  $|z| < 1$ , such that  $P'(0) = 0$  and  $|\alpha| < 1$  then from Lemma 2.7, we have

$$(3.15) \quad \int_0^{2\pi} \log \left| P(e^{i\theta}) + \frac{(\alpha - e^{i\theta})P'(e^{i\theta})}{n} \right| d\theta = \int_0^{2\pi} \log |P(e^{i\theta})| d\theta.$$

Now applying Lemma 2.5 to the integrand part of the left hand side of (3.15), by taking

$$a = \left| P(e^{i\theta}) + \frac{(\alpha - e^{i\theta})P'(e^{i\theta})}{n} \right|$$

with  $p, q > 0$ , we get

$$(3.16) \quad (p - q) \log \left| P(e^{i\theta}) + \frac{(\alpha - e^{i\theta})P'(e^{i\theta})}{n} \right| \\ \leq \left[ \left| P(e^{i\theta}) + \frac{(\alpha - e^{i\theta})P'(e^{i\theta})}{n} \right|^{\frac{p-q}{2}} - \left| P(e^{i\theta}) + \frac{(\alpha - e^{i\theta})P'(e^{i\theta})}{n} \right|^{\frac{q-p}{2}} \right].$$

Now taking integral on both sides of (3.16) with proper limits of integration we get

$$(3.17) \quad (p - q) \int_0^{2\pi} \log \left| P(e^{i\theta}) + \frac{(\alpha - e^{i\theta})P'(e^{i\theta})}{n} \right| d\theta \\ \leq \int_0^{2\pi} \left[ \left| P(e^{i\theta}) + \frac{(\alpha - e^{i\theta})P'(e^{i\theta})}{n} \right|^{\frac{p-q}{2}} - \left| P(e^{i\theta}) + \frac{(\alpha - e^{i\theta})P'(e^{i\theta})}{n} \right|^{\frac{q-p}{2}} \right] d\theta.$$

By noting that  $D_\alpha P(z) = P(z) + (\alpha - z)P'(z)$  and inequality (3.17) in conjunction with (3.15) gives the desired inequality (3.14).  $\square$

In a paper by Govil and McTume [12], a result on polynomials having all zeros in  $|z| \leq K$ ,  $K \geq 1$ , was proved, which states that, if  $P(z)$  is a polynomial of degree  $n$  having all its zeros in  $|z| \leq K$ ,  $K \geq 1$ , then for any real or complex  $\alpha$  with  $|\alpha| \geq 1 + K + K^n$ ,

$$(3.18) \quad \max_{|z|=1} |D_\alpha P(z)| \geq n \left( \frac{|\alpha| - K}{1 + K^n} \right) \max_{|z|=1} |P(z)| + n \left( \frac{|\alpha| - (1 + K + K^n)}{1 + K^n} \right) m,$$

where  $m = \min_{|z|=K} |P(z)|$ . The inequality without the term containing ‘ $m$ ’ in (3.18) was proved by Aziz and Rather [3]. Here we noted that the condition  $|\alpha| \geq 1 + K + K^n$ , may be relaxed to  $|\alpha| \geq K$ , for the class of polynomials having all zeros in  $|z| \leq K$ ,  $K \geq 1$ , with certain restriction. In other words, does there exist

a similar result but for  $|\alpha| \leq 1 + K + K^n$ ? We will answer it partially in the result given below.

**THEOREM 3.3.** *If  $P(z)$  is a polynomial of degree  $n$  having all its zeros in  $|z| \leq K$ ,  $K \geq 1$ , then for any complex  $\alpha$  with  $1 + K + K^n \geq |\alpha| \geq K$ , satisfying  $\max_{|z|=1} |P(z)| \geq \frac{1+K^n}{|\alpha|-K}m$ , we have*

$$(3.19) \quad \max_{|z|=1} |D_\alpha P(z)| \geq n \left( \frac{|\alpha| - K}{1 + K^n} \right) \max_{|z|=1} |P(z)| + n \left( \frac{(1 + K + K^n) - |\alpha|}{1 + K^n} \right) m,$$

where  $m = \min_{|z|=K} |P(z)|$ .

**PROOF.** We can assume without loss of generality that  $P(z)$  has all its zeros in  $|z| < K$ ,  $K \geq 1$ , for if  $P(z)$  has a zero on  $|z| = K$ , then  $m = 0$  and, in view of inequality (3.12) in Corollary 3.2, the theorem holds trivially. Observe that for  $|\alpha| = K$ , the result is quite evident and hence we assume that  $|\alpha| > K$ . Since  $P(z)$  has all its zeros in  $|z| < K$ ,  $K \geq 1$ , it is straightforward that, for every  $\lambda$  with  $|\lambda| < 1$ , the polynomial  $P(z) + \lambda m$  also has all its zeros in  $|z| < K$ ,  $K \geq 1$ . Thus inequality (3.12) when applied to  $P(z) + \lambda m$  gives

$$\max_{|z|=1} |D_\alpha [P(z) + \lambda m]| \geq n \left( \frac{|\alpha| - K}{1 + K^n} \right) \max_{|z|=1} |P(z) + \lambda m|$$

Let  $z_0$  be a point on  $|z| = 1$  such that  $|D_\alpha P(z_0)| = \max_{|z|=1} |D_\alpha [P(z) + \lambda m]|$ . Then

$$(3.20) \quad |D_\alpha P(z_0) + n\lambda m| \geq n \left( \frac{|\alpha| - K}{1 + K^n} \right) |P(z_0) + \lambda m|.$$

If we choose the argument of  $\lambda$  such that  $|D_\alpha P(z_0) + n\lambda m| = |D_\alpha P(z_0)| - n|\lambda|m$ , then from (3.20), we get

$$||D_\alpha P(z_0)| - n|\lambda|m| \geq n \left( \frac{|\alpha| - K}{1 + K^n} \right) \left( \max_{|z|=1} |P(z)| - |\lambda|m \right).$$

Since  $\max_{|z|=1} |P(z)| \geq \frac{1+K^n}{|\alpha|-K}m$ , and in view of (3.12) it follows that  $|D_\alpha P(z_0)| \geq nm$ . But then for any  $\lambda$  with  $|\lambda| < 1$ , it is true that  $|D_\alpha P(z_0)| > n|\lambda|m$ . Therefore

$$\max_{|z|=1} |D_\alpha P(z)| - n|\lambda|m \geq n \left( \frac{|\alpha| - K}{1 + K^n} \right) \left( \max_{|z|=1} |P(z)| - |\lambda|m \right),$$

which is further equivalent to

$$\max_{|z|=1} |D_\alpha P(z)| \geq n \left( \frac{|\alpha| - K}{1 + K^n} \right) \max_{|z|=1} |P(z)| + n|\lambda| \left( \frac{(1 + K + K^n) - |\alpha|}{1 + K^n} \right) m.$$

Letting  $|\lambda| \rightarrow 1$ , in the above inequality we get that

$$\max_{|z|=1} |D_\alpha P(z)| \geq n \left( \frac{|\alpha| - K}{1 + K^n} \right) \max_{|z|=1} |P(z)| + n \left( \frac{(1 + K + K^n) - |\alpha|}{1 + K^n} \right) m.$$

which is the desired inequality (3.19).  $\square$



As a consequence of Laguerre’s Theorem [15] we have a result that, if  $P(z)$  is a polynomial of degree  $n$  such that  $P(z) \neq 0$ , in  $|z| < 1$ , then the polar derivative of  $P(z)$  with respect to  $\alpha$  with  $|\alpha| < 1$  given by  $D_\alpha P(z) = nP(z) - (z - \alpha)P'(z) \neq 0$  in  $|z| < 1$ . Szego [23] proved this first and several other proofs can be found in [18].

In this work interestingly we found that this property holds for the polynomial  $nP(z) + (1 - \alpha)zP'(z)$  as well, with hypothesis that  $|\alpha| \leq 1$ . Even though the result seems to be simple, but the proof involves some interesting facts on real part of it.

**THEOREM 3.4.** *If  $P(z) = a_n \prod_{k=1}^n (z - z_k)$  is a polynomial of degree  $n$  having no zeros in the disc  $|z| < 1$  then the polynomial  $Q(z) = nP(z) + (\alpha - 1)zP'(z)$  has no zeros in  $|z| < 1$  for every  $\alpha$  with  $|\alpha| \leq 1$ .*

**PROOF.** We will assume that  $\alpha \neq 1$ , otherwise the case is trivial. Let  $w_j = 1/z_j$ ,  $1 \leq j \leq n$ . Then for  $|z| < 1$  we have

$$\begin{aligned} \frac{zP'(z)}{P(z)} - \frac{n}{1 - \alpha} &= \sum_{k=1}^n \frac{z}{z - z_k} - \frac{n}{1 - \alpha} = \sum_{k=1}^n \frac{zw_k}{zw_k - 1} - \frac{n}{1 - \alpha} \\ &= \frac{1}{2} \sum_{k=1}^n \left( 1 - \frac{1 + zw_k}{1 - zw_k} \right) - \frac{n}{1 - \alpha} \\ &= -\frac{1}{2} \sum_{k=1}^n \left( \frac{1 + zw_k}{1 - zw_k} \right) + \frac{n}{2} - \frac{n}{1 - \alpha} \\ &= -\frac{1}{2} \sum_{k=1}^n \left( \frac{1 + zw_k}{1 - zw_k} \right) - \frac{n(1 + \alpha)}{2(1 - \alpha)}. \end{aligned}$$

Since  $|zw_k| < 1$ ,

$$\operatorname{Re} \frac{1 + zw_k}{1 - zw_k} > 0$$

and since  $|\alpha| \leq 1$

$$\operatorname{Re} \frac{1 + \alpha}{1 - \alpha} \geq 0.$$

Hence for  $|z| < 1$ , and  $|\alpha| \leq 1$ ,

$$\operatorname{Re} \left( \frac{zP'(z)}{P(z)} + \frac{n}{\alpha - 1} \right) < 0.$$

Therefore the conclusion of the theorem follows. □

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