PUBLICATIONS DE L'INSTITUT MATHÉMATIQUE
Nouvelle série, tome 106(120) (2019), 53–67 DOI: https://doi.org/10.2298/PIM1920053A

SOME NEW CONGRUENCES FOR (s,t)-REGULAR BIPARTITION FUNCTIONS

Chandrashekar Adiga, Abdelmejid Bayad, and Ranganatha Dasappa

Abstract. Let $B_{s,t}(n)$ denote the number of (s,t)-regular bipartitions of n. We prove several infinite families of congruences modulo t for $B_{s,t}(n)$ where $(s,t) \in \{(2,7),(5^\beta,7),(3^\beta,11),(5^\beta,11),(3^\beta,17)\}$, $\beta \geqslant 1$.

1. Introduction and statement of main results

Let $\mathbb{N}$ denote the set of natural numbers and $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$. A partition of $n \in \mathbb{N}$ is a non-increasing sequence of positive integers that sum to n. We shall set $p(0) = 1$. For $n \in \mathbb{N}$, we denote the number of unrestricted partitions of n by $p(n)$.

Surprisingly, this simple combinatorial object plays an important role in diverse areas of mathematics such as the mathematical physics, representation theory, q-series, lie theory, symmetric functions, statistical mechanics and the theory of modular forms. Several authors discussed the significant role of partition functions in statistical mechanics and the problem of finding the number of partitions of a number into integers under certain restrictions are useful for the study of the Bose-Einstein condensation of a perfect gas. In this paper, we study certain restricted partition function called regular bipartition function, which may be useful in several areas of mathematics.

For any positive integer $\ell \geqslant 2$, a partition is called ℓ-regular if none of its parts is divisible by ℓ. For convenience, define $b_\ell(0) = 1$ and for $n \geqslant 1$, let $b_\ell(n)$ denote the number of ℓ-regular partitions of n. For example, $b_3(6) = 7$ because there are 7 possible 3-regular partitions of 6:

$$5+1, \quad 4+2, \quad 4+1+1, \quad 2+2+2, \quad 2+2+1+1,$$
$$2+1+1+1+1, \quad 1+1+1+1+1+1.$$

The generating function for $b_\ell(n)$ is given by $\sum_{n=0}^{\infty} b_\ell(n)q^n = \frac{(q^\ell;q^\ell)_\infty}{(q;q)_\infty}$. Here and throughout the paper, it is assumed that $|q| < 1$. Also, as usual, for any

2010 *Mathematics Subject Classification:* Primary 11P83; Secondary 05A15; 05A17.
Key words and phrases: congruences, bipartitions, k-regular partitions.
Communicated by Gradimir Milovanović.

complex number a, we define

$$(a;q)_\infty := (1-a)(1-aq)(1-aq^2)\dots, \quad |q|<1.$$

A bipartition (λ,μ) of n is a pair of partitions (λ,μ) such that the sum of all the parts of λ and μ equals n. For positive integers $s \geqslant 2$ and $t \geqslant 2$, an (s,t)-regular bipartition of n is a bipartition (λ_1,λ_2) of n such that λ_1 is an s-regular partition and λ_2 is a t-regular partition. As noted in [**5**], the generating function of $B_{s,t}(n)$, the number of (s,t)-regular bipartitions of n is given by

$$\sum_{n=0}^{\infty} B_{s,t}(n)q^n = \frac{(q^s;q^s)_\infty (q^t;q^t)_\infty}{(q;q)_\infty^2}. \tag{1.1}$$

Congruence properties for various (s,t)-regular bipartitions have been extensively studied, see, for example, Dai [**4**], Dou [**5**], Lin [**6**–**8**] and Ranganatha [**11**]. In [**5**], Dou presented some conjectures on $B_{5,7}(n)$ and $B_{3,7}(n)$. For example, for all $n \in \mathbb{N}_0$,

$$B_{3,7}(4n+3) \equiv 0 \pmod 3, \tag{1.2}$$

$$B_{3,7}(16n+1) \equiv 0 \pmod 2, \tag{1.3}$$

$$B_{3,7}(32n+21) \equiv 0 \pmod 2. \tag{1.4}$$

Recently, Chandrashekar Adiga and Ranganatha [**1**] gave an elementary proof of (1.2) and they also proved several infinite families of congruences for $B_{3,7}(n)$ modulo 3. Most recently, Xia and Yao [**15**] confirmed the congruences (1.3) and (1.4). In the same paper, they presented a number of congruences for $B_{3,s}(n)$, $B_{5,s}(n)$ $(s \geqslant 1)$ and $B_{3,7}(n)$ modulo 3, 5 and 7, respectively.

In this paper, we prove several congruences for $B_{s,t}(n)$ modulo t where $(s,t) \in \{(2,7),(5^\beta,7),(3^\beta,11),(5^\beta,11),(3^\beta,17)\}$, $\beta \in \mathbb{N}$. Our main results of this paper can be stated as follows:

Theorem 1.1. *For all $n,\beta,\alpha \in \mathbb{N}_0$ with $\beta \geqslant 16\alpha+16$, we have*

$$B_{7,5^\beta}\Big(5^{16\alpha+16}n + 5\cdot\frac{5^{16\alpha+16}-1}{24}\Big) \equiv (-1)^{\alpha-1} B_{7,5^{\beta-16(\alpha+1)}}(n) \pmod 7.$$

Theorem 1.2. *For all $n,\beta,\alpha \in \mathbb{N}_0$ with $\beta \geqslant 16(\alpha+1)$, we have*

$$B_{7,5^\beta}\Big(5^{16\alpha+15}(5n+j) + 5\cdot\frac{5^{16\alpha+14}-1}{24}\Big) \equiv 0 \pmod 7, \tag{1.5}$$

where $j \in \{0,2,3,4\}$.

Theorem 1.3. *For all $n,\alpha \in \mathbb{N}_0$ and $j \in \{0,4\}$, we have*

$$B_{7,5^{16\alpha+15}}\Big(5^{16\alpha+15}(5n+j) + 5\cdot\frac{5^{16\alpha+14}-1}{24}\Big) \equiv 0 \pmod 7.$$

Theorem 1.4. *For all $n,\alpha,\beta \in \mathbb{N}_0$ with $\beta \geqslant 10(\alpha+1)$, we have*

$$B_{11,3^\beta}\Big(3^{10\alpha+10}n + 3\cdot\frac{3^{10\alpha+10}-1}{8}\Big) \equiv B_{11,3^{\beta-10(\alpha+1)}}(n) \pmod{11}. \tag{1.6}$$

THEOREM 1.5. *For all* $n,\alpha,\beta\in\mathbb{N}_0$ *with* $\beta\geqslant 10(\alpha+1)$, *we have*

$$B_{11,3^\beta}\Big(3^{10\alpha+10}n+3\cdot\frac{3^{10\alpha+8}-1}{8}\Big)\equiv 0 \pmod{11}, \tag{1.7}$$

$$B_{11,3^\beta}\Big(3^{10\alpha+10}n+3\cdot\frac{17\cdot 3^{10\alpha+8}-1}{8}\Big)\equiv 0 \pmod{11}. \tag{1.8}$$

THEOREM 1.6. *For all* $n,\beta,\alpha\in\mathbb{N}_0$ *with* $\beta\geqslant 22\alpha$, *we have*

$$B_{11,5^\beta}\Big(5^{22\alpha}n+3\cdot\frac{5^{22\alpha}-1}{8}\Big)\equiv a(11\alpha)B_{11,5^{\beta-22\alpha}}(n) \pmod{11}, \tag{1.9}$$

where $a(\alpha)$ *is defined by*

$$a(\alpha)=\Big(\frac{-\sqrt{22}}{22}+\frac{1}{2}\Big)\Big(5+\sqrt{22}\Big)^{\alpha}+\Big(\frac{\sqrt{22}}{22}+\frac{1}{2}\Big)\Big(5-\sqrt{22}\Big)^{\alpha}. \tag{1.10}$$

THEOREM 1.7. *For all* $n,\beta,\alpha\in\mathbb{N}_0$ *with* $\beta\geqslant 22\alpha+17$, *we have*

$$B_{11,5^\beta}\Big(5^{22\alpha+17}n+3\cdot\frac{5^{22\alpha+16}-1}{8}\Big)\equiv 0 \pmod{11}, \tag{1.11}$$

$$B_{11,5^\beta}\Big(5^{22\alpha+17}n+3\cdot\frac{9\cdot 5^{22\alpha+16}-1}{8}\Big)\equiv 0 \pmod{11}. \tag{1.12}$$

THEOREM 1.8. *Let* $p_1,p_2,\cdots,p_r\geqslant 5$ *be distinct primes with* $\big(\frac{-3}{p_r}\big)=-1$. *Then we have*

$$B_{7,2}\Big(np_r^{2\beta_r+1}\prod_{i=1}^{r-1}p_i^{2\beta_i+2}+7\cdot\frac{\prod_{i=1}^{r}p_i^{2\beta_i+2}-1}{24}\Big)\equiv 0 \pmod{7},$$

where $\beta_i\in\mathbb{N}$ *and* $n\in\mathbb{N}_0$.

THEOREM 1.9. *For all* $n,\beta,\alpha\in\mathbb{N}_0$ *with* $\beta\geqslant 4\alpha$, *we have*

$$B_{17,3^\beta}\Big(3^{4\alpha}n+5\cdot\frac{3^{4\alpha}-1}{8}\Big)\equiv H(2\alpha)B_{17,3^{\beta-4\alpha}}(n) \pmod{17}, \tag{1.13}$$

where $H(\alpha)$ *is defined by*

$$H(\alpha)=\Big(\frac{55\sqrt{241}}{482}+\frac{1}{2}\Big)\Big(\frac{17}{2}+\frac{\sqrt{241}}{2}\Big)^{\alpha}+\Big(\frac{-55\sqrt{241}}{482}+\frac{1}{2}\Big)\Big(\frac{17}{2}-\frac{\sqrt{241}}{2}\Big)^{\alpha}. \tag{1.14}$$

2. Preliminaries

In this section, we collect a number of lemmas which are essential to prove the main results of this paper.

The following 5-dissection formula for $(q;q)_\infty$ was first stated by Ramanujan [**9**, p.212], which was proved by Watson [**13**].

LEMMA 2.1. [**9**, p. 212]. *We have*

$$(q;q)_\infty=(q^{25};q^{25})_\infty\Big(\frac{1}{S(q^5)}-q-q^2S(q^5)\Big), \tag{2.1}$$

where $S(q)=\frac{(q;q^5)_\infty(q^4;q^5)_\infty}{(q^2;q^5)_\infty(q^3;q^5)_\infty}$.

LEMMA 2.2. [**3**, p. 165, eq.(7.4.14)]. *If* $T(q) := \frac{1}{S(q)}$, *then*

$$\frac{1}{(q;q)_\infty} = \frac{(q^{25};q^{25})_\infty^5}{(q^5;q^5)_\infty^6}\Big(T^4(q^5) + qT^3(q^5) + 2q^2T^2(q^5) + 3q^3T(q^5) + 5q^4 - \frac{3q^5}{T(q^5)} + \frac{2q^6}{T^2(q^5)} - \frac{q^7}{T^3(q^5)} + \frac{q^8}{T^4(q^5)}\Big). \tag{2.2}$$

LEMMA 2.3. [**2**, p. 345, Entry 1(iv)]. *We have*

$$(q;q)_\infty^3 = (q^9;q^9)_\infty^3\Big(\frac{1}{A(q^3)} - 3q + 4q^3A^2(q^3)\Big), \tag{2.3}$$

where $A(q) = \frac{(q;q)_\infty(q^6;q^6)_\infty^3}{(q^2;q^2)_\infty(q^3;q^3)_\infty^3}$.

The following lemmas play an important role in proving our main congruences for $B_{s,t}(n)$.

LEMMA 2.4. *If* $\sum_{n=0}^{\infty} a(n)q^n = (q;q)_\infty^9$, *then*

$$\sum_{n=0}^{\infty} a(5n+4)q^n = -90(q^5;q^5)_\infty^3(q;q)_\infty^6 - 625q(q^5;q^5)_\infty^9.$$

PROOF. In view of (2.1), we have

$$\begin{aligned}\sum_{n=0}^{\infty} a(n)q^n &= (q^{25};q^{25})_\infty^9\Big(\frac{1}{S(q^5)} - q - q^2S(q^5)\Big)^9\\ &= (q^{25};q^{25})_\infty^9\Big(-S^9(q^5)q^{18} - 9S^8(q^5)q^{17} - 27S^7(q^5)q^{16}\\ &\quad - 12S^6(q^5)q^{15} + 90S^5(q^5)q^{14} + 126S^4(q^5)q^{13} - 126S^3(q^5)q^{12}\\ &\quad - 288S^2(q^5)q^{11} + 117S(q^5)q^{10} + 365q^9 - \frac{117q^8}{S(q^5)} - \frac{288q^7}{S^2(q^5)} + \frac{126q^6}{S^3(q^5)}\\ &\quad + \frac{126q^5}{S^4(q^5)} - \frac{90q^4}{S^5(q^5)} - \frac{12q^3}{S^6(q^5)} + \frac{27q^2}{S^7(q^5)} - \frac{9q}{S^8(q^5)} + \frac{1}{S^9(q^5)}\Big).\end{aligned}$$

Extracting the terms involving q^n for $n \equiv 4 \pmod 5$ in the above identity, we obtain

$$\begin{aligned}\sum_{n=0}^{\infty} a(5n+4)q^n &= (q^5;q^5)_\infty^9\Big(\frac{-90}{S^5(q)} + 365q + 90q^2S^5(q)\Big)\\ &= -90(q^5;q^5)_\infty^9\Big(\frac{1}{S^5(q)} - 11q - S^5(q)q^2\Big) - 625q(q^5;q^5)_\infty^9.\end{aligned} \tag{2.4}$$

From [**14**], we recall the identity

$$\frac{1}{S^5(q)} - 11q - q^2S^5(q) = \frac{(q;q)_\infty^6}{(q^5;q^5)_\infty^6}. \tag{2.5}$$

Substituting (2.5) in (2.4), we complete the proof of Lemma 2.4. □

LEMMA 2.5. *If* $\sum_{n=0}^{\infty} g(n)q^n = (q;q)_\infty^{12}$, *then*

$$\sum_{n=0}^{\infty} g(3n+1)q^n \equiv 10(q;q)_\infty^{12} + 10q(q^3;q^3)_\infty^{12} \pmod{11}. \tag{2.6}$$

PROOF. From (2.3), we have

$$\sum_{n=0}^{\infty} g(n)q^n = (q^9;q^9)_\infty^{12}\Big(\frac{1}{A(q^3)} - 3q + 4q^3A^2(q^3)\Big)^4,$$

which implies

$$\begin{aligned}\sum_{n=0}^{\infty} g(n)q^n \equiv (q^9;q^9)_\infty^{12}\Big(&\frac{1}{A^4(q^3)} + \frac{10q}{A^3(q^3)} + \frac{7q^3}{A(q^3)} + \frac{10q^2}{A^2(q^3)}\\ &+ 3q^4 + 5q^6A^2(q^3) + 3q^5A(q^3) + 7q^7A^3(q^3) + 3q^9A^5(q^3)\\ &+ 6q^8A^4(q^3) + 2q^{10}A^6(q^3) + 3q^{12}A^8(q^3)\Big) \pmod{11}.\end{aligned} \tag{2.7}$$

Extracting the terms of the form q^{3n+1} in (2.7), we get

$$\begin{aligned}\sum_{n=0}^{\infty} g(3n+1)q^n &\equiv (q^3;q^3)_\infty^{12}\Big(\frac{10}{A^3(q)} + 3q + 7q^2A^3(q) + 2q^3A^6(q)\Big)\\ &\equiv (q^3;q^3)_\infty^{12}\Big(-\Big(\frac{1}{A(q)} + 4qA^2(q)\Big)^3 + 4q\Big) \pmod{11}.\end{aligned} \tag{2.8}$$

We can rewrite the entry 1(iv) in [**2**, pp. 345] as

$$\Big(\frac{1}{A(q)} + 4qA^2(q)\Big)^3 = \frac{(q;q)_\infty^{12}}{(q^3;q^3)_\infty^{12}} + 27q, \tag{2.9}$$

where $A(q)$ is as defined in (2.3). Employing (2.9) in (2.8), we obtain (2.6). □

The proofs of the following lemmas are analogous to the proof of Lemma 2.5.

LEMMA 2.6. *If* $\sum_{n=0}^{\infty} z(n)q^n = (q;q)_\infty^9$, *then* $\sum_{n=0}^{\infty} z(3n)q^n \equiv \frac{(q;q)_\infty^{12}}{(q^3;q^3)_\infty^3} \pmod{11}$.

LEMMA 2.7. *If* $\sum_{n=0}^{\infty} m(n)q^n = (q;q)_\infty^{15}$, *then* $\sum_{n=0}^{\infty} m(3n+2)q^n \equiv 11q(q^3;q^3)_\infty^{15} + 5(q;q)_\infty^{12}(q^3;q^3)_\infty^3 \pmod{17}$.

LEMMA 2.8. *If* $\sum_{n=0}^{\infty} g(n)q^n = (q;q)_\infty^{12}$, *then* $\sum_{n=0}^{\infty} g(3n+1)q^n \equiv 5(q;q)_\infty^{12} + 12q(q^3;q^3)_\infty^{12} \pmod{17}$.

As the proof of the following lemmas are similar to that of Lemma 2.4, we omit the details.

LEMMA 2.9. *If* $\sum_{n=0}^{\infty} y(n)q^n = (q;q)_\infty^6$, *then* $\sum_{n=0}^{\infty} y(5n+1)q^n = -6(q;q)_\infty^6 - 25q(q^5;q^5)_\infty^6$.

LEMMA 2.10. *If* $\sum_{n=0}^{\infty} c(n)q^n = q(q;q)_\infty^3$, *then* $\sum_{n=0}^{\infty} c(5n+4)q^n = 5(q^5;q^5)_\infty^3$.

LEMMA 2.11. *If* $\sum_{n=0}^{\infty} d(n)q^n = (q;q)_\infty^5$, *then* $\sum_{n=0}^{\infty} d(5n)q^n = \frac{(q;q)_\infty^6}{(q^5;q^5)_\infty}$.

3. Proofs of Theorems 1.1–1.3

We first prove the following two lemmas.

LEMMA 3.1. *For all $\alpha, \beta \in \mathbb{N}_0$ with $\beta \geqslant 2\alpha$, we have*

$$\sum_{n=0}^{\infty} B_{7,5^\beta}\Big(5^{2\alpha}n + \frac{5^{2\alpha+1}-5}{24}\Big)q^n \equiv (q^{5^{\beta-2\alpha}}; q^{5^{\beta-2\alpha}})_\infty \Big(f(\alpha)(q;q)_\infty^5 + 3qf(\alpha-1)\frac{(q^5;q^5)_\infty^6}{(q;q)_\infty}\Big) \pmod 7, \tag{3.1}$$

where $f(-1) = 0$ and

$$f(\alpha) = \Big(\frac{\sqrt{61}}{122} + \frac{1}{2}\Big)\Big(\frac{\sqrt{61}}{2} + \frac{1}{2}\Big)^\alpha + \Big(\frac{-\sqrt{61}}{122} + \frac{1}{2}\Big)\Big(\frac{-\sqrt{61}}{2} + \frac{1}{2}\Big)^\alpha.$$

PROOF. Setting $s = 7$ and $t = 5^\beta (\beta \in \mathbb{N}_0)$ in (1.1), we get

$$\sum_{n=0}^{\infty} B_{7,5^\beta}(n)q^n = \frac{(q^7;q^7)_\infty (q^{5^\beta}; q^{5^\beta})_\infty}{(q;q)_\infty^2}. \tag{3.2}$$

By the binomial theorem, it is easy to check that, for any prime p,

$$(q;q)_\infty^p \equiv (q^p;q^p) \pmod p. \tag{3.3}$$

Employing (3.3) with $p = 7$ in (3.2), we see that (3.1) is true for $\alpha = 0$. Suppose that (3.1) is true for $\alpha = m$, i.e.,

$$\sum_{n=0}^{\infty} B_{7,5^\beta}\Big(5^{2m}n + \frac{5^{2m+1}-5}{24}\Big)q^n \equiv (q^{5^{\beta-2m}}; q^{5^{\beta-2m}})_\infty \Big(f(m)(q;q)_\infty^5 + 3qf(m-1)\frac{(q^5;q^5)_\infty^6}{(q;q)_\infty}\Big) \pmod 7. \tag{3.4}$$

It is a routine to check that

$$f(\alpha+1) = f(\alpha) + 15f(\alpha-1). \tag{3.5}$$

Substituting (2.2) in (3.4), extracting the terms of the form q^{5n} in the resulting congruence and then using Lemma 2.11, we have

$$\begin{aligned} \sum_{n=0}^{\infty} B_{7,5^\beta}\Big(5^{2m+1}n + \frac{5^{2m+1}-5}{24}\Big)q^n &\equiv (q^{5^{\beta-2m-1}}; q^{5^{\beta-2m-1}})_\infty \Big(f(m)\sum_{n=0}^{\infty} d(5n)q^n + 15qf(m-1)(q^5;q^5)_\infty^5\Big) \\ &\equiv (q^{5^{\beta-2m-1}}; q^{5^{\beta-2m-1}})_\infty \Big(f(m)\frac{(q;q)_\infty^6}{(q^5;q^5)_\infty} + 15qf(m-1)(q^5;q^5)_\infty^5\Big) \pmod 7. \end{aligned} \tag{3.6}$$

Extracting the terms of the form q^{5n+1} in (3.6) and then employing Lemma 2.9, we obtain

$$\sum_{n=0}^{\infty} B_{7,5^\beta}\Big(5^{2m+1}(5n+1) + \frac{5^{2m+1}-5}{24}\Big)q^n$$

$$\begin{aligned}
&\equiv (q^{5^{\beta-2m-2}};q^{5^{\beta-2m-2}})_\infty\Big(f(m)\frac{1}{(q;q)_\infty}\sum_{n=0}^{\infty}y(5n+1)q^n+15f(m-1)(q;q)_\infty^5\Big)\\
&\equiv (q^{5^{\beta-2m-2}};q^{5^{\beta-2m-2}})_\infty\\
&\quad\times\Big((f(m)+15f(m-1))(q;q)_\infty^5+3qf(m)\frac{(q^5;q^5)_\infty^6}{(q;q)_\infty}\Big)\pmod 7.
\end{aligned}$$

Employing (3.5) in the above congruence relation, we obtain (3.1) with $\alpha=m+1$. This completes the proof. □

LEMMA 3.2. *For all integer $\alpha\in\mathbb{N}_0$, we have*

$$f(8\alpha+7)\equiv 0\pmod 7 \tag{3.7}$$

$$f(8\alpha+6)\equiv(-1)^{\alpha-1}\pmod 7. \tag{3.8}$$

PROOF. For $\alpha=0$, we have $f(7)=15841\equiv 0\pmod 7$. So, (3.7) holds for $\alpha=0$. Assume that (3.7) is true for $\alpha=m$, i.e.,

$$f(8m+7)\equiv 0\pmod 7. \tag{3.9}$$

In order to complete the proof of (3.7), we need to prove that (3.7) is true for $\alpha=m+1$. By (3.5), we see that

$$\begin{aligned}
f(8m+15)&=f(8m+14)+15f(8m+13)\\
&=16f(8m+13)+15f(8m+12)\\
&=31f(8m+12)+240f(8m+11)\\
&=271f(8m+11)+465f(8m+10)\\
&=736f(8m+10)+4065f(8m+9)\\
&=4801f(8m+9)+11040f(8m+8)\\
&=15841f(8m+8)+72015f(8m+7).
\end{aligned}\tag{3.10}$$

So, from (3.9) and (3.10), we have $f(8m+15)\equiv 0\pmod 7$. So, (3.7) is proved. Similarly, we can prove (3.8). □

Now, we turn to prove Theorems 1.1, 1.2 and 1.3. Replacing α by $8\alpha+7$ in (3.1) and then using (3.7), we obtain

$$\begin{aligned}
&\sum_{n=0}^{\infty}B_{7,5^\beta}\Big(5^{16\alpha+14}n+\frac{5^{16\alpha+15}-5}{24}\Big)q^n\\
&\qquad\equiv 3qf(8\alpha+6)(q^{5^{\beta-16\alpha-14}};q^{5^{\beta-16\alpha-14}})_\infty\frac{(q^5;q^5)_\infty^6}{(q;q)_\infty}\pmod 7.
\end{aligned}\tag{3.11}$$

Substituting (2.2) into (3.11) and then extracting the terms of the form q^{5n} in the resulting congruence, we obtain

$$\sum_{n=0}^{\infty}B_{7,5^\beta}\Big(5^{16\alpha+15}n+\frac{5^{16\alpha+15}-5}{24}\Big)q^n \tag{3.12}$$

$$\equiv qf(8\alpha+6)(q^{5^{\beta-16\alpha-15}};q^{5^{\beta-16\alpha-15}})_\infty(q^5;q^5)_\infty^5 \pmod 7.$$

Extracting the terms of the form q^{5n+1} in (3.12), we get

$$\sum_{n=0}^{\infty} B_{7,5^\beta}\Big(5^{16\alpha+16}n+\frac{5^{16\alpha+17}-5}{24}\Big)q^n \equiv f(8\alpha+6)(q^{5^{\beta-16(\alpha+1)}};q^{5^{\beta-16(\alpha+1)}})_\infty(q;q)_\infty^5 \pmod 7. \tag{3.13}$$

In view of (1.1), (3.8) and (3.13), we get Theorem 1.1.

Note that, if $\beta \geqslant 16\alpha+16$, then the powers of q on the right-hand side of (3.12) are of the form $5n+1$ and hence, equating the coefficients of q^{5n+j} for $j\in\{0,2,3,4\}$, we obtain (1.5).

Setting $\beta = 16\alpha+15$ in (3.12), we find that

$$\sum_{n=0}^{\infty} B_{7,5^{16\alpha+15}}\Big(5^{16\alpha+15}n+\frac{5^{16\alpha+15}-5}{24}\Big)q^n \equiv qf(8\alpha+6)(q;q)_\infty(q^5;q^5)_\infty^5 \pmod 7. \tag{3.14}$$

Employing (2.1) in (3.14) and then equating the coefficients of q^{5n} and q^{5n+4} in the resulting congruence, we obtain Theorem 1.3.

4. Proofs of Theorems 1.4–1.5

We first prove the following two lemmas.

Lemma 4.1. *For all $n,\alpha,\beta\in\mathbb{N}_0$ with $\beta\geqslant 2\alpha$, we have*

$$\sum_{n=0}^{\infty} B_{11,3^\beta}\Big(3^{2\alpha}n+3\cdot\frac{3^{2\alpha}-1}{8}\Big)q^n \equiv (q^{3^{\beta-2\alpha}};q^{3^{\beta-2\alpha}})_\infty\Big(h(\alpha)(q;q)_\infty^9+10qh(\alpha-1)\frac{(q^3;q^3)_\infty^{12}}{(q;q)_\infty^3}\Big) \pmod{11}, \tag{4.1}$$

where $h(-1)=0$ and

$$h(\alpha)=\Big(\frac{-5\sqrt3}{18}+\frac12\Big)\Big(5-3\sqrt3\Big)^\alpha+\Big(\frac{5\sqrt3}{18}+\frac12\Big)\Big(5+3\sqrt3\Big)^\alpha.$$

Proof. Setting $s=11$ and $t=3^\beta$ in (1.1) and then employing (3.3) with $p=11$, we obtain

$$\sum_{n=0}^{\infty} B_{11,3^\beta}(n)q^n=\frac{(q^{3^\beta};q^{3^\beta})_\infty(q^{11};q^{11})_\infty}{(q;q)_\infty^2}\equiv(q^{3^\beta};q^{3^\beta})_\infty(q;q)_\infty^9 \pmod{11},$$

which implies that (4.1) is true for $\alpha=0$. Assume that (4.1) is true for $\alpha=m$, i.e.,

$$\sum_{n=0}^{\infty} B_{11,3^\beta}\Big(3^{2m}n+3\cdot\frac{3^{2m}-1}{8}\Big)q^n \equiv (q^{3^{\beta-2m}};q^{3^{\beta-2m}})_\infty\Big(h(m)(q;q)_\infty^9+10qh(m-1)\frac{(q^3;q^3)_\infty^{12}}{(q;q)_\infty^3}\Big) \pmod{11}. \tag{4.2}$$

It is trivial to check that $h(m)$ satisfies the recurrence relation

$$h(m+1)=10h(m)+2h(m-1). \tag{4.3}$$

Using the Jacobi triple product identity, Wang [**12**] proved the identity

$$\frac{1}{(q;q)_\infty^3}=\frac{(q^9;q^9)_\infty^3}{(q^3;q^3)_\infty^{12}}\Big(P^2(q^3)+3qP(q^3)(q^9;q^9)_\infty^3+9q^2(q^9;q^9)_\infty^6\Big), \tag{4.4}$$

where $P(q)=\sum_{m=-\infty}^{\infty}(-1)^m(6m+1)q^{m(3m+1)/2}$. Substituting (4.4) in (4.2), extracting the terms of the form q^{3n} in the resulting congruence and then using Lemma 2.6, we deduce

$$\begin{aligned}(4.5)\quad &\sum_{n=0}^{\infty}B_{11,3^\beta}\Big(3^{2m+1}n+3\cdot\frac{3^{2m}-1}{8}\Big)q^n\\ &\equiv(q^{3^{\beta-2m-1}};q^{3^{\beta-2m-1}})_\infty\Big(h(m)\sum_{n=0}^{\infty}z(3n)q^n+10h(m-1)(q;q)_\infty^{12}\cdot\frac{9q(q^3;q^3)_\infty^9}{(q;q)_\infty^{12}}\Big)\\ &\equiv(q^{3^{\beta-2m-1}};q^{3^{\beta-2m-1}})_\infty\Big(h(m)\frac{(q;q)_\infty^{12}}{(q^3;q^3)_\infty^3}+2qh(m-1)(q^3;q^3)_\infty^9\Big)\pmod{11}.\end{aligned}$$

From (4.5) and Lemma 4.1, we have

$$\begin{aligned}(4.6)\quad &\sum_{n=0}^{\infty}B_{11,3^\beta}\Big(3^{2m+2}n+3\cdot\frac{3^{2m+2}-1}{8}\Big)q^n\\ &\equiv(q^{3^{\beta-2m-2}};q^{3^{\beta-2m-2}})_\infty\Big(h(m)\frac{1}{(q;q)_\infty^3}\sum_{n=0}^{\infty}g(3n+1)q^n+2h(m-1)(q;q)_\infty^9\Big)\\ &\equiv(q^{3^{\beta-2m-2}};q^{3^{\beta-2m-2}})_\infty\\ &\quad\times\Big((10h(m)+2h(m-1))(q;q)_\infty^9+10qh(m)\frac{(q^3;q^3)_\infty^{12}}{(q;q)_\infty^3}\Big)\pmod{11}.\end{aligned}$$

From (4.3) and (4.6), (4.1) holds for $\alpha=m+1$. □

In view of (4.3) and by induction on α, we can prove the following lemma.

Lemma 4.2. *For all integer $\alpha\geqslant 0$, we have*

$$h(5\alpha+4)\equiv 0\pmod{11} \tag{4.7}$$

$$h(5\alpha+5)\equiv 1\pmod{11}. \tag{4.8}$$

Now, we are ready to prove Theorems 1.4 and 1.5.

Replacing α by $5\alpha+5$ in (4.1) and then using (4.7), we obtain

$$\begin{aligned}(4.9)\quad &\sum_{n=0}^{\infty}B_{11,3^\beta}\Big(3^{10(\alpha+1)}n+3\cdot\frac{3^{10(\alpha+1)}-1}{8}\Big)q^n\\ &\equiv h(5\alpha+5)(q^{3^{\beta-10(\alpha+1)}};q^{3^{\beta-10(\alpha+1)}})_\infty(q;q)_\infty^9\pmod{11}.\end{aligned}$$

By (4.9), (1.1) and (4.8), we have (1.6). This completes the proof of Theorem 1.4.

Replacing α by $5\alpha+4$ in (4.1) and then employing (4.7), we find that

$$\sum_{n=0}^{\infty} B_{11,3^\beta}\Big(3^{10\alpha+8}n+3\cdot\frac{3^{10\alpha+8}-1}{8}\Big)q^n \equiv 10h(5\alpha+3)q(q^{3^{\beta-10\alpha-8}};q^{3^{\beta-10\alpha-8}})_\infty\frac{(q^3;q^3)_\infty^{12}}{(q;q)_\infty^3} \pmod{11}. \tag{4.10}$$

Substituting (4.4) in (4.10) and then extracting the terms of the form q^{3n} in the resulting congruence, we obtain

$$\sum_{n=0}^{\infty} B_{11,3^\beta}\Big(3^{10\alpha+9}n+3\cdot\frac{3^{10\alpha+8}-1}{8}\Big)q^n \equiv 2qh(5\alpha+3)(q^{3^{\beta-10\alpha-9}};q^{3^{\beta-10\alpha-9}})_\infty(q^3;q^3)_\infty^9 \pmod{11}. \tag{4.11}$$

Equating the coefficients of q^{3n} and q^{3n+2} in (4.11), we obtain (1.7) and (1.8), respectively. This completes the proof of Theorem 1.5.

5. Proofs of Theorems 1.6–1.7

We first prove the following two lemmas.

LEMMA 5.1. *For all $\alpha,\beta,n\in\mathbb{N}_0$ with $\beta\geqslant 2\alpha$, we have*

$$\sum_{n=0}^{\infty} B_{11,5^\beta}\Big(5^{2\alpha}n+3\cdot\frac{5^{2\alpha}-1}{8}\Big)q^n \equiv (q^{5^{\beta-2\alpha}};q^{5^{\beta-2\alpha}})_\infty\Big(a(\alpha)(q;q)_\infty^9+b(\alpha)q(q;q)_\infty^3(q^5;q^5)_\infty^6\Big) \pmod{11}, \tag{5.1}$$

where $a(\alpha)$ is as defined in (1.10) *and $b(\alpha)$ is defined by*

$$b(\alpha)=3\cdot\frac{\big(5+\sqrt{22}\big)^\alpha}{\sqrt{22}}-3\cdot\frac{\big(5-\sqrt{22}\big)^\alpha}{\sqrt{22}}. \tag{5.2}$$

PROOF. Setting $s=11$ and $t=5^\beta(\beta\in\mathbb{N}_0)$ in (1.1) and then employing (3.3) with $p=11$, we get

$$\sum_{n=0}^{\infty} B_{11,5^\beta}(n)q^n=\frac{(q^{11};q^{11})_\infty(q^{5^\beta};q^{5^\beta})_\infty}{(q;q)_\infty^2}\equiv(q^{5^\beta};q^{5^\beta})_\infty(q;q)_\infty^9 \pmod{11}.$$

This shows that (5.1) holds for $\alpha=0$. Suppose that (5.1) is true for $\alpha=m$, i.e.,

$$\sum_{n=0}^{\infty} B_{11,5^\beta}\Big(5^{2m}n+3\cdot\frac{5^{2m}-1}{8}\Big)q^n \equiv (q^{5^{\beta-2m}};q^{5^{\beta-2m}})_\infty\Big(a(m)(q;q)_\infty^9+b(m)q(q;q)_\infty^3(q^5;q^5)_\infty^6\Big) \pmod{11}. \tag{5.3}$$

It is trivial to check that $a(\alpha)$ and $b(\alpha)$ satisfy the relations

$$6a(\alpha)=b(\alpha+1)-7b(\alpha), \tag{5.4}$$

$$3b(\alpha)=a(\alpha+1)-3a(\alpha). \tag{5.5}$$

Using Lemmas 2.4 and 2.10 in (5.3), we obtain

$$(5.6)\quad \sum_{n=0}^{\infty} B_{11,5^\beta}\Big(5^{2m+1}n+\frac{7\cdot 5^{2m+1}-3}{8}\Big)q^n$$
$$\equiv (q^{5^{\beta-2m-1}};q^{5^{\beta-2m-1}})_\infty$$
$$\times \Big(a(m)\Big(9(q^5;q^5)_\infty^3(q;q)_\infty^6+2q(q^5;q^5)_\infty^9\Big)+5b(m)(q^5;q^5)_\infty^3(q;q)_\infty^6\Big)$$
$$\equiv (q^{5^{\beta-2m-1}};q^{5^{\beta-2m-1}})_\infty$$
$$\times (9a(m)+5b(m))(q^5;q^5)_\infty^3(q;q)_\infty^6+2a(m)q(q^5;q^5)_\infty^9 \pmod{11}.$$

Extracting the terms of the form q^{5n+1} in (5.6) and then using Lemma 2.9, we obtain

$$\sum_{n=0}^{\infty} B_{11,5^\beta}\Big(5^{2m+2}n+3\cdot\frac{\cdot 5^{2m+2}-1}{8}\Big)q^n$$
$$\equiv (q^{5^{\beta-2m-2}};q^{5^{\beta-2m-2}})_\infty$$
$$\times \Big((9a(m)+5b(m))(q;q)_\infty^3\Big(5(q;q)_\infty^6+8q(q^5;q^5)_\infty^6\Big)+2a(m)(q;q)_\infty^9\Big)$$
$$\equiv (q^{5^{\beta-2m-2}};q^{5^{\beta-2m-2}})_\infty$$
$$\times \Big((3a(m)+3b(m))(q;q)_\infty^9+(6a(m)+7b(m))q(q;q)_\infty^3(q^5;q^5)_\infty^6\Big) \pmod{11}.$$

Employing (5.4) and (5.5) in the above congruence, we see that (5.1) also holds for $\alpha=m+1$. □

Lemma 5.2. *For all integers $\alpha\in\mathbb{N}_0$, we have*

$$(5.7)\quad a(11\alpha+8)\equiv 0 \pmod{11},$$
$$(5.8)\quad b(11\alpha)\equiv 0 \pmod{11},$$

where $a(\alpha)$ and $b(\alpha)$ are as defined in (1.10) *and* (5.2), *respectively.*

Proof. Note that $a(\alpha)$ and $b(\alpha)$ satisfies the following recurrence relations:

$$(5.9)\quad a(\alpha+1)=10a(\alpha)-3a(\alpha-1)$$

and

$$(5.10)\quad b(\alpha+1)=10b(\alpha)-3b(\alpha-1).$$

Using (5.9), (5.10) and by induction on α, we obtain (5.7) and (5.8), respectively. □

Now, we turn to prove Theorems 1.6 and 1.7.

Replacing α by 11α in (5.1) and then using (5.8), we deduce

$$(5.11)\quad \sum_{n=0}^{\infty} B_{11,5^\beta}\Big(5^{22\alpha}n+3\cdot\frac{5^{22\alpha}-1}{8}\Big)q^n\equiv a(11\alpha)(q^{5^{\beta-2\alpha}};q^{5^{\beta-2\alpha}})_\infty(q;q)_\infty^9 \pmod{11},$$

Congruence (1.9) follows from (5.11) and (1.1). Hence the proof of Theorem 1.6.

Changing α to $11\alpha+8$ in (5.1) and then using (5.7), we have

$$\sum_{n=0}^{\infty} B_{11,5^\beta}\Big(5^{22\alpha+16}n+3\cdot\frac{5^{22\alpha+16}-1}{8}\Big)q^n$$
$$\equiv b(11\alpha+8)q(q^{5^{\beta-22\alpha-16}};q^{5^{\beta-22\alpha-16}})_\infty(q;q)_\infty^3(q^5;q^5)_\infty^6 \pmod{11}.$$

In view of (2.1), we can rewrite the above congruence relation as

$$(5.12)\quad \sum_{n=0}^{\infty} B_{11,5^\beta}\Big(5^{22\alpha+16}n+3\cdot\frac{5^{22\alpha+16}-1}{8}\Big)q^n$$
$$\equiv b(11\alpha+8)(q^{5^{\beta-22\alpha-16}};q^{5^{\beta-22\alpha-16}})_\infty(q^5;q^5)_\infty^6(q^{25};q^{25})_\infty^3$$
$$\times\Big(\frac{q}{R^3(q^5)}+\frac{8q^2}{R^2(q^5)}+5q^4+8q^6R^2(q^5)+10q^7R^3(q^5)\Big) \pmod{11}.$$

Equating the coefficients of q^{5n} and q^{5n+3} in (5.12), we obtain (1.11) and (1.12), respectively. This completes the proof of Theorem 1.7.

6. Proof of Theorem 1.8

Setting $s=7$ and $t=2$ in (1.1) and then applying (3.3) with $p=7$, we get

$$(6.1)\quad \sum_{n=0}^{\infty} B_{7,2}(n)q^n=\frac{(q^7;q^7)_\infty(q^2;q^2)_\infty}{(q;q)_\infty^2}\equiv\frac{(q;q)_\infty^5}{(q^2;q^2)_\infty^2}(q^2;q^2)_\infty^3 \pmod{7}.$$

From [**3**, Theorem 1.3.9 and Corollary 1.3.21], we recall the identities

$$(6.2)\quad (q;q)_\infty^3=\sum_{n=0}^{\infty}(-1)^n(2n+1)q^{n(n+1)/2}$$

$$(6.3)\quad \frac{(q;q)_\infty^5}{(q^2;q^2)_\infty^2}=\sum_{n=-\infty}^{\infty}(6n+1)q^{n(3n+1)/2}.$$

From (6.1), (6.2) and (6.3), we have

$$\sum_{n=0}^{\infty} B_{7,2}(n)q^n\equiv\sum_{m=0}^{\infty}\sum_{k=-\infty}^{\infty}(-1)^m(2m+1)(6k+1)q^{m(m+1)/2+k(3k+1)/2} \pmod{7}.$$

And then, q replaced by q^{24} and then multiplied by q^7 on both sides of the above congruence, we arrive at

$$\sum_{n=0}^{\infty} B_{7,2}(n)q^{24n+7}\equiv\sum_{m=0}^{\infty}\sum_{k=-\infty}^{\infty}(-1)^m(2m+1)(6k+1)q^{3(2m+1)^2+(6k+1)^2} \pmod{7}.$$

If $24n+7$ is not of the form $3(2m+1)^2+(6k+1)^2$ for $n,m\geqslant 0$ and $-\infty<k<\infty$, then $B_{7,2}(n)\equiv 0 \pmod{7}$. Let $p_1,p_2,\cdots,p_r\geqslant 5$ be distinct primes with $\big(\frac{-3}{p_r}\big)=-1$. Let $v_{p_r}(N)$ denote the power of prime p_r in the unique prime factorisation of

N. If $N = x^2 + 3y^2$, then $v_{p_r}(N)$ must be even because $\left(\frac{-3}{p_r}\right) = -1$. Hence, if $p_r \nmid n$, then

$$v_{p_r}\left(24\left(np_r^{2\beta_r+1}\prod_{i=1}^{r-1}p_i^{2\beta_i+2} + 7\cdot\frac{\prod_{i=1}^{r}p_i^{2\beta_i+2}-1}{24}\right)+7\right) = 2\beta_r+1,$$

and so $24\big(np_r^{2\beta_r+1}\prod_{i=1}^{r-1}p_i^{2\beta_i+2} + 7\cdot\frac{\prod_{i=1}^{r}p_i^{2\beta_i+2}-1}{24}\big)+7$ is not of the form x^2+3y^2. Hence

$$B_{7,2}\left(np_r^{2\beta_r+1}\prod_{i=1}^{r-1}p_i^{2\beta_i+2} + 7\cdot\frac{\prod_{i=1}^{r}p_i^{2\beta_i+2}-1}{24}\right) \equiv 0 \pmod 7.$$

This completes the proof of Theorem 1.8.

7. Proof of Theorem 1.9

LEMMA 7.1. *For all $n,\alpha,\beta\in\mathbb{N}_0$ with $\beta\geqslant 2\alpha$, we have*

$$\sum_{n=0}^{\infty}B_{17,3^\beta}\left(3^{2\alpha}n+5\cdot\frac{3^{2\alpha}-1}{8}\right)q^n \equiv (q^{3^{\beta-2\alpha}};q^{3^{\beta-2\alpha}})_\infty \times\left(H(\alpha)(q;q)_\infty^{15}+\frac{H(\alpha+1)-2H(\alpha)}{2}q(q;q)_\infty^3(q^3;q^3)_\infty^{12}\right) \pmod{17}, \tag{7.1}$$

where $H(\alpha)$ is as defined in (1.14).

PROOF. Setting $s=17$ and $t=3^\beta$ in (1.1) and then employing (3.3) with $p=17$, we see that

$$\sum_{n=0}^{\infty}B_{17,3^\beta}(n)q^n = \frac{(q^{17};q^{17})_\infty(q^{3^\beta};q^{3^\beta})_\infty}{(q;q)_\infty^2} \equiv (q;q)_\infty^{15}(q^{3^\beta};q^{3^\beta})_\infty \pmod{17}, \tag{7.2}$$

which is the same as (7.1) with $\alpha=0$. Assume that (7.1) is true for some $\alpha\geqslant 1$. It is a routine to check that $H(\alpha)$ satisfies the recurrence relation

$$H(\alpha+1) = 17H(\alpha)-12H(\alpha-1). \tag{7.3}$$

In view of Lemma 2.7, (7.1) and (2.3), we deduce that

$$\begin{aligned}
&\sum_{n=0}^{\infty}B_{17,3^\beta}\left(3^{2\alpha}(3n+2)+5\cdot\frac{3^{2\alpha}-1}{8}\right)q^n \qquad (7.4)\\
&\equiv (q^{3^{\beta-2\alpha-1}};q^{3^{\beta-2\alpha-1}})_\infty\\
&\quad\times\left(H(\alpha)\sum_{n=0}^{\infty}m(3n+2)q^n+\frac{H(\alpha+1)-2H(\alpha)}{2}\cdot(-3)(q^3;q^3)_\infty^3(q;q)_\infty^{12}\right)\\
&\equiv (q^{3^{\beta-2\alpha-1}};q^{3^{\beta-2\alpha-1}})_\infty\\
&\quad\times\left(11H(\alpha)q(q^3;q^3)_\infty^{15}+\frac{16H(\alpha)-3H(\alpha+1)}{2}(q^3;q^3)_\infty^3(q;q)_\infty^{12}\right) \pmod{17}.
\end{aligned}$$

From (7.4), Lemma 2.8 and (7.3), we find that

$$\sum_{n=0}^{\infty} B_{17,3^\beta}\Big(3^{2\alpha+1}(3n+1)+\frac{21\cdot 3^{2\alpha}-5}{8}\Big)q^n \equiv (q^{3^{\beta-2\alpha-2}};q^{3^{\beta-2\alpha-2}})_\infty$$
$$\times\Big(11H(\alpha)(q;q)_\infty^{15}+\frac{16H(\alpha)-3H(\alpha+1)}{2}(q;q)_\infty^{3}\sum_{n=0}^{\infty}g(3n+1)q^n\Big)$$
$$\equiv (q^{3^{\beta-2\alpha-2}};q^{3^{\beta-2\alpha-2}})_\infty$$
$$\times\Big(H(\alpha+1)(q;q)_\infty^{15}+\frac{H(\alpha+2)-2H(\alpha+1)}{2}q(q;q)_\infty^{3}(q^3;q^3)_\infty^{12}\Big) \pmod{17}.$$

That is, (7.1) holds for $\alpha+1$. This completes the proof of (7.1) by induction on α. □

LEMMA 7.2. *For all $\alpha \geqslant 0$, we have*

$$\frac{H(2\alpha+1)-2H(2\alpha)}{2}\equiv 0 \pmod{17}. \tag{7.5}$$

PROOF. It is trivial to check that (7.5) holds for $\alpha=0$. Suppose that (7.5) holds for some $\alpha\geqslant 1$. By (7.3), we have

$$\frac{H(2\alpha+3)-2H(2\alpha+2)}{2}=\frac{15H(2\alpha+2)-12H(2\alpha+1)}{2} \tag{7.6}$$
$$=243\cdot\frac{H(2\alpha+1)-2H(2\alpha)}{2}+153H(2\alpha).$$

From (7.5) and (7.6), we see that (7.5) holds for $\alpha+1$. □

PROOF OF THEOREM 1.9. Replacing α by 2α in (7.1) and then employing (7.5), we find that

$$\sum_{n=0}^{\infty} B_{17,3^\beta}\Big(3^{4\alpha}n+5\cdot\frac{3^{4\alpha}-1}{8}\Big)q^n \equiv H(2\alpha)(q^{3^{\beta-4\alpha}};q^{3^{\beta-4\alpha}})_\infty(q;q)_\infty^{15} \pmod{17}.$$

In view of (7.2), we can rewrite the above congruence relation as follows:

$$\sum_{n=0}^{\infty} B_{17,3^\beta}\Big(3^{4\alpha}n+5\cdot\frac{3^{4\alpha}-1}{8}\Big)q^n \equiv H(2\alpha)\sum_{n=0}^{\infty}B_{17,3^{\beta-4\alpha}}(n)q^n \pmod{17}. \tag{7.7}$$

Equating the coefficients of q^n on both sides of (7.7), we obtain congruence (1.13). □

Acknowledgment. The third author's research was partially supported by Grant no. CUK/ACAD-II/R&D/F-3625/19-20/22 of the Central University of Karnataka, Kalaburagi.

References

1. C. Adiga, D. Ranganatha, *A simple proof of a conjecture of Dou on* $(3,7)$*-regular bipartitions modulo* 3, Integers **17** (2017), Paper A14, 8 pp.
2. B. C. Berndt, *Ramanujan's Notebooks, Part III*, Springer, New York, 1991.
3. ______, *Number Theory in the Spirit of Ramanujan*, American Mathematical Society, Providence, 2006.
4. H. B. Dai, *Congruences for the number of partitions and bipartitions with distinct even parts*, Discrete Math. **338** (2015), 133–138.
5. D. Q. J. Dou, *Congruences for* $(3,11)$*-regular bipartitions modulo* 11, Ramanujan J. **40** (2016), 535–540.
6. B. L. S. Lin, *Arithmetic properties of bipartitions with even parts distinct*, Ramanujan J. **33** (2014), 269–279.
7. ______, *Arithmetic of the* 7*-regular bipartition function modulo* 3, Ramanujan J. **37** (2015), 469–478.
8. ______, *An infinite family of congruences modulo* 3 *for* 13*-regular bipartitions*, Ramanujan J. **39** (2016), 169–178.
9. S. Ramanujan, *Collected Papers*, Cambridge University Press, Cambridge, 1927.
10. ______, *Some propeties of* $p(n)$*, the number of partitions of* n, Proc. Camb. Philos. Soc. **19** (1919), 207–210.
11. D. Ranganatha, *Ramanujan-type congruences modulo powers of* 5 *and* 7, Indian J. Pure Appl. Math. **48**(3) (2017), 449–465.
12. L. Wang, *Arithmetic identities and congruences for partition triples with* 3*-cores*, Int. J. Number Theory **4** (2016), 995–1010.
13. G. N. Watson, *Ramanujan's Vermutung über Zerfällungsanzahlem*, J. Reine Angew. Math. **179** (1938), 97–128.
14. ______, *Theorems stated by Ramanujan* (VII)*: theorems on contuined fractions*, J. Lond. Math. Soc. **4** (1929), 39–48.
15. E. X. W. Xia, X. M. Yao, *Arithmetic properties for* (s,t)*-regular bipartition functions*, J. Number Theory **171** (2017), 1–17.

Department of Studies in Mathematics
University of Mysore, Manasagangotri
Karnataka
India
c_adiga@hotmail.com

(Received 23 01 2018)

Université d'Évry Val d'Essonne
Laboratoire de Mathématiques et Modélisation d'Évry (UMR8071)
I.B.G.B.I. Évry Cedex France
abayad@maths.univ-evry.fr

Department of Mathematics
School of Physical Sciences
Central University of Karnataka
Karnataka
India
ddranganatha@gmail.com