PUBLICATIONS DE L'INSTITUT MATHÉMATIQUE
Nouvelle série, tome 106(120) (2019), 39–46 DOI: https://doi.org/10.2298/PIM1920039N

A GENERALIZATION OF THE ZERO-DIVISOR GRAPH FOR MODULES

Katayoun Nozari and Shiroyeh Payrovi

ABSTRACT. Let R be a commutative ring and M a Noetherian R-module. The zero-divisor graph of M, denoted by $\Gamma(M)$, is an undirected simple graph whose vertices are the elements of $Z_R(M) \setminus \mathrm{Ann}_R(M)$ and two distinct vertices a and b are adjacent if and only if $abM = 0$. In this paper, we study diameter and girth of $\Gamma(M)$. We show that the zero-divisor graph of M has a universal vertex in $Z_R(M) \setminus r(\mathrm{Ann}_R(M))$ if and only if $R = \oplus\mathbb{Z}_2 \oplus R'$ and $M = \mathbb{Z}_2 \oplus M'$, where M' is an R'-module. Moreover, we show that if $\Gamma(M)$ is a complete graph, then one of the following statements is true:

(i) $\mathrm{Ass}_R(M) = \{\mathfrak{m}_1, \mathfrak{m}_2\}$, where $\mathfrak{m}_1, \mathfrak{m}_2$ are maximal ideals of R.
(ii) $\mathrm{Ass}_R(M) = \{\mathfrak{p}\}$, where $\mathfrak{p}^2 \subseteq \mathrm{Ann}_R(M)$.
(iii) $\mathrm{Ass}_R(M) = \{\mathfrak{p}\}$, where $\mathfrak{p}^3 \subseteq \mathrm{Ann}_R(M)$.

1. Introduction

Let R be a commutative ring. The concept of the zero-divisor graph of a commutative ring introduced in [**6**] and studied in [**2**], they associate a graph, $\Gamma(R)$, to the ring R with vertex set $Z^*(R) := Z(R) \setminus \{0\}$, the set of nonzero zero divisors of R, and two distinct vertices a and b are adjacent if and only if $ab = 0$. The zero-divisor graphs of commutative rings have been studied by many authors, see for examples [**1**, **3**, **4**]. There are many papers on assigning graphs to rings, see [**5**, **9**, **11**]. The concepts of zero-divisor elements and zero divisors graph of a ring, have been generalized to a module in many papers, see for example [**7**, **10**].

Let R be a commutative ring with identity and M an R-module. In this paper, we define a zero divisor graph for M, denoted by $\Gamma(M)$, which is an undirected simple graph whose vertices are the elements of $Z_R(M) \setminus \mathrm{Ann}_R(M)$, and two distinct vertices $a, b \in Z_R(M) \setminus \mathrm{Ann}_R(M)$ are adjacent if and only if $abM = 0$. It is clear that when $M = R$, $\Gamma(M)$ is exactly the classic zero divisor graph of R. Let M be a Noetherian R-module. In Section 2, it is shown that $c \in Z_R(M) \setminus \mathrm{Ann}_R(M)$ is a universal vertex of $\Gamma(M)$ if and only if either $Z_R(M) = \mathrm{Ann}_R(cM) \cup \{c\}$ or $Z_R(M) = \mathrm{Ann}_R(cM)$. Furthermore, it is proved that $\Gamma(M)$ is a complete bipartite

2010 *Mathematics Subject Classification:* Primary 05C25; Secondary 13C99.
Key words and phrases: zero divisor, diameter, girth, complete bipartite graph.
Communicated by Gradimir Milovanović.

graph if and only if $|\operatorname{Ass}_R(M)| = 2$, whenever $r(\operatorname{Ann}_R(M)) = \operatorname{Ann}_R(M)$. In Section 3, we study connectivity, diameter and the girth of $\Gamma(M)$. Moreover, we show that for a Noetherian R-module M, the following statements are true:

(i) If $r(\operatorname{Ann}_R(M)) = \operatorname{Ann}_R(M)$ and there exists $b \in Z_R(M) \setminus \operatorname{Ann}_R(M)$ such that $\operatorname{Ann}_M(b) \subseteq \bigcap_{P \in m-\operatorname{Ass}(M)} P$, then $\Gamma(M)$ is a disconnected graph.

(ii) If $\Gamma(M)$ is a disconnected graph, then there exists $b \in Z_R(M) \setminus \operatorname{Ann}_R(M)$ such that $\operatorname{Ann}_M(b) \subseteq \bigcap_{P \in m-\operatorname{Ass}(M)} P$.

Let G be a graph with the vertex set $V(G)$ and the edge set $E(G)$. For each pair of vertices $u, v \in V(G)$, if u is adjacent to v, then we write $u - v$. A graph with no edge is called null graph. Recall that G is connected if there is a path between any two distinct vertices of G. For vertices x and y of G, let $d(x, y)$ be the length of a shortest path from x to y ($d(x, x) = 0$ and $d(x, y) = \infty$ if there is no such path). The diameter of G is $\operatorname{diam}(G) = \sup\{d(x, y) \mid x, y \text{ are vertices of } G\}$. The girth of G, denoted by $\operatorname{gr}(G)$, is the length of the shortest cycle in G ($\operatorname{gr}(G) = \infty$ if G contains no cycles). A graph G is complete if any two distinct vertices are adjacent. The complete graph with n vertices will be denoted by K_n. A complete bipartite graph is a graph G which may be partitioned into two disjoint nonempty vertex sets A and B such that two distinct vertices are adjacent if and only if they are in distinct vertex sets. If one of the vertex set is a singleton, then we call G a star graph. A clique of G is a complete subgraph of G and the number of vertices in the largest clique of G, denoted by $\omega(G)$, is called the clique number.

Throughout, R denotes a commutative ring with nonzero identity and M is a unitary R-module, $Z_R(M)$ its set of zero-divisors. Let $\operatorname{Ass}_R(M) = \{\mathfrak{p} \in \operatorname{Spec}(R) : \mathfrak{p} = \operatorname{Ann}_R(m) \text{ for some } 0 \neq m \in M\}$ denote the set of associated primes of M. Let $\operatorname{Spec}_R(M)$ denote the set of prime submodules of M and $m - \operatorname{Ass}_R(M) = \{P \in \operatorname{Spec}_R(M) : P = \operatorname{Ann}_M(a) \text{ for some } 0 \neq a \in R\}$, where $\operatorname{Ann}_M(a) = \{m \in M : am = 0\}$, for all $a \in R$. For notations and terminologies not given in this article, the reader is referred to [**12**].

2. Properties of zero-divisor graph of a module

In this section we define an undirected graph $\Gamma(M)$ and study the relations between module theoretic properties of M and graph theoretic properties of $\Gamma(M)$.

Definition 2.1. Let M be an R-module. The zero-divisor graph of M is the undirected graph $\Gamma(M)$ with vertices $Z_R(M) \setminus \operatorname{Ann}_R(M)$ and two distinct vertices x, y are adjacent if and only if $xyM = 0$.

Lemma 2.1. *Let M be an R-module. Then $\Gamma(M)$ is a null graph if and only if $\operatorname{Ann}_R(M)$ is a prime ideal of R.*

Proof. Let $\Gamma(M)$ be a null graph and $x, y \in R$ such that $xyM = 0$ and $xM \neq 0$. So $y \in Z_R(M)$. If $y \notin \operatorname{Ann}_R(M)$, then $x, y \in Z_R(M) \setminus \operatorname{Ann}_R(M)$ and $xyM = 0$ which is a contradiction. Thus $y \in \operatorname{Ann}_R(M)$. So $\operatorname{Ann}_R(M)$ is a prime ideal of R. Now let $\operatorname{Ann}_R(M)$ be a prime ideal of R. If $\Gamma(M)$ is not a null graph, then there exist $x, y \in Z_R(M) \setminus \operatorname{Ann}_R(M)$ such that $xyM = 0$. So $x \in \operatorname{Ann}_R(M)$ or $y \in \operatorname{Ann}_R(M)$ which contracts to definition of vertices. ☐

Lemma 2.2. *Let R be a nontrivial commutative ring. Then R is field if and only if $\Gamma(M)$ is a null graph, for each R-module M.*

Proof. Suppose that $\Gamma(M)$ is a null graph, for each R-module M. Let $\mathfrak{m}$ be a nonzero maximal ideal of R and $0 \neq a \in \mathfrak{m}$. Set $M := R/\mathfrak{m} \oplus R$. So we have $(0,a)(1+\mathfrak{m},0)M = 0$. Hence, $(0,a)$ and $(1+\mathfrak{m},0)$ are adjacent which is a contradiction. So $\mathfrak{m} = 0$ and R is a field. The converse is true by Lemma 2.1. □

Example 2.1. Consider $M = \mathbb{Z}_{p^\infty} = \{a/p^n + \mathbb{Z} : a \in \mathbb{Z}, n \geqslant 1\}$ as a $\mathbb{Z}$-module. Then $\{p, p^2, p^3, \dots\} \subseteq Z(M) \smallsetminus \mathrm{Ann}_{\mathbb{Z}}(M)$ and $\Gamma(M)$ is a null graph since $\mathrm{Ann}_{\mathbb{Z}}(M) = 0$ is a prime ideal of $\mathbb{Z}$.

Theorem 2.1. *Let $c \in Z_R(M) \smallsetminus \mathrm{Ann}_R(M)$. Then c is a universal vertex of $\Gamma(M)$ if and only if either $Z_R(M) = \mathrm{Ann}_R(cM) \cup \{c\}$ or $Z_R(M) = \mathrm{Ann}_R(cM)$.*

Proof. Let c be a universal vertex of $\Gamma(M)$. Then $c(Z_R(M) \smallsetminus \{c\}) \subseteq \mathrm{Ann}_R(M)$. So $Z_R(M) \smallsetminus \{c\} \subseteq \mathrm{Ann}_R(M) :_R c = \mathrm{Ann}_R(cM)$. If $c \notin r(\mathrm{Ann}_R(M))$, then $c = c^n$ for all $n \geqslant 1$ otherwise $c^{n+1}M = 0$ which contradicts the assumption. Thus $c^2M \neq 0$ and $\mathrm{Ann}_R(cM) \subseteq Z_R(M) \smallsetminus \{c\}$. Hence $\mathrm{Ann}_R(cM) = Z_R(M) \smallsetminus \{c\}$ and so $Z_R(M) = \mathrm{Ann}_R(cM) \cup \{c\}$. Now let $c \in r(\mathrm{Ann}_R(M))$; then $c \neq c^2$, otherwise $c = c^n$ for all $n \geqslant 1$ and so $c \in \mathrm{Ann}_R(M)$ which is a contradiction. Thus $c^3M = 0$. If $c^2M \neq 0$, then $Z_R(M) = \mathrm{Ann}_R(cM) \cup \{c\}$ otherwise $Z_R(M) = \mathrm{Ann}_R(cM)$. The reverse is obvious. □

Theorem 2.2. *Let $c \in Z_R(M) \smallsetminus r(\mathrm{Ann}_R(M))$ be a universal vertex of $\Gamma(M)$. Then $R = \mathbb{Z}_2 \oplus R'$ and $M = \oplus\mathbb{Z}_2 \oplus M'$, where R' is a subring of R, M' is an R-submodule of M and $Z_R(M) = \{(1,0)\} \cup (\{0\} \oplus R')$.*

Proof. Suppose that $c \in Z_R(M) \smallsetminus r(\mathrm{Ann}_R(M))$ is a universal vertex of $\Gamma(M)$. By assumption $c^2 = c$. So $R = cR \oplus (1-c)R$ and $M = cM \oplus (1-c)M$. Let $R_1 = cR$ and $R' = (1-c)R$. Then R_1, R' are subrings of R. In addition, $M_1 = cM$ is an R_1-module and $M' = (1-c)M$ is an R'-module since M_1, M' are R-submodules of M. Moreover, if $r = (r_1, r')$ and $m = (m_1, m')$, then $rm = (cr + (1-c)r)(cm + (1-c)m) = c^2rm + (1-c)^2rm = r_1m_1 + r'm' = (r_1m_1, r'm')$. With regard to above decomposition, we have $c = (1,0)$. So $(1,0)$ is a universal vertex.

Let $0 \neq b \in Z_{R'}(M')$. Then there exists $0 \neq m' \in M'$ such that $bm' = 0$. So $(1,b)(0,m') = (0,0)$ but $(1,b)(M_1 \oplus M') = M_1 \oplus bM' \neq 0$ which means that $(1,b) \in Z_R(M_1 \oplus M') \smallsetminus \mathrm{Ann}_R(M_1 \oplus M')$. Finally by assumption $(1,0)(1,b)M_1 \oplus M' = M_1 \oplus 0 = 0$ which is a contradiction. So $Z_{R'}(M') = 0$. Also, if $0 \neq a \in Z_{R_1}(M_1)$, then there exists $0 \neq m_1 \in M_1$ such that $am_1 = 0$. Thus $(a,1)(m_1,0) = (0,0)$ and so $(a,1) \in Z_R(M_1 \oplus M') \smallsetminus \mathrm{Ann}_R(M_1 \oplus M')$. Since $(1,0)$ is a universal vertex, $(1,0)(a,1)(M_1 \oplus M') = aM_1 \oplus 0 = 0$. Thus $a \in \mathrm{Ann}_{R_1}(M_1)$ which means that $Z_{R_1}(M_1) \subseteq \mathrm{Ann}_{R_1}(M_1)$. So $Z_{R_1}(M_1) = \mathrm{Ann}_{R_1}(M_1)$. If $1 \neq a \in R_1 \smallsetminus \mathrm{Ann}_{R_1}(M_1)$, then $(a,0) \in Z_R(M_1 \oplus M') \smallsetminus \mathrm{Ann}_R(M_1 \oplus M')$. Thus $(1,0)(a,0)(M_1 \oplus M') = 0$ which is a contradiction. Hence $R_1 \smallsetminus \{1\} \subseteq \mathrm{Ann}_{R_1}(M_1)$ and so $R_1 \smallsetminus \{1\} = \mathrm{Ann}_{R_1}(M_1)$. In the following, we show that R_1 has characteristic 2. Assume that $(1,0) \neq (-1,0)$. Thus $(1,0)(-1,0)(M_1 \oplus M') = -M_1 = 0$ which is a contradiction. Hence $1 = -1 \in R_1$ and so R_1 has characteristic 2. If $|R_1| \geqslant 4$, then $R_1 = \{0, 1, a, 1+a, \cdots\}$. Thus

$aM_1 = (1+a)M_1 = 0$ so $M_1 = 0$ that is a contradiction. Hence $R_1 \cong \mathbb{Z}_2$ and $M_1 \cong \oplus\mathbb{Z}_2$ by [**12**, Theorem 10.8]. Also, $R_1 \setminus \{1\} = \mathbb{Z}_2 \setminus \{1\} = \mathrm{Ann}_{R_1}(M_1) = \{0\}$. Hence, $Z_R(M_1 \oplus M') = \{(1,0)\} \cup (\{0\} \oplus R')$. □

COROLLARY 2.1. *Let M be an R-module. Then the graph $\Gamma(M)$ has a universal vertex in $Z_R(M) \setminus r(\mathrm{Ann}_R(M))$ if and only if $R = \mathbb{Z}_2 \oplus R'$ and $M = \oplus\mathbb{Z}_2 \oplus M'$, where M' is an R'-module and $Z_R(M) = \{(1,0)\} \cup (\{0\} \oplus R')$.*

COROLLARY 2.2. *Let $r(\mathrm{Ann}_R(M)) = 0$. Then $\Gamma(M)$ is a complete graph if and only if $\Gamma(M) = K_2$.*

PROOF. By Corollary 2.1, $(1,0)$ and $(0,1)$ are universal vertices. So $R = \mathbb{Z}_2 \oplus \mathbb{Z}_2$. □

THEOREM 2.3. *Let M be a Noetherian R-module and $\Gamma(M)$ a complete graph. Then one of the following statements is true:*

(i) $\mathrm{Ass}_R(M) = \{\mathfrak{m}_1, \mathfrak{m}_2\}$ *where* $\mathfrak{m}_1, \mathfrak{m}_2$ *are maximal ideals of* R.
(ii) $\mathrm{Ass}_R(M) = \{\mathfrak{p}\}$ *where* $\mathfrak{p}^2 \subseteq \mathrm{Ann}_R(M)$.
(iii) $\mathrm{Ass}_R(M) = \{\mathfrak{p}\}$ *where* $\mathfrak{p}^3 \subseteq \mathrm{Ann}_R(M)$.

PROOF. (i) If there exists $c \in Z_R(M) \setminus \mathrm{Ann}_R(M)$ such that $c = c^2$, then by the proof of Theorem 2.2 it follows that $R = \mathbb{Z}_2 \oplus \mathbb{Z}_2$ and $M = M_1 \oplus M_2$ where $M_1 = \oplus\mathbb{Z}_2$ and $M_2 = \oplus\mathbb{Z}_2$ are $\mathbb{Z}_2$-modules. In this case $\mathrm{Ass}_R(M) = \{\mathfrak{m}_1 = \mathbb{Z}_2 \oplus 0, \mathfrak{m}_2 = 0 \oplus \mathbb{Z}_2\}$.

(ii) Suppose that $c \neq c^2$ for all $c \in Z_R(M) \setminus \mathrm{Ann}_R(M)$ and a, b are two distinct elements of $Z_R(M) \setminus \mathrm{Ann}_R(M)$. Since $\Gamma(M)$ is a complete graph, $abM = 0$. So $ab \in \mathrm{Ann}_R(M)$ and $\{ab : a, b \text{ are distinct elements of } Z_R(M) \setminus \mathrm{Ann}_R(M)\} \subseteq \mathrm{Ann}_R(M)$. If for all $c \in Z_R(M) \setminus \mathrm{Ann}_R(M)$, $c^2M = 0$, then $\{ab : a, b \in Z_R(M)\} \subseteq \mathrm{Ann}_R(M)$. Let $\mathfrak{p}, \mathfrak{p}_1 \in \mathrm{Ass}_R(M)$. Thus $\mathfrak{p}^2 \subseteq \mathfrak{p}_1$ and $\mathfrak{p}_1^2 \subseteq \mathfrak{p}$. So $\mathfrak{p} = \mathfrak{p}_1$. Hence, $\mathrm{Ass}_R(M) = \{\mathfrak{p}\}$ and $Z_R(M) = r(\mathrm{Ann}_R(M)) = \mathfrak{p}$ which implies that $\mathfrak{p}^2 \subseteq \mathrm{Ann}_R(M)$.

(iii) Now assume that there exists $c \in Z_R(M) \setminus \mathrm{Ann}_R(M)$ such that $c^2M \neq 0$. In this case, we have $c^3M = 0$. Thus $\{abc : a, b, c \in Z_R(M)\} \subseteq \mathrm{Ann}_R(M)$. Let $\mathfrak{p}, \mathfrak{p}_1 \in \mathrm{Ass}_R(M)$. Then $\mathfrak{p}^3 \subseteq \mathfrak{p}_1$ and $\mathfrak{p}_1^3 \subseteq \mathfrak{p}$. So $\mathfrak{p} = \mathfrak{p}_1$. Hence, $\mathrm{Ass}_R(M) = \{\mathfrak{p}\}$ and $Z_R(M) = r(\mathrm{Ann}_R(M)) = \mathfrak{p}$ which implies that $\mathfrak{p}^3 \subseteq \mathrm{Ann}_R(M)$. □

COROLLARY 2.3. *Let M be a Noetherian R-module, $\mathrm{Ass}_R(M) = \{\mathfrak{p}\}$ and $\mathfrak{p}^2 \subseteq \mathrm{Ann}_R(M)$. Then $\Gamma(M)$ is a complete graph. In particular, if R is a Noetherian ring, then $\Gamma(R/\mathfrak{p}^2)$ is a complete graph, where $\mathfrak{p}$ is a prime ideal of R.*

PROOF. Let $a \in Z_R(M) \setminus \mathrm{Ann}_R(M)$. Then for each $b \in Z_R(M) \setminus \mathrm{Ann}_R(M)$, we have $ab \in \mathfrak{p}^2 \subseteq \mathrm{Ann}_R(M)$ since $\mathrm{Ass}_R(M) = \{\mathfrak{p}\}$ and $Z_R(M) = \mathfrak{p}$. So $abM = 0$. For the second part it is obvious that $\mathrm{Ass}_R(R/\mathfrak{p}^2) = \{\mathfrak{p}\}$ and $\mathfrak{p}^2 \subseteq \mathrm{Ann}_R(R/\mathfrak{p}^2)$. According to the above, the proof is completed. □

EXAMPLE 2.2. Let p be a prime number and $M = \mathbb{Z}/p^3\mathbb{Z}$. Then $\mathrm{Ass}_{\mathbb{Z}}(M) = \{p\mathbb{Z}\}$ and $p^3\mathbb{Z} \subseteq \mathrm{Ann}_R(M)$ but $\Gamma(M)$ is not a complete graph.

COROLLARY 2.4. *Let R be a local ring and $\Gamma(M)$ be a complete graph. Then $\mathrm{Ass}_R(M) = \{\mathfrak{p}\}$ where either $\mathfrak{p}^2 \subseteq \mathrm{Ann}_R(M)$ or $\mathfrak{p}^3 \subseteq \mathrm{Ann}_R(M)$.*

PROOF. In this case for all $a \in Z_R(M) \setminus \mathrm{Ann}_R(M)$, $a \neq a^2$. So the result follows by the proof of Theorem 2.3. □

THEOREM 2.4. *Let M be a Noetherian R-module and $r(\mathrm{Ann}_R(M)) = \mathrm{Ann}_R(M)$. Then $\Gamma(M)$ is a complete bipartite graph if and only if $|\mathrm{Ass}_R(M)| = 2$.*

PROOF. Suppose that $I = \mathrm{Ann}_R(M)$. Then $r(\mathrm{Ann}_R(M))/I = r(\mathrm{Ann}_{R/I}(M))$ since M is an R/I-module. Thus by hypothesis $r(\mathrm{Ann}_{R/I}(M)) = 0$. Moreover, for each $a \in R$, we have $a \in Z_R(M) \setminus \mathrm{Ann}_R(M)$ if and only if $0 \neq a + I \in Z_{R/I}(M)$ and $\mathfrak{p} \in \mathrm{Ass}_R(M)$ if and only if $\mathfrak{p}/I \in \mathrm{Ass}_{R/I}(M)$ since $\mathfrak{p} = \mathrm{Ann}_R(m)$ if and only if $p/I = \mathrm{Ann}_R(m)/I = \mathrm{Ann}_{R/I}(m)$ for all $0 \neq m \in M$. So we can and do assume $r(\mathrm{Ann}_R(M)) = 0$ and we have to show that $\Gamma(M)$ is a complete bipartite graph if and only if $|\mathrm{Ass}_R(M)| = 2$.

($\Rightarrow$) Let $\Gamma(M)$ be a complete bipartite graph and V_1,V_2 be two distinct sets of vertices. We prove that $V_i \cup \{0\}$, for $i = 1, 2$, is a prime ideal of R. To show this let $a, b \in \bar{V_1} = V_1 \cup \{0\}$. If either $a = 0 = b$ or $a \in V_1$ and $b = 0$, then $a + b \in \bar{V_1}$. Now, suppose that $a, b \in V_1$. Thus there exist $x, y \in V_2$ such that $xaM = 0 = ybM$. Hence, $(a + b)xyM = 0$ and $xyM \neq 0$ implies $a + b \in Z_R(M)$. If $a + b \in \bar{V_1}$ we are done. Otherwise $a + b \in V_2$ where $a(a + b)M = 0$ and $b(a + b)M = 0$. So $(a + b)^2M = 0$ which means that $a + b \in r(\mathrm{Ann}_R(M)) = 0$ and this is a contradiction. So $a + b \in \bar{V_1}$. Let $a, b \in R$ and $ab \in \bar{V_1}$. We show that $a \in \bar{V_1}$ or $b \in \bar{V_1}$. If $a = 0$ or $b = 0$, then the proof is completed. So let $0 \neq a, b \in R$. If $abM = 0$, then by hypothesis $a \in V_1$ or $b \in V_1$. Now, suppose that $abM \neq 0$ and $a, b \in V_2$. Thus $a^2bM = 0$ and so $(ab)^2M = 0$ which implies that $ab \in r(\mathrm{Ann}_R(M)) = 0$ and this is a contradiction.

($\Leftarrow$) Let $\mathrm{Ass}_R(M) = \{\mathfrak{p}_1, \mathfrak{p}_2\}$. Then $r(\mathrm{Ann}_R(M)) = \mathfrak{p}_1 \cap \mathfrak{p}_2 = 0$. Assume that $a, b \in \mathfrak{p}_1 \setminus \{0\}$ and $abM = 0$. Thus $ab \in \mathfrak{p}_2$ and so either $a \in \mathfrak{p}_2$ or $b \in \mathfrak{p}_2$ which implies that $a = 0$ or $b = 0$ and this is a contradiction. Hence, two elements of $\mathfrak{p}_1 \setminus \{0\}$ are not adjacent. By a similar argument, we can show that the elements of $\mathfrak{p}_2 \setminus \{0\}$ are not adjacent. Let $a \in \mathfrak{p}_1^* = \mathfrak{p}_1 \setminus \{0\}$ and $b \in \mathfrak{p}_2^* = \mathfrak{p}_2 \setminus \{0\}$. Then $ab \in \mathfrak{p}_1\mathfrak{p}_2 \subseteq \mathfrak{p}_1 \cap \mathfrak{p}_2 = 0$. So $abM = 0$ which means that two elements of $\mathfrak{p}_1^*$ and $\mathfrak{p}_2^*$ are adjacent. Hence, $\Gamma(M)$ is a complete bipartite graph. □

THEOREM 2.5. *Let M be a Noetherian R-module and $r(\mathrm{Ann}_R(M)) = \mathrm{Ann}_R(M)$. If $\Gamma(M)$ is a complete r-partite graph, $r \geqslant 3$, then at most one of the parts has more than one vertex.*

PROOF. Suppose that $V_1, \ldots, V_r$ are distinct parts of $\Gamma(M)$. Let V_t and V_s have more than one vertex. Choose $x \in V_t$ and $y \in V_s$. Let V_l be a part of $\Gamma(M)$ such that $V_l \neq V_t$ and $V_1 \neq V_s$. Let $z \in V_l$. Since $\Gamma(M)$ is a complete r-partite graph, $\mathrm{Ann}_R(xM) = \bigcup_{\substack{1 \leqslant i \leqslant r \\ i \neq t}} V_i \cup \{0\}$, $\mathrm{Ann}_R(yM) = \bigcup_{\substack{1 \leqslant i \leqslant r \\ i \neq s}} V_i \cup \{0\}$ and $\mathrm{Ann}_R(zM) = \bigcup_{\substack{1 \leqslant i \leqslant r \\ i \neq l}} V_i \cup \{0\}$. Hence, $\mathrm{Ann}_R(zM) \subseteq \mathrm{Ann}_R(xM) \cup \mathrm{Ann}_R(yM)$. So $\mathrm{Ann}_R(zM) \subseteq \mathrm{Ann}_R(xM)$ or $\mathrm{Ann}_R(zM) \subseteq \mathrm{Ann}_R(yM)$. Let $\mathrm{Ann}_R(zM) \subseteq \mathrm{Ann}_R(xM)$ and let $x' \in V_t$ be such that $x' \neq x$. Then $x' \in \mathrm{Ann}_R(zM) \setminus \mathrm{Ann}_R(xM)$ this is a contradiction. □

3. Study of the zero-divisor graph of a module by annihilator submodules

In this section we study the relations between the set of annihilator prime submodules of a module and zero-divisor graph of a module. Let M be an R-module. A proper submodule P of M is said to be prime submodule whenever for $r \in R$ and $m \in M$, $rm \in P$ implies $m \in P$ or $r \in \mathrm{Ann}_R(M/P)$. Let $\mathrm{Spec}_R(M)$ denote the set of prime submodules of M and $m - \mathrm{Ass}_R(M) = \{P \in \mathrm{Spec}_R(M) : P = \mathrm{Ann}_M(a)$ for some $0 \neq a \in R\}$, where $\mathrm{Ann}_M(a) = \{m \in M : am = 0\}$, for $a \in R$.

Lemma 3.1. *Let M be an R-module, $a, b, c \in R$ and $\mathrm{Ann}_M(a)$ be a prime submodule of M. Then the following statements are true:*

(i) *If $\mathrm{Ann}_M(b)$ is a prime submodule of M, then $abM = 0$.*
(ii) *If $\mathrm{Ann}_M(b) \nsubseteq \mathrm{Ann}_M(a)$, then $abM = 0$.*
(iii) *If $b \in r(\mathrm{Ann}_R(M)) \setminus \mathrm{Ann}_R(M)$, then $abM = 0$.*
(iv) *If $abM \neq 0$, then $\mathrm{Ann}_M(b)$ is not a prime submodule of M, $\mathrm{Ann}_M(b) \subseteq \mathrm{Ann}_M(a)$ and $b \notin r(\mathrm{Ann}_R(M))$.*
(v) *If $\mathrm{Ann}_M(b) \subseteq \mathrm{Ann}_M(a)$ and $a \notin r(\mathrm{Ann}_R(M))$, then $abM \neq 0$.*
(vi) *If $bcM = 0$, then either $abM = 0$ or $acM = 0$.*

Proof. (i) Let $P_1 = \mathrm{Ann}_M(a)$, $P_2 = \mathrm{Ann}_M(b)$ be two distinct prime submodules of M. Assume that $m \in P_1 \setminus P_2$. Thus $ma = 0 \in P_2$ which implies that $aM \subseteq P_2 = \mathrm{Ann}_M(b)$ since $m \notin P_2$. Hence, $abM = 0$ and so a, b are adjacent in $\Gamma(M)$.

(ii) It follows by a similar argument to that of (i).

(iii) Let $b \in r(\mathrm{Ann}_R(M)) \setminus \mathrm{Ann}_R(M)$. Then there exists $t \in \mathbb{N}$ such that $b^t M = 0$. So $b^t M = 0 \subseteq \mathrm{Ann}_M(a)$ which implies that $bM \subseteq \mathrm{Ann}_M(a)$ and $abM = 0$.

(iv) It is contrapositive of (i), (ii) and (iii).

(v) If $abM = 0$, then $aM \subseteq \mathrm{Ann}_M(b) \subseteq \mathrm{Ann}_M(a)$. So $a \in r(\mathrm{Ann}_R(M))$ which contradicts the assumption. Thus $abM \neq 0$.

(vi) Suppose that $bcM = 0$ and $m \in M \setminus P_1 = \mathrm{Ann}_M(a)$. Then $bcm \in P_1$ which implies that $bc \in \mathrm{Ann}_R(M/P_1)$ such that $\mathrm{Ann}_R(M/P_1)$ is a prime ideal of R. Hence, either $b \in \mathrm{Ann}_R(M/P_1)$ or $c \in \mathrm{Ann}_R(M/P_1)$. So either $abM = 0$ or $acM = 0$. □

Theorem 3.1. *Let M be a Noetherian R-module. Then the following statements are true:*

(i) *If $r(\mathrm{Ann}_R(M)) = \mathrm{Ann}_R(M)$ and there exists $b \in Z_R(M) \setminus \mathrm{Ann}_R(M)$ such that $\mathrm{Ann}_M(b) \subseteq \bigcap_{P \in m-\mathrm{Ass}(M)} P$, then $\Gamma(M)$ is a disconnected graph.*
(ii) *If $\Gamma(M)$ is a disconnected graph, then there exists $b \in Z_R(M) \setminus \mathrm{Ann}_R(M)$ such that $\mathrm{Ann}_M(b) \subseteq \bigcap_{P \in m-\mathrm{Ass}(M)} P$.*

Proof. (i) We show that b is an isolated vertex of $\Gamma(M)$. Let $a \in Z_R(M) \setminus \mathrm{Ann}_R(M)$ and let a be adjacent to b in $\Gamma(M)$. We know that $X = \{\mathrm{Ann}_M(c) : c \notin \mathrm{Ann}_R(M)\}$ has a maximal element and it is easy to see that the maximal element of X is a prime submodule. So there exists $c \in Z_R(M) \setminus \mathrm{Ann}_R(M)$ such that $\mathrm{Ann}_M(a) \subseteq \mathrm{Ann}_M(c)$ and $\mathrm{Ann}_M(c)$ is a prime submodule of M. Now, $abM = 0$ implies $bcM = 0$ which contradicts the Lemma 3.1(v). Hence, b is an isolated vertex of $\Gamma(M)$.

(ii) Let $b \in Z_R(M) \setminus \mathrm{Ann}_R(M)$ be an isolated vertex of $\Gamma(M)$. If $|m - \mathrm{Ass}(M)| = 1$, then by the proof of (i) we have $\mathrm{Ann}_M(b) \subseteq \bigcap_{P \in m-\mathrm{Ass}(M)} P$. If $|m - \mathrm{Ass}(M)| \geqslant 2$, then $\mathrm{Ann}_M(b)$ is not a prime submodule, see Lemma 3.1(i). So $\mathrm{Ann}_M(b) \subseteq \bigcap_{P \in m-\mathrm{Ass}(M)} P$ by Lemma 3.1(ii). □

THEOREM 3.2. *If $\Gamma(M)$ is a connected graph, then* $\mathrm{diam}(\Gamma(M)) \leqslant 3$.

PROOF. Let $x, y \in Z_R(M) \setminus \mathrm{Ann}_R(M)$ be distinct vertices of $\Gamma(M)$. If $xyM = 0$, then $d(x, y) = 1$. Suppose that xyM is nonzero. Then there exist $a, b \in Z_R(M) \setminus \mathrm{Ann}_R(M) \cup \{x, y\}$ with $axM = byM = 0$. If $a = b$, then $x - a - y$ is a path of length 2. Thus we may assume that $a \neq b$. If $abM = 0$, then $x - a - b - y$ is a path of length 3 so $d(x, y) \leqslant 3$. If $abM \neq 0$, then $x - ab - y$ is a path of length 2. Hence $d(x, y) = 2$. Thus $d(x, y) \leqslant 3$ and so $\mathrm{diam}(\Gamma(M)) \leqslant 3$. □

THEOREM 3.3. *Let M be a Noetherian R-module and let $\Gamma(M)$ have a cycle. Then* $\mathrm{gr}(\Gamma(M)) \leqslant 4$.

PROOF. Let $c_1 - c_2 - \cdots - c_7$ be a cycle such that $c_i = \mathrm{Ann}_M(b_i)$ for each $1 \leqslant i \leqslant 7$. If $|m - \mathrm{Ass}_R(M)| \geqslant 3$, then $\mathrm{gr}(\Gamma(M)) \leqslant 3$ by Lemma 3.1(i). Let $|m - \mathrm{Ass}_R(M)| \leqslant 2$ and $P = \mathrm{Ann}_M(a) \in m - \mathrm{Ass}_R(M)$. From $b_1 b_2 M = 0$ it follows that either $b_1 a M = 0$ or $b_2 a M = 0$, by Lemma 3.1(vi). Let $b_1 a M = 0$. If $b_2 a M = 0$, then $\mathrm{gr}(\Gamma(M)) \leqslant 3$. Otherwise $b_2 b_3 M = 0$ implies $ab_3 M = 0$. So $\mathrm{gr}(\Gamma(M)) \leqslant 4$. If a be one of the b_i's, then $\mathrm{gr}(\Gamma(M)) \leqslant 4$. □

THEOREM 3.4. *Let M be a Noetherian R-module and* $r(\mathrm{Ann}_R(M)) = \mathrm{Ann}_R(M)$. *If $\omega(\Gamma(M))$ is finite, then* $\omega(\Gamma(M)) = |m - \mathrm{Ass}(M)|$.

PROOF. Let $I = \mathrm{Ann}_R(M)$. Then it is easy to see that M is an R/I-module and $r(\mathrm{Ann}_R(M))/I = r(\mathrm{Ann}_{R/I}(M))$. Thus by hypothesis $r(\mathrm{Ann}_{R/I}(M)) = 0$. Moreover, for each $a \in R$, we have $\mathrm{Ann}_M(a) = \mathrm{Ann}_M(a + I)$. So we can and do assume that $r(\mathrm{Ann}_R(M)) = 0$. Let $|m - \mathrm{Ass}(M)| = n$. By Lemma 3.1(i), all elements of $m - \mathrm{Ass}(M)$ are adjacent. Now, let $\omega(\Gamma(M)) = k > n$. Then there exist $a, b \in Z_R(M) \setminus \mathrm{Ann}_R(M)$ such that $abM = 0$ and there is $P = \mathrm{Ann}_M(c) \in m - \mathrm{Ass}(M)$ where $\mathrm{Ann}_M(a), \mathrm{Ann}_M(b) \subseteq \mathrm{Ann}_M(c)$. If $abM = 0$ then $bM \subseteq \mathrm{Ann}_M(a) \subseteq \mathrm{Ann}_M(c)$. So $bcM = 0$. Similarly, $acM = 0$. Also, $bcM = 0$ implies $c^2 M = 0$ which is a contradiction. So a, b are not adjacent. Therefore $\omega(\Gamma(M)) = |m - \mathrm{Ass}(M)| = n$. □

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Imam Khomeini International University
Qazvin
Iran
k.nozari@edu.ikiu.ac.ir
shpayrovi@sci.ikiu.ac.ir

(Received 13 12 2018)