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ON CHEBYSHEV CENTERS IN METRIC SPACES

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ABSTRACT. A Chebyshev center of a set A in a metric space (X, d) is a point of X best approximating the set A i.e., it is a point $x_0 \in X$ such that $\sup_{y\in A} d(x_0, y) = \inf_{x\in X} \sup_{y\in A} d(x, y)$. We discuss the existence and uniqueness of such points in metric spaces thereby generalizing and extending several known result on the subject.

1. Introduction

Let (X, d) be a metric space and A a subset of X. We consider the problem of choosing an element of X which best represents the set A. If x is any particular element of X chosen to represent the set A, the error incurred will be $\sup\{d(x, y) : y \in A\}$. In order that this quantity is finite, it is necessary and sufficient that the set A is bounded. So, we make this assumption. Then $x_0 \in X$ will best represent the set A when this error is minimum i.e., $\sup_{y\in A} d(x_0, y) = \inf_{x\in X} \sup_{y\in A} d(x, y)$. Such an element $x_0 \in X$ is called a Chebyshev center of the set A. The notion of Chebyshev centers is an important concept in the theory of optimization. A systematic study of Chebyshev centers was initiated by Garakavi [**4**]. Subsequently, this study has been taken up by many researchers. Several results are known on the existence and uniqueness of Chebyshev centers (see [**2–6**], [**8–11**], and references cited therein).

In this paper, we also discuss some results on the existence and uniqueness of Chebyshev centers when the underlying spaces are metric spaces thereby extending and generalizing several known results on the subject. We start with few definitions and basic facts.

Let A be a bounded subset of a metric space (X, d). A *Chebyshev center* (or *center*) of A is the center of the minimal closed ball containing A i.e., it is an element $x_A \in X$ for which $\sup_{y\in A} d(x_A, y) = \inf_{x\in X} \sup_{y\in A} d(x, y)$. The number $r(A) \equiv \inf_{x\in X} \sup_{y\in A} d(x, y)$, is called the *Chebyshev radius* of A. This $r(A)$ is

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the radius of the smallest ball in X (if one exists) which contains the set A. The Chebyshev radius $r(A)$ of A always satisfies $r(A) \geqslant \frac{1}{2} \operatorname{diam} A$. The centers of all such balls are just the centers of A. We denote the collection of such centers by $Z(A)$ i.e., $Z(A) = \{x_A \in X : \sup_{y \in A} d(x_A, y) = r(A)\}$ is the set of centers of balls of minimal radius covering A. It is easy to see that $Z(A)$ is a bounded closed subset of X, $r(A) > 0$ if and only if A contains at least two points, $r(A) = r(\overline{A})$ and $Z(A) = Z(\overline{A})$.

Example 1.1. The Chebyshev radius of the line segment $[a, b]$ in the real line (with usual metric) is $\frac{1}{2}(b-a)$ and its Chebyshev center is $\frac{1}{2}(a+b)$.

Example 1.2. [**3**] In the Euclidian space, the Chebyshev radius of a circle or sphere is its ordinary radius and its Chebyshev center is the ordinary center. The center of an acute triangle is the center of its circumscribing circle.

Example 1.3. [**3**] In the space $(\mathbb{R}^2, \|.\|_\infty)$, the Centers of the set $\{(x,0) : |x| \leqslant 1\}$ is the set $\{(0,y) : |y| \leqslant 1\}$.

A sequence $\{y_n\}$ in a subset A of X is called a *maximizing sequence* for $x \in X$ if $\lim d(x, y_n) = \sup\{d(x,y) : y \in A\}$. The set A is said to be *nearly compact* [**6**] if every maximizing sequence in A for any $x \in X$ has a convergent subsequence in A.

Clearly, every compact set in a metric space is nearly compact but converse is not true [**6**].

The set A is said to be *remotal* if for each $x \in X$ there exists an element $y_0 \in A$ farthest from x i.e., $d(x, y_0) = \sup\{d(x,y) : y \in A\}$.

A nearly compact set (compact set) in a metric space is remotal.

Let (X, d) be a metric space and $I = [0,1]$ be the closed unit interval. A continuous mapping $W : X \times X \times I \to X$ is said to be a *convex structure* on X [**14**] if for all $x, y \in X, \lambda \in I,\ d(W(x,y,\lambda), u) \leqslant \lambda d(x,u) + (1-\lambda) d(y,u)$ for all $u \in X$. The metric space together with a convex structure is called a *convex metric space* [**14**] and is denoted by (X, d, W).

A convex metric space (X, d, W) is said to be *strictly convex* [**7**] (see also [**1**]) if for every $x, y \in X$ and $r > 0,\ d(x,p) \leqslant r,\ d(y,p) \leqslant r$ imply $d(W(x,y,\lambda), p) < r$ unless $x = y$, where p is arbitrary but fixed point of X.

A convex metric space (X, d, W) is said to be *uniformly convex* [**13**] (see also [**1, 12**]) if for any $\varepsilon > 0$ there exists $\delta > 0$ such that for all $r > 0$ and $x, y, p \in X$ with $d(x,p) \leqslant r,\ d(y,p) \leqslant r$ and $d(W(x,y,\frac{1}{2}), p) > r - \delta$, we have $d(x,y) < \varepsilon$.

A subset A of a convex metric space (X, d, W) is said to be *convex* [**14**] if for every $x, y \in K$ and $0 \leqslant \lambda \leqslant 1,\ W(x,y,\lambda) \in K$.

2. Main Results

Regarding the existence and uniqueness of Chebyshev centers, the following result of Smith [**5**] is well known.

If X is a strictly convex Banach space, then every compact set in X has at most one center in X. If X is an E-space, then each compact convex set in X has a unique center.

It is not known whether the E-propery also suffices for the uniqueness of Chebyshev centers for arbitrary bounded subsets. However, it is known (see [**5**]) that every bounded subset of a uniformly convex Banach space has a unique center.

The following generalization of the result of Smith was proved by the author in [**9**]:

If X is a reflexive strictly convex dual Banach space, then every convex remotal set A in X has a unique center.

In this paper, we further generalize and extend the result of Smith to strictly convex metric spaces and also show that for bounded subsets of uniformly convex metric spaces, Chebyshev center is also unique. For this, we start with proving few lemmas.

LEMMA 2.1. *If A is a bounded subset of a convex metric space (X, d, W), then $Z(A)$ is a convex subset of X.*

PROOF. Suppose $x, y \in Z(A)$ and $0 \leqslant \lambda \leqslant 1$. Then $\sup_{z\in A} d(x, z) = r(A) = \sup_{z\in A} d(y, z)$. Consider

$$\begin{aligned}\sup_{z\in A} d(W(x, y, \lambda), z) &\leqslant \sup_{z\in A}[\lambda d(x, z) + (1-\lambda)d(y, z)]\\ &\leqslant \lambda \sup_{z\in A} d(x, z) + (1-\lambda)\sup_{z\in A} d(y, z) \quad \text{as } 0 \leqslant \lambda \leqslant 1\\ &\leqslant \lambda r(A) + (1-\lambda)r(A) = r(A) = \inf_{u\in X}\sup_{z\in A} d(u, z)\\ &\leqslant \sup_{z\in A} d(W(x, y, \lambda), z).\end{aligned}$$

This gives $\sup_{z\in A} d(W(x, y, \lambda), z) = r(A)$ and so $W(x, y, \lambda) \in Z(A)$. Hence $Z(A)$ is convex. □

LEMMA 2.2. *Let A be a bounded subset of a metric space (X, d). Then the mapping $f_A : X \to \mathbb{R}$ defined by $f_A(x) = \sup_{z\in A} d(x, z)$ is (Lipschitz) continuous on X.*

PROOF. Let $x, y \in X$. For any $z \in A$, the inequality $d(x, z) \leqslant d(x, y) + d(y, z)$ gives

$$\sup_{z\in A} d(x, z) \leqslant d(x, y) + \sup_{z\in A} d(y, z)$$

i.e., $f_A(x) \leqslant d(x, y) + f_A(y)$. This implies $|f_A(x) - f_A(y)| \leqslant d(x, y)$ for all $x, y \in X$ and the result follows. □

LEMMA 2.3. *Let A be a bounded subset of a metric space (X, d) such that the continuous map $f_A : X \to \mathbb{R}$ attains its infimum at some point of X; then $Z(A) \neq \phi$.*

PROOF. Let $x^* \in X$ be such that $f_A(x^*) = \inf_{x\in X} f_A(x)$ i.e., $\sup_{y\in A} d(y, x^*) = \inf_{x\in X} \sup_{y\in A} d(y, x)$. Therefore, $x^* \in Z(A)$. □

The following theorem deals with the uniqueness of Chebyshev centers in strictly convex metric spaces.

Theorem 2.1. *Let A be a remotal subset of a strictly convex metric space (X, d, W). Then $Z(A)$ is at most a singleton.*

Proof. Suppose $x, y \in Z(A)$, $x \neq y$. Then $W(x, y, \frac{1}{2}) \in Z(A)$ by Lemma 2.1 i.e., $\sup_{z \in A} d(W(x, y, \frac{1}{2}), z) = r(A)$. Since A is a remotal subset of X, there exists some $u \in A$ such that

$$d(W(x, y, \tfrac{1}{2}), u) = r(A). \tag{2.1}$$

Now $x \in Z(A) \Rightarrow \sup_{z \in A} d(x, z) = r(A) \equiv \inf_{v \in X} \sup_{z \in A} d(v, z)$. This gives $d(x, u) \leqslant r(A)$. Similarly, $y \in Z(A)$ will give $d(y, u) \leqslant r(A)$. Since X is strictly convex, $d(W(x, y, \frac{1}{2}), u) < r(A)$ unless $x = y$. This contradicts (2.1) and hence $Z(A)$ is a singleton. □

If there exists $x^* \in X$ such that $f_A(x^*) = \inf_{x \in X} f_A(x)$, then $x^* \in X$ is a Chebyshev center of A. So, the problem of existence of Chebyshev centers reduces to the problem of minimizing f_A on X.

Therefore using Lemma 2.3 and Theorem 2.1, we obtain

Theorem 2.2. *Let A be a remotal subset of a strictly convex metric space (X, d, W) such that $f_A : X \to \mathbb{R}$ attains its infimum on X; then $Z(A)$ is exactly a singleton.*

Since a compact (nearly compact) subset of a metric space is remotal, we obtain

Corollary 2.1. *Let A be a a compact (nearly compact) subset of a strictly convex metric space (X, d, W) such that $f_A : X \to \mathbb{R}$ attains its infimum on X; then $Z(A)$ is exactly a singleton.*

Since a continuous mapping defined on a compact space always attains its infimum, using Lemma 2.2, we obtain

Corollary 2.2. *Let A be a remotal subset of a strictly convex compact metric space (X, d, W); then $Z(A)$ is exactly a singleton.*

The following theorem deals with uniqueness of Chebyshev centers for bounded subsets of metric spaces. For Banach spaces this result is given in [**5**].

Theorem 2.3. *Let A be a bounded subset of a uniformly convex metric space (X, d, W); then $Z(A)$ is at most a singleton.*

Proof. Suppose $x, y \in Z(A)$, $x \neq y$ i.e., $\sup_{z \in A} d(x, z) = \sup_{z \in A} d(y, z) = r(A)$. Then by Lemma 2.1, $W(x, y, \frac{1}{2}) \in Z(A)$ i.e.,

$$\sup_{z \in A} d(W(x, y, \tfrac{1}{2}, z) = r(A) \equiv \inf_{u \in X} \sup_{z \in A} d(u, z).$$

So, there exists a sequence $\{z_n\}$ in A such that

$$d(W(x, y, \tfrac{1}{2}, z_n)) \to r(A). \tag{2.2}$$

Also, $d(x, z_n) \leqslant r(A)$, $d(y, z_n) \leqslant r(A)$ for all n. Now (2.2) implies that for every $\delta > 0$ there exists a positive integer m such that $d(W(x, y, \frac{1}{2}, z_n)) > r(A) - \delta$ for all $n \geqslant m$.

Let $d(x, y) = \varepsilon > 0$. Since (X, d, W) is uniformly convex, there exists a $\delta > 0$ such that $d(x, z_m) \leqslant r(A)$, $d(y, z_m) \leqslant r(A)$ and $d(W(x, y, \frac{1}{2}), z_m) > r(A) - \delta$ imply $d(x, y) < \varepsilon$, a contradiction and hence $x = y$. $\square$

Using Lemma 2.3, we obtain:

THEOREM 2.4. *Let A be a bounded subset of a uniformly convex metric space (X, d, W) such that the map $f_A : X \to \mathbb{R}$ attains its infimum on X; then $Z(A)$ is exactly a singleton.*

REMARK. Let A be a bounded subset of a metric space (X, d) and $\varepsilon > 0$. An *ε-Chebyshev center* (or *ε-center*) of A is an element $x_A \in X$ such that

$$\sup_{y \in A} d(x_A, y) \leqslant \inf_{x \in X} \sup_{y \in A} d(x, y) + \varepsilon$$

It will be interesting to study ε-Chebyshev centers in different abstract spaces.

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