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SOME CONVOLUTION INEQUALITIES IN REALIZED HOMOGENEOUS BESOV AND TRIEBEL–LIZORKIN SPACES

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Abstract. Using the realizations, we study some convolution inequalities in the realized homogeneous Besov spaces $\dot{\tilde{B}}^s_{p,q}(\mathbb{R}^n)$ and the realized homogeneous Triebel–Lizorkin spaces $\dot{\tilde{F}}^s_{p,q}(\mathbb{R}^n)$. We also deduce for the homogeneous Sobolev spaces $\dot{W}^m_p(\mathbb{R}^n)$ in certain sense.

1. Introduction

We study some properties of the convolution in homogeneous Besov spaces $\dot{B}^s_{p,q}(\mathbb{R}^n)$ and homogeneous Triebel–Lizorkin spaces $\dot{F}^s_{p,q}(\mathbb{R}^n)$. This type of properties has been studied by Peetre [**13**, Chapter 8] considering $\dot{B}^s_{p,q}(\mathbb{R}^n)$, see also Bourdaud [**3**]. As these spaces are defined modulo polynomials, since $\|f\|_{\dot{B}^s_{p,q}} = \|f\|_{\dot{F}^s_{p,q}} = 0$ if and only if, f is a polynomial on $\mathbb{R}^n$, then in our investigation, we will consider *realized* homogeneous Besov spaces $\dot{\tilde{B}}^s_{p,q}(\mathbb{R}^n)$ and *realized* homogeneous Triebel–Lizorkin spaces $\dot{\tilde{F}}^s_{p,q}(\mathbb{R}^n)$, which are defined in the tempered distributions space $\mathcal{S}'(\mathbb{R}^n)$. We will employ the notation $\dot{A}^s_{p,q}(\mathbb{R}^n)$ for either $\dot{B}^s_{p,q}(\mathbb{R}^n)$ or $\dot{F}^s_{p,q}(\mathbb{R}^n)$, the notation $\dot{\tilde{A}}^s_{p,q}(\mathbb{R}^n)$ for either $\dot{\tilde{B}}^s_{p,q}(\mathbb{R}^n)$ or $\dot{\tilde{F}}^s_{p,q}(\mathbb{R}^n)$ and their initials B and F, respectively. Also, we will omit the symbol $\mathbb{R}^n$ in notations since all function spaces which occur in this work are defined on $\mathbb{R}^n$. We will also use the following two notations:

- If $f \in \mathcal{S}'$, $[f]_{\mathcal{P}}$ denotes the equivalence class of f modulo all polynomials on $\mathbb{R}^n$.
- $\mathcal{E}'$ is the set of distributions with compact support in $\mathbb{R}^n$.

So in the convolution, we essentially prove an estimate in $\dot{\tilde{A}}^s_{p,q}$ (see below, Theorem 2.2 and Remark 2.2 in which we explain why we work with the realized spaces) using the convergence in $\mathcal{S}'_\nu$ (the space of tempered distributions modulo polynomials $\mathcal{P}_\nu$

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of degree $< \nu$), where for any 4-tuples (n,s,p,q) and throughout this paper the number $\nu \in \mathbb{N}_0$ is defined by

$$\nu := ([s-n/p]+1)_+ \text{ if } s-n/p \notin \mathbb{N}_0 \text{ or } q>1 \text{ in } B\text{-case } (p>1 \text{ in } F\text{-case})$$
$$\nu := s-n/p \text{ if } s-n/p \in \mathbb{N}_0 \text{ and } 0<q\leqslant 1 \text{ in } B\text{-case } (0<p\leqslant 1 \text{ in } F\text{-case}),$$

(see [**4**]), with $[t]$ denotes the greatest integer less than or equal to $t \in \mathbb{R}$.

Notation and plan of the paper. As usual, $\mathbb{N}$ denotes the set of natural numbers, $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$, $\mathbb{Z}$ the integers and $\mathbb{R}$ the real numbers. For $a \in \mathbb{R}$ we put $a_+ := \max(0,a)$. The symbol $\hookrightarrow$ indicates a continuous embedding. For $0<p\leqslant\infty$ we denote by $\|\cdot\|_p$ the quasi-norm of L_p. We will use the parameters s,p and q as $s \in \mathbb{R}$ and $0<p,q\leqslant\infty$ ($p<\infty$ in the F-case) along the paper unless otherwise stated. For a function θ defined on $\mathbb{R}^n$, we set $\theta_\lambda := \lambda^{-n}\theta(\lambda^{-1}(\cdot))$ for all $\lambda>0$ and $\check{\theta}(x) := \theta(-x)$. The standard norms in the Schwartz space $\mathcal{S}$ are defined by

$$\zeta_m(f) := \sup_{|\alpha|\leqslant m} \sup_{x\in\mathbb{R}^n} (1+|x|)^m |f^{(\alpha)}(x)|, \quad (m \in \mathbb{N}_0).$$

For $f \in L_1$,

$$\mathcal{F}f(x) = \hat{f}(\xi) := \int_{\mathbb{R}^n} \mathrm{e}^{-\mathrm{i}x\cdot\xi} f(x)\mathrm{d}x$$

is the Fourier transform and $\mathcal{F}^{-1}f(x) := (2\pi)^{-n}\hat{f}(-x)$ is the inverse Fourier transform. The operators $\mathcal{F}$ and $\mathcal{F}^{-1}$ are extended to the whole $\mathcal{S}'$ in the usual way.

For $k \in \mathbb{N}_0 \cup \{\infty\}$, $\mathcal{P}_k$ denotes the set of all polynomials on $\mathbb{R}^n$ of degree $<k$ (in particular $\mathcal{P}_0 = \{0\}$, $\mathcal{P}_1 = \{c\}, \dots, \mathcal{P}_\infty$ the set of all polynomials). $\mathcal{S}_k$ will be used for the set of all $\varphi \in \mathcal{S}$ such that $\langle u,\varphi\rangle = 0$ ($\forall u \in \mathcal{P}_k$), its topological dual is $\mathcal{S}'_k$. The mapping which takes any $[f]_{\mathcal{P}}$ to the restriction of f to $\mathcal{S}_k$ is an isomorphism from $\mathcal{S}'/\mathcal{P}_k$ onto $\mathcal{S}'_k$.

The constants $c, c_1, \dots$ are strictly positive, depend only on the fixed parameters $n,s,p,q,\dots$, their values may change from line to line.

This work is organized as follows. In Section 2 we state the main results. In Section 3 we collect some needed tools. Section 4 is devoted to the proofs. In the last section, we give applications and an extension to Sobolev homogeneous spaces.

2. Statement of the main results

The Littlewood–Paley decomposition plays a major role here, then once and for all, we fix two functions ρ and γ, where ρ is a positive C^∞ and radial such that $0\leqslant\rho\leqslant 1$, $\rho(\xi)=1$ for $|\xi|\leqslant 1$ and $\rho(\xi)=0$ for $|\xi|\geqslant 3/2$, and $\gamma(\xi) := \rho(\xi)-\rho(2\xi)$ which is supported by $1/2\leqslant|\xi|\leqslant 3/2$. Then we define the operators Q_j and S_j ($\forall j \in \mathbb{Z}$) by $\widehat{Q_j f} := \gamma(2^{-j}(\cdot))\hat{f}$ and $\widehat{S_j f} := \rho(2^{-j}(\cdot))\hat{f}$. We also fix a positive and radial function $\tilde{\gamma} \in \mathcal{D}(\mathbb{R}^n \setminus \{0\})$ such that $\tilde{\gamma}\gamma = \gamma$. We associate $\tilde{Q}_j$ ($\forall j \in \mathbb{Z}$) defined by $\widehat{\tilde{Q}_j f} := \tilde{\gamma}(2^{-j}(\cdot))\hat{f}$. Now, for brevity we set $\omega := \omega(p,q)$ such that

$$(2.1) \qquad 1/\omega = 1/p - 1/q \;\text{ if }\; p\leqslant\min(1,q), \quad \omega = q' \;\text{ if }\; p>1 \;\text{ or }\; q\leqslant p\leqslant 1,$$

where, here and throughout the paper, $q' := q/(q-1)$ if $q>1$ and $q' := \infty$ if $0<q\leqslant 1$. So we have our first result:

THEOREM 2.1. *Let ω be given as in* (2.1). *We put $r := \min(1,p)$ and $\mu := -s + (n/p - n)_+$. Let $f \in \dot{A}^s_{p,q}$ and $\theta \in \dot{A}^\mu_{r,\omega}$. Then $\sum_{j\in\mathbb{Z}} \tilde{Q}_j\theta * Q_j f$ converges in $\mathcal{S}'_\nu$ to an element denoted by $\theta \circledast f$, such that*

$$\|\theta \circledast f\|_p \leqslant c\|\theta\|_{\dot{A}^\mu_{r,\omega}}\|f\|_{\dot{A}^s_{p,q}} \tag{2.2}$$

holds, where the constant $c > 0$ is independent of f and θ (with $c = 1$ if $p \geqslant 1$).

REMARK 2.1. By taking θ_λ instead of θ in the above theorem (recall $\theta_\lambda := \lambda^{-n}\theta(\lambda^{-1}(\cdot))$), we obtain a generalization of [**13**, Theorem 1, p. 156] given in B-case for $p \geqslant 1$ and $q = \infty$, see Proposition 2.1 below. Note that owing to (2.1), $\omega \geqslant r$, then $\dot{B}^\mu_{r,r} \hookrightarrow \dot{A}^\mu_{r,\omega}$, in particular $\dot{B}^{-s}_{1,1} \hookrightarrow \dot{A}^{-s}_{1,q'}$ for $p \geqslant 1$ which covers the result given in the previous reference.

Secondly and similarly to (2.2), we wish to give an inequality for the usual convolution. Since in (2.2) taking $\theta * f$ instead of $\theta \circledast f$ is not true in general (see Subsection 5.1 below), we then pass to $\dot{\tilde{A}}^s_{p,q}$, where the distributions vanishing at infinity play an important role.

DEFINITION 2.1. We say that a distribution $f \in \mathcal{S}'$ vanishes at infinity if $\lim_{\lambda\to 0} f(\lambda^{-1}(\cdot)) = 0$ in $\mathcal{S}'$. The set of all such distributions is denoted by $\tilde{C}_0$.

Examples of such distributions are:
(i) $f \in \tilde{C}_0$ if $f \in L_p$ $(1 \leqslant p < \infty)$; (ii) $\partial_j f \in \tilde{C}_0$ if either $f \in L_\infty$ or $f \in \tilde{C}_0$.

Using the notion of the realization, see e.g., [**2**], we now recall the definition of $\dot{\tilde{A}}^s_{p,q}$ according to [**4**] or [**11**]:

The space $\dot{\tilde{A}}^s_{p,q}$ is the set of $f \in \mathcal{S}'$ such that $[f]_\mathcal{P} \in \dot{A}^s_{p,q}$ and $f^{(\alpha)} \in \tilde{C}_0$ $(\forall|\alpha| = \nu)$, and one of the following three conditions:

(1) There is no supplementary condition if either $s < n/p$, or $s = n/p$ and $0 < q \leqslant 1$ in B-case ($0 < p \leqslant 1$ in F-case); here $\nu = 0$.
(2) f is of class $C^{\nu-1}$ and $f^{(\beta)}(0) = 0$ for $|\beta| \leqslant \nu - 1$, if either $s - n/p \in \mathbb{R}^+ \setminus \mathbb{N}_0$, or $s - n/p \in \mathbb{N}$ and $0 < q \leqslant 1$ in B-case ($0 < p \leqslant 1$ in F-case); here either $\nu = [s - n/p] + 1$ or $\nu = s - n/p$, respectively; here $\nu \geqslant 1$.
(3) f is of class $C^{\nu-1}$ with $f^{(\beta)}(0) = \sum_{j>0}(Q_j f)^{(\beta)}(0)$, $|\beta| \leqslant \nu - 1$, if $s - n/p \in \mathbb{N}_0$ and $q > 1$ in B-case ($p > 1$ in F-case); here $\nu = s - n/p + 1 \geqslant 1$.

$\dot{\tilde{A}}^s_{p,q}$ endowed with $\|f\|_{\dot{\tilde{A}}^s_{p,q}} := \|[f]_\mathcal{P}\|_{\dot{A}^s_{p,q}}$ is a quasi-Banach space. Then we have the following statement:

THEOREM 2.2. *Let r, μ and ω be real numbers given as in Theorem* 2.1. *Then there exists a constant $c > 0$ (with $c = 1$ if $p \geqslant 1$) such that*

$$\|\theta * f\|_p \leqslant c\|[\theta]_\mathcal{P}\|_{\dot{A}^\mu_{r,\omega}}\|[f]_\mathcal{P}\|_{\dot{A}^s_{p,q}} \tag{2.3}$$

holds, for all $f \in \dot{\tilde{A}}^s_{p,q}$ and all θ satisfying $[\theta]_\mathcal{P} \in \dot{A}^\mu_{r,\omega}$ and either $\theta \in \mathcal{S}$ or $\theta \in \mathcal{E}'$.

REMARK 2.2. The condition on f guarantees a "good" representative. Indeed, if we replace the assumption $f \in \dot{\tilde{A}}^s_{p,q}$ by only $[f]_\mathcal{P} \in \dot{A}^s_{p,q}$, it is possible to fall on a wrong choice of representative which yields a contradiction. For instance, assume that (2.3) is valid in that case. Let f be a nonzero polynomial on $\mathbb{R}^n$, then $\|[f]_{\mathcal{P}}\|_{\dot{A}^s_{p,q}} = 0$. We take $\theta := \delta$ (Dirac distribution at the origin), it is not difficult to get $[\delta]_{\mathcal{P}} \in \dot{B}^{n/p-n}_{p,\infty}$ $(0 < p \leqslant \infty)$, see e.g., the beginning of Subsection 5.1, then

– if $0 < q \leqslant p \leqslant 1$, then $\|[\theta]_{\mathcal{P}}\|_{\dot{B}^{n/p-n}_{p,\infty}} \|[f]_{\mathcal{P}}\|_{\dot{B}^0_{p,q}} = 0$,

– if $1 < p \leqslant \infty$ and $0 < q \leqslant 1$, then $\|[\theta]_{\mathcal{P}}\|_{\dot{B}^0_{1,\infty}} \|[f]_{\mathcal{P}}\|_{\dot{B}^0_{p,q}} = 0$,

however $\theta * f = f$, thus it is impossible to satisfy (2.3) since its left-hand side is ∞.

Remark 2.3. If $\theta \in \mathcal{S}_\infty$, then Theorem 2.2 holds with only $[f]_{\mathcal{P}} \in \dot{A}^s_{p,q}$. Indeed, by Lemma 3.1 (see below) $\theta * f \in \mathcal{S}'$, and if $[f_1]_{\mathcal{P}} = [f_2]_{\mathcal{P}} = f$, then $f_1 - f_2 = \mathcal{P} \in \mathcal{P}_\infty$ and $\mathcal{P} * \theta = 0$. Recall that $\mathcal{F}(x^\alpha * \theta) = c\hat{\theta}\delta^{(\alpha)} = 0$, since $\hat{\theta}^{(\beta)}(0) = 0$ for all $\alpha, \beta \in \mathbb{N}^n_0$.

In connection with the assertion in [**13**, pp. 156–159] given for the homogeneous Besov spaces, we have:

Proposition 2.1. *Let r, μ and ω be real numbers given as in Theorem 2.1.*

(i) *There exists a constant $c > 0$ (with $c = 1$ if $p \geqslant 1$) such that*

$$\lambda^{-s}\Big\|\sum_{j\in\mathbb{Z}} \tilde{Q}_j\theta_\lambda * Q_j f\Big\|_p \leqslant c\|\theta\|_{\dot{A}^\mu_{r,\omega}} \|f\|_{\dot{A}^s_{p,q}}, \quad (\forall\lambda > 0) \tag{2.4}$$

holds, for all $f \in \dot{A}^s_{p,q}$ and all $\theta \in \dot{A}^\mu_{r,\omega}$

(ii) *There exists a constant $c > 0$ (with $c = 1$ if $p \geqslant 1$) such that*

$$\lambda^{-s}\|\theta_\lambda * f\|_p \leqslant c\|[\theta]_{\mathcal{P}}\|_{\dot{A}^\mu_{r,\omega}} \|[f]_{\mathcal{P}}\|_{\dot{A}^s_{p,q}}, \quad (\forall\lambda > 0) \tag{2.5}$$

holds, for all $f \in \dot{\tilde{A}}^s_{p,q}$ and all θ satisfying $[\theta]_{\mathcal{P}} \in \dot{A}^\mu_{r,\omega}$ and either $\theta \in \mathcal{S}$ or $\theta \in \mathcal{E}'$.

In (2.4)–(2.5), the particular case $\lambda := 2^{-k}$ $(k \in \mathbb{Z})$ yields an interesting situation when the $\ell_q(\mathbb{Z})$ quasi-norm is taken of these affirmations, according to [**13**, Remark 1, p. 159] at least for the B-case, see also [**3**] in case $p \geqslant 1$; ($\ell_q(\mathbb{Z})$ is the set of all sequences $(a_k)_{k\in\mathbb{Z}} \subset \mathbb{C}$ such that $\|(a_k)\|_{\ell_q} := (\sum_{k\in\mathbb{Z}} |a_k|^q)^{1/q} < \infty$). Namely we have the following two results, where we start with the B-spaces.

Theorem 2.3. *Let $r := \min(1, p)$, $t := \min(1, p, q)$ and $\mu := -s + (n/p - n)_+$.*

(i) *Let $f \in \dot{B}^s_{p,q}$ and $\theta \in \dot{B}^\mu_{r,t}$. Then $\sum_{j\in\mathbb{Z}} \tilde{Q}_j\theta_{2^{-k}} * Q_j f$ converges in $\mathcal{S}'_\nu$ to an element denoted by $\theta_{2^{-k}} \circledast f$ for all $k \in \mathbb{Z}$. Moreover, there exists a constant $c > 0$ (with $c = 1$ if $p \geqslant 1$) such that*

$$\Big(\sum_{k\in\mathbb{Z}} 2^{ksq}\|\theta_{2^{-k}} \circledast f\|_p^q\Big)^{1/q} \leqslant c\|\theta\|_{\dot{B}^\mu_{r,t}} \|f\|_{\dot{B}^s_{p,q}}$$

holds, for all such f and θ.

(ii) *Let $f \in \dot{\tilde{B}}^s_{p,q}$. Let θ be such that $[\theta]_{\mathcal{P}} \in \dot{B}^\mu_{r,t}$ and either $\theta \in \mathcal{S}$ or $\theta \in \mathcal{E}'$. Then*

$$\Big(\sum_{k\in\mathbb{Z}} 2^{ksq}\|\theta_{2^{-k}} * f\|_p^q\Big)^{1/q} \leqslant c\|[\theta]_{\mathcal{P}}\|_{\dot{B}^\mu_{r,t}} \|[f]_{\mathcal{P}}\|_{\dot{B}^s_{p,q}}$$

holds. The positive constant c does not depend on f and θ; (if $p \geqslant 1$ then $c = 1$).

For the case of the F-spaces, we use poised homogeneous spaces of Besov $\dot{B}^{s,a}_{p,q}$, introduced in e.g., [**12**], see Section 3 below.

THEOREM 2.4. *Let $a > n/\min(p,q)$ and $t := \min(1,q)$.*

(i) *Let $f \in \dot{F}^s_{p,q}$ and $\theta \in \dot{B}^{-s,a}_{1,t}$. Then $\sum_{j\in\mathbb{Z}} \tilde{Q}_j \theta_{2^{-k}} * Q_j f$ converges in $\mathcal{S}'_\nu$ to an element denoted by $\theta_{2^{-k}} \circledast f$ for all $k \in \mathbb{Z}$. Moreover, there exists a constant $c > 0$ such that*

$$\Big\| \Big(\sum_{k\in\mathbb{Z}} 2^{ksq} |\theta_{2^{-k}} \circledast f|^q \Big)^{1/q} \Big\|_p \leqslant c \|\theta\|_{\dot{B}^{-s,a}_{1,t}} \|f\|_{\dot{F}^s_{p,q}}$$

holds, for all such f and θ.

(ii) *Let $f \in \dot{\tilde{F}}^s_{p,q}$. Let θ be such that $[\theta]_{\mathcal{P}} \in \dot{B}^{-s,a}_{1,t}$ and either $\theta \in \mathcal{S}$ or $\theta \in \mathcal{E}'$. Then*

$$\Big\| \Big(\sum_{k\in\mathbb{Z}} 2^{ksq} |\theta_{2^{-k}} * f|^q \Big)^{1/q} \Big\|_p \leqslant c \|[\theta]_{\mathcal{P}}\|_{\dot{B}^{-s,a}_{1,t}} \|[f]_{\mathcal{P}}\|_{\dot{F}^s_{p,q}}$$

holds. The positive constant c does not depend on f and θ.

3. Preliminaries

This section contains preparations, definitions and a characterization for realized homogeneous spaces. We first recall that the operators Q_j and S_j take values in the space of analytical functions of exponential type, see Paley–Wiener theorem, e.g., [**15**, Theorem 29.2, p. 311] or [**16**, Remark 2.3.1/2, p. 45]. They are defined on $\mathcal{S}'_\infty$ and $\mathcal{S}'$, respectively, since $Q_j f(x) = 0$ if and only if, $f \in \mathcal{P}_\infty$. For brevity, we make use of the following convention:

$$\text{If } \ f \in \mathcal{S}'_\infty \ \text{ we define } \ Q_j f := Q_j f_1 \ \text{ for all } \ f_1 \ \text{ such that } \ [f_1]_{\mathcal{P}} = f.$$

We will exploit the following assertions:

LEMMA 3.1. *Let $k \in \mathbb{N}_0 \cup \{\infty\}$. If $f \in \mathcal{S}'_k$ ($\mathcal{S}_k$, respectively) and $\theta \in \mathcal{S}_k$ ($\mathcal{E}'$, respectively), then $\theta * f \in \mathcal{S}'$ ($\mathcal{S}_k$, respectively).*

PROOF. Assume that $f \in \mathcal{S}'_k$ and $\theta \in \mathcal{S}_k$. For all $\varphi \in \mathcal{D}$ we obtain $\langle \theta * f, \varphi\rangle = \langle f, \bar{\bar{\theta}} * \varphi\rangle$. Clearly, $\bar{\bar{\theta}} * \varphi \in \mathcal{S}_k$ since $\mathcal{F}(\bar{\bar{\theta}} * \varphi) = (2\pi)^n (\mathcal{F}^{-1}\theta)\hat{\varphi}$ and $(\mathcal{F}^{-1}\theta)^{(\alpha)}(0) = 0$ for all $|\alpha| < k$. The assertion follows since $\mathcal{D}$ is dense in $\mathcal{S}$.

Now suppose that $f \in \mathcal{S}_k$ and $\theta \in \mathcal{E}'$. It suffices to observe that $\mathcal{F}(\theta * f) = \hat{\theta}\,\hat{f}$ and both $\hat{f}^{(\alpha)}(0) = 0$ if $|\alpha| < k$ and $\hat{\theta}$ is a function of class C^∞ with $|\hat{\theta}^{(\beta)}(\xi)| \leqslant c_{\beta,m}(1+|\xi|)^m$ for all $m \in \mathbb{N}_0$ and all $\beta \in \mathbb{N}_0^n$, see e.g., [**9**, Theorem 1.7.5, p. 20]. □

LEMMA 3.2. *Let $N \in \mathbb{N}_0$. There exist constants $c_1, c_2 > 0$ and a number $m \in \mathbb{N}_0$, such that for all $\varphi \in \mathcal{D}(\mathbb{R}^n \setminus \{0\})$ and all $\psi \in \mathcal{D}$, (we put $\hat{\varphi}_j := \varphi(2^{-j}(\cdot))$ and $\hat{\psi}_j := \psi(2^{-j}(\cdot))$), it holds:*

(i) *$\|\varphi_j * f\|_p \leqslant c_1 2^{-jN} \zeta_m(f)$ for all $f \in \mathcal{S}$ and all $j \in \mathbb{N}_0$.*

(ii) *$\|\varphi_j * f\|_p + \|\psi_j * f\|_p \leqslant c_2 2^{jN} \zeta_m(f)$ for all $f \in \mathcal{S}_N$ and all $j \in \mathbb{Z} \setminus \mathbb{N}$.*

PROOF. For (i), it suffices in $\varphi_j * f(x) = \int_{\mathbb{R}^n} f(x-y)\varphi_j(y)\mathrm{d}y$ to apply the N-th degree Taylor formula with integral remainder of $y \to f(x-y)$ at x. However for (ii), we change the roles between f and φ_j as $\int_{\mathbb{R}^n} f(y)\varphi_j(x-y)\mathrm{d}y$, then again the N-th degree Taylor formula of $y \to \varphi_j(x-y)$ at x. Similarly for $\psi_j f$. See [**11**] for more details. □

This lemma yields the convergence of the Littlewood–Paley decomposition in the following sense: we have $f = \sum_{j\in\mathbb{Z}} Q_j f$ in either $\mathcal{S}_\infty$ or $\mathcal{S}'_\infty$, and $f = S_k f + \sum_{j>k} Q_j f$ in either $\mathcal{S}$ or $\mathcal{S}'$. On the other hand, we will make use of the following classical inequalities (see e.g., [**16**, Remark 1, p. 18, Remark 2, p. 28]):

PROPOSITION 3.1. (i) *If $0 < p \leqslant q \leqslant \infty$ and $R > 0$, then it holds $\|f\|_q \leqslant cR^{n/p-n/q}\|f\|_p$ for all $f \in L_p$ satisfying $\operatorname{supp}\hat{f} \subseteq \{\xi \in \mathbb{R}^n : |\xi| \leqslant R\}$.*

(ii) *If $0 < p \leqslant 1$ and $R > 0$, then $\|f*g\|_p \leqslant cR^{n/p-n}\|f\|_p\|g\|_p$ for all $f, g \in \mathcal{S}'$ satisfying that $\operatorname{supp}\hat{f}$ and $\operatorname{supp}\hat{g}$ are subsets of $\{\xi \in \mathbb{R}^n : |\xi| \leqslant R\}$. If $p = 1$ then $c = 1$, that is Young's inequality.*

For more details about $\dot{B}^s_{p,q}$ and $\dot{F}^s_{p,q}$, we refer to e.g., [**6**, **10**, **13**], however we recall definitions and some properties.

DEFINITION 3.1. The homogeneous Besov space $\dot{B}^s_{p,q}$ and Triebel–Lizorkin space $\dot{F}^s_{p,q}$ are the sets of $f \in \mathcal{S}'_\infty$ such that $\|f\|_{\dot{B}^s_{p,q}} := (\sum_{j\in\mathbb{Z}} 2^{jsq}\|Q_j f\|_p^q)^{1/q} < \infty$ and $\|f\|_{\dot{F}^s_{p,q}} := \|(\sum_{j\in\mathbb{Z}} 2^{jsq}|Q_j f|^q)^{1/q}\|_p < \infty$, respectively.

$\dot{A}^s_{p,q}$ are quasi-Banach spaces for the above quasi-seminorms, and do not depend on functions ρ and γ. We have in particular:

- $\mathcal{S}_\infty \hookrightarrow \dot{A}^s_{p,q} \hookrightarrow \mathcal{S}'_\infty$,
- the homogeneity property $\|f\|_{\dot{A}^s_{p,q}} \equiv \lambda^{n/p-s}\|f(\lambda(\cdot))\|_{\dot{A}^s_{p,q}}$ ($\forall\lambda > 0$, $\forall f \in \dot{A}^s_{p,q}$),
- an equivalent quasi-seminorm $\|f\|_{\dot{A}^s_{p,q}} \equiv \sum_{|\alpha|=m} \|f^{(\alpha)}\|_{\dot{A}^{s-m}_{p,q}}$ ($\forall m \in \mathbb{N}$, $\forall f \in \dot{A}^s_{p,q}$) (see [**5**, Proposition 5] and [**6**, Proposition 8]),
- and some embeddings $\dot{A}^s_{p,q_1} \hookrightarrow \dot{A}^s_{p,q_2}$ if $q_1 < q_2$, $\dot{B}^s_{p,\min(p,q)} \hookrightarrow \dot{F}^s_{p,q} \hookrightarrow \dot{B}^s_{p,\max(p,q)}$. If $s_1 > s_2$, $0 < p_1 < p_2 < \infty$, $0 < q, r \leqslant \infty$ and $s_1 - n/p_1 = s_2 - n/p_2$, then $\dot{B}^{s_1}_{p_1,q} \hookrightarrow \dot{B}^{s_2}_{p_2,q} \hookrightarrow \dot{B}^{s_2-n/p_2}_{\infty,q}$, $\dot{F}^{s_1}_{p_1,q} \hookrightarrow \dot{B}^{s_2}_{p_2,p_1}$ and $\dot{F}^{s_1}_{p_1,q} \hookrightarrow \dot{F}^{s_2}_{p_2,r}$, see [**10**, Theorem 2.1].

DEFINITION 3.2. Let $a \geqslant 0$. The poised homogeneous space of Besov $\dot{B}^{s,a}_{p,q}$ is the set of $f \in \mathcal{S}'_\infty$ such that $\|f\|_{\dot{B}^{s,a}_{p,q}} := (\sum_{j\in\mathbb{Z}} 2^{jsq}\|(1+2^j|\cdot|)^a Q_j f\|_p^q)^{1/q} < \infty$.

The most properties of $\dot{B}^s_{p,q}$ are hold for the $\dot{B}^{s,a}_{p,q}$-case, e.g., $\dot{B}^{s,0}_{p,q} = \dot{B}^s_{p,q}$, $\dot{B}^{s,a}_{p,q} \hookrightarrow \dot{B}^{s,a}_{p,q_1}$ if $q \leqslant q_1$, $\dot{B}^{s,a}_{p,q} \hookrightarrow \dot{B}^{s_1,a}_{p_1,q}$ if $s - n/p = s_1 - n/p_1$ and $p < p_1, \ldots$

In connection with the number ν (see Section 1) we have the following characterization, see [**4**, Proposition 4.6]:

PROPOSITION 3.2. *For all* $f \in \dot{A}^s_{p,q}$, *the series* $\sum_{j\in\mathbb{Z}} Q_j f$ *converges in* $\mathcal{S}'_\nu$ *to an element denoted by* $\sigma_\nu(f)$ *which satisfies* $f = [\sigma_\nu(f)]_{\mathcal{P}}$ *in* $\mathcal{S}'_\infty$ *and* $\partial^\alpha \sigma_\nu(f) \in \tilde{C}_0$ *for all* $|\alpha| = \nu$.

By this proposition we have a continuous linear mapping $f \mapsto \sigma_\nu(f)$ from $\dot{A}^s_{p,q}$ to $\mathcal{S}'_\nu$. In this context we recall the notion of realization.

DEFINITION 3.3. Let $k \in \mathbb{N}_0 \cup \{\infty\}$. Let E be a vector subspace of $\mathcal{S}'_\infty$ endowed with a quasi-norm such that $E \hookrightarrow \mathcal{S}'_\infty$. A realization of E in $\mathcal{S}'_k$ is a continuous linear mapping $\sigma : E \to \mathcal{S}'_k$ such that $[\sigma(f)]_{\mathcal{P}} = f$ $(\forall f \in E)$. The image set $\sigma(E)$ is called the realized space of E.

4. Proofs

We begin by the following assertion, an estimate of Nikol'skij-type representation method.

PROPOSITION 4.1. *Let* r, μ *and* ω *be real numbers given as in Theorem* 2.1. *Let* a_1, a_2, b_1, b_2 *be positive numbers such that* $0 < a_1 < b_1$ *and* $0 < a_2 < b_2$. *Let* $(u_j)_{j\in\mathbb{Z}}$ *and* $(v_j)_{j\in\mathbb{Z}}$ *be sequences in* $\mathcal{S}'$, *such that*

- $\widehat{u_j}$ *and* $\widehat{v_j}$ *have compact supports in the annulus* $a_1 2^j \leqslant |\xi| \leqslant b_1 2^j$ *and* $a_2 2^j \leqslant |\xi| \leqslant b_2 2^j$, *respectively,*
- $A := (\sum_{j\in\mathbb{Z}} 2^{jsq}\|u_j\|_p^q)^{1/q} < \infty$ *and* $B := (\sum_{j\in\mathbb{Z}} 2^{j\mu\omega}\|v_j\|_r^\omega)^{1/\omega} < \infty$.

(i) *Then* $\sum_{j\in\mathbb{Z}} u_j * v_j$ *converges in* $\mathcal{S}'_\nu$ *to an element denoted by* $u \circledast v$, *satisfying*

$$\|u \circledast v\|_p \leqslant cAB, \tag{4.1}$$

where the positive constant c *depends only on the parameters* $n, s, p, q, a_1, a_2, b_1$ *and* b_2, *with* $c = 1$ *if* $p \geqslant 1$.

(ii) *If* $0 < p < \infty$, *then we can replace* A *and* B *by* $A_1 := \|(\sum_{j\in\mathbb{Z}} 2^{jsq}|u_j|^q)^{1/q}\|_p$ *and* $B_1 := \|(\sum_{j\in\mathbb{Z}} 2^{j\mu\omega}|v_j|^\omega)^{1/\omega}\|_r$, *with* $A_1 < \infty$ *and* $B_1 < \infty$, *respectively, in the preceding statement.*

PROOF. *Step* 1: *proof of* (i). *Substep* 1.1: *convergence in* $\mathcal{S}'_\nu$. We first note that if $b_1 < a_2$ or $b_2 < a_1$ then $u_j * v_j = 0$, so these cases are excluded. We introduce a radial and positive function $\eta \in \mathcal{D}(\mathbb{R}^n \setminus \{0\})$ supported in $\min(a_1/2, a_2/2) \leqslant |\xi| \leqslant \max(2b_1, 2b_2)$, and $\eta(\xi) = 1$ for $\min(a_1, a_2) \leqslant |\xi| \leqslant \max(b_1, b_2)$. We define the operators $\eta_j := \eta(2^{-j}D)$ $(\forall j \in \mathbb{Z})$, i.e., $\widehat{\eta_j f} := \eta(2^{-j}(\cdot))\hat{f}$, then we have $u_j * v_j = \eta_j(u_j * v_j)$. Taking η's properties into account, we can write $\langle u_j * v_j, \varphi\rangle = \langle u_j * v_j, \eta_j\varphi\rangle$ for all $\varphi \in \mathcal{S}_\nu$, and prove

$$\sum_{j\in\mathbb{Z}} |\langle u_j * v_j, \eta_j\varphi\rangle| < \infty. \tag{4.2}$$

The case $1 < p \leqslant \infty$: We begin by $|\langle u_j * v_j, \eta_j\varphi\rangle| \leqslant \|u_j * v_j\|_p \|\eta_j\varphi\|_{p'}$. By Lemma 3.2, $\|\eta_j\varphi\|_{p'} \leqslant c\zeta_m(\varphi)\min(2^{-jN}, 2^{j\nu})$. Then using Young's inequality, we get

$$\sum_{j\in\mathbb{Z}}|\langle u_j*v_j,\eta_j\varphi\rangle|\leqslant c_1\zeta_m(\varphi)\sum_{j\in\mathbb{Z}}\min(2^{-jN},2^{j\nu})\big(2^{js}\|u_j\|_p\big)\big(2^{-sj}\|v_j\|_1\big)\leqslant c_2AB. \tag{4.3}$$

The case $0<p\leqslant 1$: We estimate the first term in (4.2) as $|\langle u_j*v_j,\eta_j\varphi\rangle|\leqslant\|u_j*v_j\|_1\|\eta_j\varphi\|_\infty$. Then, $\|\eta_j\varphi\|_\infty\leqslant c\zeta_m(\varphi)\min(2^{-jN},2^{j\nu})$ (see Lemma 3.2). Also, by definition of ν we have $\nu+n/p-n>0$. Then choosing an integer $N>n/p-n$ and using both Bernstein and convolution inequalities, we obtain

$$\begin{aligned}\sum_{j\in\mathbb{Z}}|\langle u_j*v_j,\eta_j\varphi\rangle|&\leqslant c_1\zeta_m(\varphi)\sum_{j\in\mathbb{Z}}2^{(n/p-n)j}\min(2^{-jN},2^{j\nu})\|u_j*v_j\|_p\\&\leqslant c_2\sum_{j\in\mathbb{Z}}2^{(n/p-n)j}\min(2^{-jN},2^{j\nu})\big(2^{js}\|u_j\|_p\big)\big(2^{j(n/p-n-s)}\|v_j\|_p\big)\\&\leqslant c_3AB\sum_{j\in\mathbb{Z}}2^{(n/p-n)j}\min(2^{-jN},2^{j\nu})\leqslant c_4AB.\end{aligned}\tag{4.4}$$

Substep 1.2: *proof of*(4.1). First, assume that $1\leqslant p\leqslant\infty$. By inequalities of Young and Hölder with exponents q and q' for $q>1$, we have

$$\sum_{j\in\mathbb{Z}}\|u_j*v_j\|_p\leqslant\sum_{j\in\mathbb{Z}}(2^{js}\|u_j\|_p)(2^{-js}\|v_j\|_1)\leqslant AB. \tag{4.5}$$

For $0<q\leqslant 1$, we use the following elementary inequality

$$\Big(\sum_j a_j\Big)^d\leqslant\sum_j a_j^d,\quad\forall a_j\geqslant 0,\quad 0<d\leqslant 1, \tag{4.6}$$

in the second term of (4.5) with $d:=q$, then

$$\sum_{j\in\mathbb{Z}}(2^{js}\|u_j\|_p)(2^{-js}\|v_j\|_1)\leqslant\Big(\sum_{j\in\mathbb{Z}}(2^{js}\|u_j\|_p)^q\Big)^{1/q}\sup_{k\in\mathbb{Z}}(2^{-ks}\|v_k\|_1)\leqslant AB. \tag{4.7}$$

Clearly that (4.5) and (4.7) describe the behaviour of constant c in the right-hand side of (4.1), that is $c=1$.

Second, let $0<p<1$. By using (4.6) with $d:=p$ and convolution inequality, we get

$$\|u\circledast v\|_p\leqslant\Big(\sum_{j\in\mathbb{Z}}\|u_j*v_j\|_p^p\Big)^{1/p}\leqslant c\Big(\sum_{j\in\mathbb{Z}}(2^{js}\|u_j\|_p)^p(2^{j(n/p-n-s)}\|v_j\|_p)^p\Big)^{1/p}. \tag{4.8}$$

• If $p<q$, we apply Hölder inequality with exponents q/p and ω/p, the last term in (4.8) is bounded by cAB.

• If $q\leqslant p<1$, we use (4.6) with $d:=q/p$, and the last term in (4.8) is bounded by

$$c_1\Big(\sum_{j\in\mathbb{Z}}(2^{js}\|u_j\|_p)^q(2^{j(n/p-n-s)}\|v_j\|_p)^q\Big)^{1/q}\leqslant c_2AB.$$

Step 2*: proof of* (ii). *Substep* 2.1: *convergence in* $\mathcal{S}'_\nu$. We use the notations of Substep 1.1. From (4.3), if $1<p<\infty$, as $(2^{js}\|u_j\|_p)(2^{-sj}\|v_j\|_1)\leqslant A_1B_1$

($\forall j \in \mathbb{Z}$), then (4.2) holds. Similar if $0 < p \leqslant 1$, i.e., as in (4.4) since it holds $(2^{js}\|u_j\|_p)(2^{j(n/p-n-s)}\|v_j\|_p) \leqslant A_1 B_1$ ($\forall j \in \mathbb{Z}$).

Substep 2.2: *proof of* (4.1) *with* A_1 *and* B_1. Assume that $1 \leqslant p < \infty$. By Hölder inequality with exponents q and q' for $q > 1$, and by Minkowski inequality with respect to L_p, we have

$$\begin{aligned} (4.9)\quad \Big\|\sum_{j\in\mathbb{Z}} |u_j * v_j|\Big\|_p &\leqslant \Big\|\int_{\mathbb{R}^n}\sum_{j\in\mathbb{Z}} \big(2^{js}|u_j(\cdot - y)|\big)\big(2^{-js}|v_j(y)|\big)\mathrm{d}y\Big\|_p \\ &\leqslant \int_{\mathbb{R}^n} \Big\|\Big(\sum_{j\in\mathbb{Z}} 2^{jsq}|u_j(\cdot - y)|^q\Big)^{1/q}\Big\|_p \Big(\sum_{j\in\mathbb{Z}} 2^{-jsq'}|v_j(y)|^{q'}\Big)^{1/q'}\mathrm{d}y = A_1B_1. \end{aligned}$$

For $0 < q \leqslant 1$, we use (4.6) with $d := q$ in the first line of (4.9), then

$$\begin{aligned} (4.10)\quad \Big\|\sum_{j\in\mathbb{Z}} |u_j * v_j|\Big\|_p &\leqslant \Big\|\int_{\mathbb{R}^n}\Big(\sum_{j\in\mathbb{Z}} 2^{jsq}|u_j(\cdot - y)|^q\Big)^{1/q}\sup_{k\in\mathbb{Z}} 2^{-ks}|v_k(y)|\mathrm{d}y\Big\|_p \\ &\leqslant \int_{\mathbb{R}^n} \Big\|\Big(\sum_{j\in\mathbb{Z}} 2^{jsq}|u_j(\cdot - y)|^q\Big)^{1/q}\Big\|_p \sup_{k\in\mathbb{Z}} 2^{-ks}|v_k(y)|\mathrm{d}y = A_1B_1. \end{aligned}$$

As above in (4.5) and (4.7), inequalities (4.9)–(4.10) show that the constant c in the right-hand side of (4.1) is equal to 1.

Now, the case $0 < p < 1$. We first suppose $p < q$ (here $q \in]0,\infty]$). From (4.8) and by Hölder inequality with exponents q/p and ω/p, we get

$$(4.11)\quad \Big(\sum_{j\in\mathbb{Z}} \|u_j * v_j\|_p^p\Big)^{1/p} \leqslant c\Big(\sum_{j\in\mathbb{Z}} (2^{js}\|u_j\|_p)^q\Big)^{1/q}\Big(\sum_{j\in\mathbb{Z}} (2^{j(n/p-n-s)}\|v_j\|_p)^\omega\Big)^{1/\omega}.$$

Since $p \leqslant \omega$, by Minkowski inequality, the right-hand side of (4.11) is bounded by

$$c_1\Big\|\Big(\sum_{j\in\mathbb{Z}} (2^{js}|u_j|)^q\Big)^{1/q}\Big\|_p\Big\|\Big(\sum_{j\in\mathbb{Z}} (2^{j(n/p-n-s)}|v_j|)^\omega\Big)^{1/\omega}\Big\|_p \leqslant c_2A_1B_1.$$

Second, suppose that $q \leqslant p < 1$. Again, from (4.8) and since $\ell_q(\mathbb{Z}) \hookrightarrow \ell_p(\mathbb{Z})$, we have

$$\begin{aligned} \Big(\sum_{j\in\mathbb{Z}} \|u_j * v_j\|_p^p\Big)^{1/p} &\leqslant c_1\Big(\sum_{j\in\mathbb{Z}} (2^{js}\|u_j\|_p)^p\Big)^{1/p}\sup_{k\in\mathbb{Z}} 2^{k(n/p-n-s)}\|v_k\|_p \\ &\leqslant c_1B_1\Big\|\Big(\sum_{j\in\mathbb{Z}} (2^{js}|u_j|)^p\Big)^{1/p}\Big\|_p \leqslant c_2B_1\Big\|\Big(\sum_{j\in\mathbb{Z}} (2^{js}|u_j|)^q\Big)^{1/q}\Big\|_p = c_2A_1B_1. \end{aligned}$$

Hence the proof is complete. □

Proof of Theorem 2.1. It suffices to apply Proposition 4.1 with both $u_j := Q_j f$ and $v_j := \tilde{Q}_j\theta$. □

Proof of Theorem 2.2. Let $f \in \dot{\tilde{A}}^s_{p,q}$ and $\theta \in \mathcal{E}'$ ($\mathcal{S}$, respectively) be such that $[\theta]_{\mathcal{P}} \in \dot{A}^\mu_{r,\omega}$.

Step 1: *preparation*. We first note that $\theta \in \mathcal{E}'$ ($\mathcal{S}$, respectively) and $f \in \mathcal{S}'$ imply $\theta * f \in \mathcal{S}'$ (see [**9**, p. 21] and [**15**, Theorem 30.2, p. 317]). Also, for all $\varphi \in \mathcal{S}_\nu$, we have $\theta * \check{\tilde{\varphi}} \in \mathcal{S}_\nu$; indeed, in case $\theta \in \mathcal{E}'$ this follows by Lemma 3.1, in case $\theta \in \mathcal{S}$ it suffices to apply the Fourier's properties. Then, by Proposition 3.2, for all $\varphi \in \mathcal{S}_\nu$, it holds

$$\langle \theta * f, \varphi\rangle = \langle f, \theta * \check{\tilde{\varphi}}\rangle = \sum_{j\in\mathbb{Z}} \langle Q_j f, \theta * \check{\tilde{\varphi}}\rangle = \sum_{j\in\mathbb{Z}} \langle Q_j f, \tilde{Q}_j\theta * \check{\tilde{\varphi}}\rangle = \sum_{j\in\mathbb{Z}} \langle Q_j f * \tilde{Q}_j\theta, \varphi\rangle. \tag{4.12}$$

Hence $\theta * f = \theta \circledast f$ in $\mathcal{S}'_\nu$, and Proposition 4.1 with $u_j := Q_j f$ and $v_j := \tilde{Q}_j\theta$ gives

$$\|\theta \circledast f\|_p \leqslant c\|[\theta]_{\mathcal{P}}\|_{\dot{A}^\mu_{r,\omega}} \|[f]_{\mathcal{P}}\|_{\dot{A}^s_{p,q}}.$$

Step 2. We set $g_k := \theta * f - \theta * (S_{-k} f)$ for $k \in \mathbb{N}_0$. Then the sequence $(g_k)_{k\in\mathbb{N}_0}$ has the following properties:

- Since $g_k = \sum_{j>-k} Q_j f * \tilde{Q}_j\theta$ in $\mathcal{S}'_\nu$ (the proof is easy as in (4.12)), by Step 1,

$$\|g_k\|_p \leqslant c\|[\theta]_{\mathcal{P}}\|_{\dot{A}^\mu_{r,\omega}} \|[f]_{\mathcal{P}}\|_{\dot{A}^s_{p,q}} \quad (\forall k \in \mathbb{N}_0). \tag{4.13}$$

- g_k tends to $\theta * f$ pointwise; indeed, assume that $p \geqslant 1$, then by Young and Bernstein inequalities, we have $\|Q_j f * \tilde{Q}_j\theta\|_\infty \leqslant \|Q_j f\|_p \|\tilde{Q}_j\theta\|_{p'} \leqslant c2^{jn/p}\|Q_j f\|_p\|\tilde{Q}_j\theta\|_1$, then

$$\begin{aligned} |g_k(x) - \theta * f(x)| &\leqslant c_1 \sum_{j\leqslant -k} (2^{sj}\|Q_j f\|_p)(2^{-sj}\|\tilde{Q}_j\theta\|_1)2^{jn/p} \\ &\leqslant c_2 2^{-kn/p}\|[\theta]_{\mathcal{P}}\|_{\dot{B}^{-s}_{1,\infty}} \|[f]_{\mathcal{P}}\|_{\dot{B}^s_{p,\infty}}, \qquad \forall x \in \mathbb{R}^n, \end{aligned} \tag{4.14}$$

then we use the embeddings $\dot{A}^s_{p,q} \hookrightarrow \dot{B}^s_{p,\infty}$ and $\dot{A}^{-s}_{1,q'} \hookrightarrow \dot{B}^{-s}_{1,\infty}$, on the one hand. On the other, suppose $0 < p < 1$, and again by Young and Bernstein inequalities, we have $\|Q_j f * \tilde{Q}_j\theta\|_\infty \leqslant \|Q_j f\|_\infty\|\tilde{Q}_j\theta\|_1 \leqslant c2^{n(2/p-1)j}\|Q_j f\|_p\|\tilde{Q}_j\theta\|_p$, then

$$\begin{aligned} |g_k(x) - \theta * f(x)| &\leqslant c_1 \sum_{j\leqslant -k} (2^{sj}\|Q_j f\|_p)(2^{(n/p-n-s)j}\|\tilde{Q}_j\theta\|_p)2^{jn/p} \\ &\leqslant c_2 2^{-kn/p}\|[\theta]_{\mathcal{P}}\|_{\dot{B}^{n/p-n-s}_{p,\infty}} \|[f]_{\mathcal{P}}\|_{\dot{B}^s_{p,\infty}}, \qquad \forall x \in \mathbb{R}^n, \end{aligned} \tag{4.15}$$

and use both $\dot{A}^s_{p,q} \hookrightarrow \dot{B}^s_{p,\infty}$ and $\dot{A}^{n/p-n-s}_{p,\omega} \hookrightarrow \dot{B}^{n/p-n-s}_{p,\infty}$. Now, it suffices to take $k \to +\infty$ in, both, (4.14) and (4.15), to obtain the desired result.

Finally, by writing (4.13) as $\int_{\mathbb{R}^n} |g_k(x)|^p \mathrm{d}x \leqslant c\|[\theta]_{\mathcal{P}}\|^p_{\dot{A}^\mu_{r,\omega}} \|[f]_{\mathcal{P}}\|^p_{\dot{A}^s_{p,q}}$ if $p < \infty$, and applying Fatou's lemma to the sequence $(|g_k|^p)_{k\in\mathbb{N}_0}$ (recall that $|g_k|^p$ tends to $|\theta * f|^p$ also pointwise), inequality (2.3) follows. However, if $p = \infty$, we take an arbitrary $\varepsilon > 0$, then there exists a number $k_0 \in \mathbb{N}_0$ such that

$$|\theta * f(x)| \leqslant |g_k(x) - \theta * f(x)| + \|g_k\|_\infty \leqslant \varepsilon + \|g_k\|_\infty \quad (\forall k \geqslant k_0, \forall x \in \mathbb{R}^n);$$

but $\|g_k\|_\infty \leqslant c\|[\theta]_{\mathcal{P}}\|_{\dot{B}^{-s}_{1,q'}} \|[f]_{\mathcal{P}}\|_{\dot{B}^s_{\infty,q}}$ for all $k \in \mathbb{N}_0$ (see again (4.13)). By arbitrariness of ε, we deduce estimate (2.3). The proof is complete. □

Proof of Proposition 2.1. It suffices to apply the homogeneity argument (see Section 3) and Theorems 2.1–2.2. □

Proof of Theorem 2.3. The convergence of $\sum_{j\in\mathbb{Z}} \tilde{Q}_j\theta_{2^{-k}} * Q_j f$ in $\mathcal{S}'_\nu$ can be done as in the proof of Proposition 4.1/Substep 1.1. The details will be omitted. As above, its limit will be denoted by $\theta_{2^{-k}} \circledast f$.

Step 1: *proof of* (i). *Substep* 1.1: *the case* $p \geqslant 1$. Applying Young's inequality, we obtain $\|\theta_{2^{-k}} \circledast f\|_p$ is bounded by $\sum_{j\in\mathbb{Z}} \|Q_j f\|_p \|\tilde{Q}_j\theta_{2^{-k}}\|_1$. By the identity

$$\tilde{Q}_j\theta_{2^{-k}} = 2^{kn}\tilde{Q}_{j-k}\theta(2^k(\cdot)), \tag{4.16}$$

we have $2^{ks}\|\theta_{2^{-k}} \circledast f\|_p$ is bounded by $\sum_{j\in\mathbb{Z}} 2^{js}2^{-(j-k)s}\|Q_j f\|_p\|\tilde{Q}_{j-k}\theta\|_1$; we set $l := j-k$, then

$$\Big(\sum_{k\in\mathbb{Z}} 2^{ksq}\|\theta_{2^{-k}} \circledast f\|_p^q\Big)^{1/q} \leqslant \Big(\sum_{k\in\mathbb{Z}}\Big(\sum_{l\in\mathbb{Z}} 2^{(l+k)s}2^{-ls}\|Q_{l+k}f\|_p\|\tilde{Q}_l\theta\|_1\Big)^q\Big)^{1/q}. \tag{4.17}$$

– If $q > 1$, then by Minkowski inequality, we estimate (4.17) by

$$\sum_{l\in\mathbb{Z}}\Big(\sum_{k\in\mathbb{Z}} 2^{(l+k)sq}\|Q_{l+k}f\|_p^q\Big)^{1/q}\|\tilde{Q}_l\theta\|_1 2^{-ls} \leqslant \|f\|_{\dot{B}^s_{p,q}}\|\theta\|_{\dot{B}^{-s}_{1,1}}.$$

– If $0 < q \leqslant 1$, then by using (4.6) we again estimate (4.17) by

$$\Big(\sum_{l\in\mathbb{Z}} 2^{-lsq}\|\tilde{Q}_l\theta\|_1^q \sum_{k\in\mathbb{Z}} 2^{(l+k)sq}\|Q_{l+k}f\|_p^q\Big)^{1/q} \leqslant \|f\|_{\dot{B}^s_{p,q}}\|\theta\|_{\dot{B}^{-s}_{1,q}}.$$

Observe that $c = 1$ in the right-hand side of above inequalities.

Substep 1.2: *the case* $0 < p < 1$. By using (4.6), the convolution in L_p and (4.16), we have $2^{ks}\|\theta_{2^{-k}} \circledast f\|_p \leqslant c(\sum_{j\in\mathbb{Z}} 2^{jsp}2^{(j-k)(n/p-n-s)p}\|Q_j f\|_p^p\|\tilde{Q}_{j-k}\theta\|_p^p)^{1/p}$ is bounded by

$$c\Big(\sum_{l\in\mathbb{Z}} 2^{(l+k)sp}2^{l(n/p-n-s)p}\|Q_{l+k}f\|_p^p\|\tilde{Q}_l\theta\|_p^p\Big)^{1/p}. \tag{4.18}$$

Now, we estimate $(\sum_{k\in\mathbb{Z}} 2^{ksq}\|\theta_{2^{-k}} \circledast f\|_p^q)^{1/q}$ as the following (we separate the estimate with respect to q into two cases):

– If $p \leqslant q$ (here $q \in]0,\infty]$), by both (4.18) and Minkowski inequality, we have the bound

$$c_1\Big(\sum_{l\in\mathbb{Z}} 2^{l(n/p-n-s)p}\|\tilde{Q}_l\theta\|_p^p\Big\{\sum_{k\in\mathbb{Z}} 2^{(l+k)sq}\|Q_{l+k}f\|_p^q\Big\}^{p/q}\Big)^{1/p} \leqslant c_2\|f\|_{\dot{B}^s_{p,q}}\|\theta\|_{\dot{B}^{n/p-n-s}_{p,p}}.$$

– If $q < p$, by using again both (4.18) and (4.6) with $d := q/p$, we have the bound

$$c_1\Big(\sum_{l\in\mathbb{Z}} 2^{l(n/p-n-s)q}\|\tilde{Q}_l\theta\|_p^q \sum_{k\in\mathbb{Z}} 2^{(l+k)sq}\|Q_{l+k}f\|_p^q\Big)^{1/q} \leqslant c_2\|f\|_{\dot{B}^s_{p,q}}\|\theta\|_{\dot{B}^{n/p-n-s}_{p,q}}.$$

Therefore, the desired estimates hold.

Step 2: *proof of* (ii). This can be done as in Step 2 of the proof of Theorem 2.2. We briefly outline it. We fix $k \in \mathbb{Z}$ and introduce the sequence $(g_{l,k})_{l\in\mathbb{N}_0}$ defined by $g_{l,k} := \theta_{2^{-k}} * f - \theta_{2^{-k}} * (S_{-l}f)$ for $l \in \mathbb{N}_0$, which satisfies (as above) both

$$\Big(\sum_{k\in\mathbb{Z}} 2^{ksq}\|g_{l,k}\|_p^q\Big)^{1/q} \leqslant c\|[\theta]_{\mathcal{P}}\|_{\dot{B}^{\mu}_{r,t}}\|[f]_{\mathcal{P}}\|_{\dot{B}^s_{p,q}} \quad (\forall l \in \mathbb{N}_0),$$

and $g_{l,k}$ tends to $\theta_{2^{-k}} * f$ pointwise as $l \to +\infty$ for all $k \in \mathbb{Z}$. Hence it suffices to apply twice the Fatou lemma in the last inequality. □

Proof of Theorem 2.4. Using the Peetre-maximal function

$$Q_j^{*,a}f(x) := \sup_{y\in\mathbb{R}^n}(1 + |2^jy|)^{-a}|Q_jf(x-y)| \qquad (x \in \mathbb{R}^n, j \in \mathbb{Z}, a > 0),$$

we have at our disposal the following characterization of F-spaces (see e.g., [**8**, p. 45]):

Proposition 4.2. *Let* $a > n/\min(p,q)$. *Then the expression* $\|f\|^*_{\dot{F}^s_{p,q}} := \|(\sum_{j\in\mathbb{Z}} 2^{jsq}|Q_j^{*,a}f|^q)^{1/q}\|_p$ *is an equivalent quasi-seminorm in* $\dot{F}^s_{p,q}$.

For the convergence of $\sum_{j\in\mathbb{Z}} \tilde{Q}_j\theta_{2^{-k}} * Q_jf$ in $\mathcal{S}'_\nu$ to $\theta_{2^{-k}} \circledast f$, the same technique used in the proof of Proposition 4.1 will be applied here, but some changes are needed; so we use the same notations. For all $\varphi \in \mathcal{S}_\nu$, we have $|\langle\tilde{Q}_j\theta_{2^{-k}} * Q_jf, \eta_j\varphi\rangle|$ is bounded by $c\zeta_m(\varphi)\min(2^{-jN}, 2^{j\nu})\|\tilde{Q}_j\theta_{2^{-k}} * Q_jf\|_p$ $(\forall j \in \mathbb{Z})$, where η_j is defined such that $\eta_j(\tilde{Q}_j\theta_{2^{-k}} * Q_jf) = \tilde{Q}_j\theta_{2^{-k}} * Q_jf$ $(\forall j \in \mathbb{Z})$. Now, by (4.16) and for a real $a > n/\min(p,q)$, it suffices to observe that $\|\tilde{Q}_j\theta_{2^{-k}} * Q_jf\|_p$ is bounded by

$$\begin{aligned}\Big\|\int_{\mathbb{R}^n}\tilde{Q}_{j-k}\theta(y)Q_jf(\cdot - 2^{-k}y)\mathrm{d}y\Big\|_p &\leqslant \|Q_j^{*,a}f\|_p\int_{\mathbb{R}^n}|\tilde{Q}_{j-k}\theta(y)|(1 + 2^{j-k}|y|)^a\mathrm{d}y\\ &\leqslant c2^{-ks}\|f\|_{\dot{F}^s_{p,\infty}}\|\theta\|_{\dot{B}^{-s,a}_{1,\infty}}.\end{aligned}$$

Hence the convergence of the series $\sum_{j\in\mathbb{Z}}|\langle\tilde{Q}_j\theta_{2^{-k}} * Q_jf, \varphi\rangle|$ for all $k \in \mathbb{Z}$.

Step 1: *proof of* (i). We estimate $(\sum_{k\in\mathbb{Z}} 2^{ksq}|\sum_{j\in\mathbb{Z}}\tilde{Q}_j\theta_{2^{-k}} * Q_jf|^q)^{1/q}$ as the following: if $q > 1$, using the identity (4.16) and Minkowski inequality (twice), then it is bounded by

$$\begin{aligned}&\int_{\mathbb{R}^n}\Big(\sum_{k\in\mathbb{Z}}\Big(\sum_{j\in\mathbb{Z}} 2^{ks}|Q_jf(\cdot - 2^{-k}y)\tilde{Q}_{j-k}\theta(y)|\Big)^q\Big)^{1/q}\mathrm{d}y\\ &\leqslant \int_{\mathbb{R}^n}\sum_{l\in\mathbb{Z}} 2^{-ls}|\tilde{Q}_l\theta(y)|\Big(\sum_{k\in\mathbb{Z}} 2^{(k+l)sq}|Q_{k+l}f(\cdot - 2^{-k}y)|^q\Big)^{1/q}\mathrm{d}y \quad (m := k + l)\\ &\leqslant \Big(\sum_{m\in\mathbb{Z}} 2^{msq}|Q_m^{*,a}f|^q\Big)^{1/q}\sum_{l\in\mathbb{Z}} 2^{-ls}\|(1 + 2^l|\cdot|)^a\tilde{Q}_l\theta\|_1;\end{aligned}$$

if $0 < q \leqslant 1$, by (4.16) and (4.6) with $d := q$, then as above the desired term is bounded by

$$\Big(\sum_{k\in\mathbb{Z}}\sum_{j\in\mathbb{Z}}\Big\{\int_{\mathbb{R}^n}2^{ks}|Q_jf(\cdot-2^{-k}y)\tilde{Q}_{j-k}\theta(y)|\mathrm{d}y\Big\}^q\Big)^{1/q} \leqslant \Big(\sum_{j\in\mathbb{Z}}2^{jsq}|Q_j^{*,a}f|^q\sum_{l\in\mathbb{Z}}\{2^{-ls}\|(1+2^l|\cdot|)^a\tilde{Q}_l\theta\|_1\}^q\Big)^{1/q}.$$

Then we calculate the L_p quasi-norm of $(\sum_{k\in\mathbb{Z}}2^{ksq}|\sum_{j\in\mathbb{Z}}\tilde{Q}_j\theta_{2^{-k}}*Q_jf|^q)^{1/q}$ and the desired estimate is obtained.

Step 2: *proof of* (ii). Similar to Step 2/proof of Theorem 2.3. □

5. Concluding Remarks

5.1. Applications. 1. For any function f, we define the differences $\Delta_h^mf := \sum_{j=0}^m\binom{m}{j}(-1)^{m-j}f(\cdot+jh) = \sum_{j=0}^m\binom{m}{j}(-1)^{m-j}\delta_{-jh}*f$ (where δ_{-jh} is the Dirac distribution at the point $x := -jh$). As $Q_k\delta_{-jh} = 2^{kn}\mathcal{F}^{-1}\gamma(2^k(\cdot+jh))$, then $[\delta_{-jh}]_{\mathcal{P}}\in\dot{B}^{n/u-n}_{u,\infty}$ $(0<u\leqslant\infty)$. We now see when $[\delta_{-jh}]_{\mathcal{P}}\in\dot{B}^{\mu}_{r,\omega}$:

• For $p\geqslant 1$; here $r:=1$ and $\mu:=-s$. We have $-s=n-n$, $\omega=\infty$ (i.e., $0<q\leqslant 1$).

• For $0<p<1$; here $r:=p$ and $\mu:=-s-n+n/p=n/p-n$, $\omega=\infty$ (i.e., $q\leqslant p<1$).

Consequently, $\forall q\in]0,1]$, $\forall p\geqslant q$ and $\forall m\in\mathbb{N}$, by Theorem 2.2 it holds

$$\|\Delta_h^mf\|_p\leqslant c\|[f]_{\mathcal{P}}\|_{\dot{B}^0_{p,q}}\quad(\forall f\in\dot{\tilde{B}}^0_{p,q},\forall h\in\mathbb{R}^n).$$

This estimate fails with only the assumption $[f]_{\mathcal{P}}\in\dot{B}^0_{p,q}$. Indeed, let $f(x):=x_1^m$, then $\|[f]_{\mathcal{P}}\|_{\dot{B}^0_{p,q}}=0$, while $\Delta_h^mf(x)=m!h_1^m$ $(\forall x,h\in\mathbb{R}^n)$, implies $\|\Delta_h^mf\|_p=\infty$.

2. Let ϱ be a C^∞ function on $\mathbb{R}$ such that $\varrho(t)=1$ for $t\leqslant e^{-3}$ and $\varrho(t)=0$ for $t\geqslant e^{-2}$. For $\alpha>-n$ and $\beta\geqslant 0$, we set $\theta_{\alpha,\beta}(x):=|x|^\alpha(-\log|x|)^{-\beta}\varrho(|x|)$, $x\in\mathbb{R}^n$. This type of functions have been studied in e.g., [**7**, p. 82]. We have $\theta_{\alpha,\beta}\in\mathcal{E}'$; indeed, let $\varphi\in C^\infty$, then using polar coordinates and as $\sup_{|x|\leqslant e^{-2}}|\varphi(x)|\leqslant c<\infty$, we find

$$\int_{S^{n-1}}\int_0^{e^{-2}}r^{n+\alpha-1}(-\log r)^{-\beta}|\varrho(r)||\varphi(ry)|\mathrm{d}r\mathrm{d}y\leqslant c2^{-\beta}e^{-2(n+\alpha)}\|\varrho\|_\infty.$$

To continue, we need to introduce inhomogeneous Besov $B^s_{p,q}$ and Triebel–Lizorkin $F^s_{p,q}$ spaces ($p<\infty$ in $F^s_{p,q}$-case). We denote by $A^s_{p,q}$ for either $B^s_{p,q}$ or $F^s_{p,q}$, and use the abbreviations B, F to indicate them.

Definition 5.1. The spaces $B^s_{p,q}$ and $F^s_{p,q}$ are the sets of $f\in\mathcal{S}'$ such that

$$\|f\|_{B^s_{p,q}}:=\|S_0f\|_p+\Big(\sum_{j\geqslant 1}2^{jsq}\|Q_jf\|_p^q\Big)^{1/q}<\infty,$$

$$\|f\|_{F^s_{p,q}}:=\|S_0f\|_p+\Big\|\Big(\sum_{j\geqslant 1}2^{jsq}|Q_jf|^q\Big)^{1/q}\Big\|_p<\infty,$$

respectively.

PROPOSITION 5.1. (See e.g., [**17**, p. 98]) *Let s be such that $s > (n/p - n)_+$. Then $\|f\|_p + \|[f]_{\mathcal{P}}\|_{\dot{A}^s_{p,q}}$ is an equivalent quasi-norm in $A^s_{p,q}$.*

Assume that $\beta > 0$. By [**14**, Lemmas 1-2, pp. 44-47] we have, e.g., $\theta_{\alpha,\beta} \in A^{\alpha+n/r}_{r,\omega}$ for $\alpha \neq 0$, $\alpha + n/r > (n/r - n)_+$ and $\beta\omega > 1$ ($\beta r > 1$ in F-case); also for $\alpha = 0$ and $(\beta+1)\omega > 1$ ($(\beta+1)r > 1$ in F-case). But in that case, by Proposition 5.1, $A^{\alpha+n/r}_{r,\omega} \hookrightarrow \dot{A}^{\alpha+n/r}_{r,\omega}$; (at now $r \in]0,\infty]$, $r \neq \infty$ in F-case, and $\omega \in]0,\infty]$).

Now, for all $s < 0$, $\alpha := -s - n$, $r := \min(1,p)$ and ω as in (2.1), it holds $\|\theta_{-n-s,\beta} * f\|_p \leqslant c\|[f]_{\mathcal{P}}\|_{\dot{A}^s_{p,q}}$ ($\forall f \in \dot{\tilde{A}}^s_{p,q}$), where $\beta\omega > 1$ ($\beta r > 1$ in F-case) for $\alpha \neq 0$, and $(\beta+1)\omega > 1$ ($(\beta+1)r > 1$ in F-case) for $0 < p < \infty$ and $\alpha = 0$.

REMARK 5.1. We note that other cases on the parameters (e.g., the case $\theta_{\alpha,\beta} \in A^{\mu}_{r,\omega}$ with $\mu > \alpha + n/r$) can be obtained from the properties of $\theta_{\alpha,\beta}$, see [**14**], etc.

3. Let $\mathcal{X}$ be the characteristic function of the unit cube $[-1,1]^n$ in $\mathbb{R}^n$. Clearly that $\hat{\mathcal{X}}(\xi) = \mathrm{i}^{-n}\prod_{j=1}^n (e^{\mathrm{i}\xi_j} - e^{-\mathrm{i}\xi_j})/\xi_j$. Using the development of $\prod_{j=1}^n (e^{\mathrm{i}\xi_j} - e^{-\mathrm{i}\xi_j})$ (see [**1**, I §8.1/(13), p. 98]), and we define a function $\psi \in \mathcal{S}_\infty$ by $\hat{\psi}(\xi) := \gamma(\xi)/(\xi_1 \dots \xi_n)$, we find

$$\begin{aligned} Q_k\mathcal{X}(x) = 2^{-kn}(2\mathrm{i}\pi)^{-n} \int_{\mathbb{R}^n} e^{\mathrm{i}x\cdot\xi} \Big\{ & e^{\mathrm{i}(\xi_1+\cdots+\xi_n)} - \sum_{j=1}^n e^{\mathrm{i}(\xi_1+\cdots-\xi_j+\cdots+\xi_n)} \\ & + (-1)^2 \sum_{j=1}^{n-1} \sum_{\substack{j_1=2\\ j_1>j}}^{n} e^{\mathrm{i}(\xi_1+\cdots-\xi_j-\xi_{j_1}+\cdots+\xi_n)} + (-1)^3 \sum_{j=1}^{n-2} \sum_{\substack{j_1=2\\ j_1>j}}^{n-1} \sum_{\substack{j_2=3\\ j_2>j_1}}^{n} \\ & e^{\mathrm{i}(\xi_1+\cdots-\xi_j-\xi_{j_1}-\xi_{j_2}+\cdots+\xi_n)} + \cdots + (-1)^{-n} e^{-\mathrm{i}(\xi_1+\cdots+\xi_n)} \Big\} \hat{\psi}(2^{-k}\xi)\,\mathrm{d}\xi, \end{aligned}$$

which implies

$$\begin{aligned} \mathrm{i}^n Q_k\mathcal{X}(x) &= \psi(2^k(x_1+1,\dots,x_n+1)) - \sum_{j=1}^n \psi(2^k(x_1+1,\dots,x_j-1,\dots,x_n+1)) \\ &+ (-1)^2 \sum_{j=1}^{n-1} \sum_{\substack{j_1=2\\ j_1>j}}^{n} \psi(2^k(x_1+1,\dots,x_j-1,x_{j_1}-1,\dots,x_n+1)) \\ &+ (-1)^3 \sum_{j=1}^{n-2} \sum_{\substack{j_1=2\\ j_1>j}}^{n-1} \sum_{\substack{j_2=3\\ j_2>j_1}}^{n} \psi(2^k(x_1+1,\dots,x_j-1,x_{j_1}-1,x_{j_2}-1,\dots,x_n+1)) \\ &+ \cdots + (-1)^{-n}\psi(2^k(x_1-1,\dots,x_n-1)) \quad \text{(there are } 2^n \text{ terms).} \end{aligned}$$

Consequently, we have $\|Q_k\mathcal{X}\|_u \leqslant 2^{n/\alpha}2^{-kn/u}\|\psi\|_u$ for all $u \in]0,\infty]$, $\alpha := \min(1,u)$, and all $k \in \mathbb{Z}$. On the other hand, as $\mathcal{X} \in L_1 \cap L_u$, then $S_0\mathcal{X} = \mathcal{X} - \sum_{k\geqslant 1} Q_k\mathcal{X}$ implies

$$\|S_0\mathcal{X}\|_u = \|\mathcal{X}\|_u + \Big(\sum_{k\geqslant 1} \|Q_k\mathcal{X}\|_u^\alpha\Big)^{1/\alpha} \leqslant \|\mathcal{X}\|_u + c_1\Big(\sum_{k\geqslant 1} 2^{-kn\alpha/u}\Big)^{1/\alpha} \leqslant c_2.$$

All these facts give $\mathcal{X} \in B^t_{u,v}$ with, either $t < n/u$, or $t = n/u$ and $v = \infty$.

We now turn to the application of the above results, looking for $[\mathcal{X}]_{\mathcal{P}} \in \dot{B}^{\mu}_{r,\omega}$:

• For $p > 1$; here $\mu := -s$ and $r := 1$. By Proposition 5.1, we have $[\mathcal{X}]_{\mathcal{P}}$ belongs to $\dot{B}^{-s}_{1,q'}$ for $-n < s < 0$ and $0 < q \leqslant \infty$, belongs to $\dot{B}^{n}_{1,\infty}$ with $0 < q \leqslant 1$.

• For $0 < p \leqslant 1$; here $\mu := -s - n + n/p$ and $r := p$. We have $[\mathcal{X}]_{\mathcal{P}}$ belongs to $\dot{B}^{-s-n+n/p}_{p,\omega}$ for $-n < s < 0$ and $0 < q \leqslant p$ or $p \leqslant \min(1, q)$, belongs to $\dot{B}^{n/p}_{p,\infty}$ with $0 < q \leqslant p \leqslant 1$.

We conclude that for, either $-n < s < 0$ and $0 < q \leqslant \infty$, or $s = -n$ and $0 < q \leqslant 1$, it holds $\|\mathcal{X} * f\|_p \leqslant c\|[f]_{\mathcal{P}}\|_{\dot{B}^s_{p,q}}$ $(\forall f \in \dot{\tilde{B}}^s_{p,q})$, where the constant c depends only on n, s, p, q.

5.2. An extension to homogeneous Sobolev spaces. The homogeneous Sobolev spaces $\dot{W}^m_p$ $(1 \leqslant p \leqslant \infty,\ m \in \mathbb{N}_0)$ is the set of distributions f such that $f^{(\alpha)} \in L_p$ for all $|\alpha| = m$ and endowed with the seminorm $\|f\|_{\dot{W}^m_p} := \sum_{|\alpha|=m} \|f^{(\alpha)}\|_p$. The quotient $\dot{W}^m_p/\mathcal{P}_m$ is a Banach space in $\mathcal{S}'_m$ for this norm.

THEOREM 5.1. *Let $1 \leqslant p < \infty$ and $m \in \mathbb{N}_0$. There exists a constant $c > 0$ such that $\|\theta * f\|_{\dot{W}^m_p} \leqslant c\|[\theta]_{\mathcal{P}}\|_{\dot{B}^0_{1,1}}\|f\|_{\dot{W}^m_p}$ for all $f \in \dot{W}^m_p$ and all θ satisfying $[\theta]_{\mathcal{P}} \in \dot{B}^0_{1,1}$ and either $\theta \in \mathcal{S}$ or $\theta \in \mathcal{E}'$.*

PROOF. We note that $\theta * f$ is well defined, see Lemma 3.1 or [**9**, p. 21] or [**15**, p. 317].

First, we have $\dot{W}^m_p \hookrightarrow \dot{B}^m_{p,\infty}$. Indeed, since $\|Q_j g\|_p \leqslant c\|g\|_p$ $(\forall j \in \mathbb{Z}$ and $\forall g \in L_p)$, then $L_p \hookrightarrow \dot{B}^0_{p,\infty}$; now $\|f\|_{\dot{B}^m_{p,\infty}} = \sum_{|\alpha|=m} \|f^{(\alpha)}\|_{\dot{B}^0_{p,\infty}}$ yields the desired embedding. Let now $f \in \dot{W}^m_p$. We have $f^{(\alpha)} \in \tilde{C}_0$ $(\forall|\alpha| = m)$, see example (i) just after Definition 2.1. Consequently $f^{(\alpha)} \in \dot{\tilde{B}}^0_{p,\infty}$ $(\forall|\alpha| = m)$, and by Theorem 2.2 it holds $\|\theta * f^{(\alpha)}\|_p \leqslant c\|[\theta]_{\mathcal{P}}\|_{\dot{B}^0_{1,1}}\|f\|_{\dot{B}^m_{p,\infty}}$. Since $\theta * f^{(\alpha)} = (\theta * f)^{(\alpha)}$, then the desired estimate follows. □

REMARK 5.2. Similar to Theorem 5.1's proof, $\|\theta * f\|_{\dot{W}^{m+k}_\infty} \leqslant c\|[\theta]_{\mathcal{P}}\|_{\dot{B}^k_{1,1}}\|f\|_{\dot{W}^m_\infty}$ $(k = 1, 2, \ldots)$ for all $f \in \dot{W}^m_\infty$ and all θ satisfying $[\theta]_{\mathcal{P}} \in \dot{B}^k_{1,1}$ and either $\theta \in \mathcal{S}$ or $\theta \in \mathcal{E}'$. On the other hand, Subsection 5.1 can be adapted according to Theorems 2.3, 2.4 and 5.1.

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