PUBLICATIONS DE L'INSTITUT MATHÉMATIQUE
Nouvelle série, tome 106(120) (2019), 29–37 DOI: https://doi.org/10.2298/PIM1920029L

DEPTH OF AN IDEAL ON ZD-MODULES

M. Lotfi Parsa

ABSTRACT. Let R be a Noetherian ring, I an ideal of R and M a ZD-module. Let S be a Serre subcategory of the category of R-modules satisfying the condition C_I, and let I contain a maximal S-sequence on M. We show that all maximal S-sequences on M in I, have the same length. If this common length is denoted by S-depth(I, M), then $S\text{-depth}(I, M) = \inf\{i : \operatorname{Ext}^i_R(R/I, M) \notin S\} = \inf\{i : H^i_I(M) \notin S\}$. Also some properties of this notion are investigated. It is proved that $S\text{-depth}(I, M) = \inf\{\operatorname{depth} M_\mathfrak{p} : \mathfrak{p} \in V(I) \text{ and } R/\mathfrak{p} \notin S\} = \inf\{S\text{-depth}(\mathfrak{p}, M) : \mathfrak{p} \in V(I) \text{ and } R/\mathfrak{p} \notin S\}$ whenever S is a Serre subcategory closed under taking injective hulls, and M is a ZD-module.

1. Introduction

Throughout this paper, R is a commutative Noetherian ring with nonzero identity, I is an ideal of R, M is an R-module and s and t are two integers. For notations and terminologies not given in this paper, the reader is referred to [**5**] and [**6**] if necessary.

The notion of ZD-module was introduced by Evans [**8**]. An R-module M is said to be ZD-module (zero-divisor module) if for any submodule N of M, the set of zero divisors of M/N is a union of finitely many prime ideals in $\operatorname{Ass}_R M/N$. According to [**7**, Example 2.2], the class of ZD-modules is much larger than that of finitely generated modules. Some authors studied these extended modules; for example see [**2**] and [**13**].

Let S be a Serre subcategory of the category of R-modules. Aghapournahr and Melkersson [**1**] defined the notion of S-sequence on M, as a generalization of the regular sequences. An element a of R is called S-regular on M, if $0 :_M a \in S$. A sequence $a_1, \dots, a_t$ is an S-sequence on M, if a_i is S-regular on $M/(a_1, \dots, a_{i-1})M$ for $i = 1, \dots, t$. Depending on which S is chosen, we obtain various sequences studied in the literature such as regular sequences, filter-regular sequences, generalized regular sequences, etc. Let S satisfy the condition C_I. They proved that all maximal S-sequences on M in I have the same length, whenever M is a finitely

2010 *Mathematics Subject Classification:* Primary 13C60; Secondary 13C15.
Key words and phrases: depth, regular sequence, Serre subcategory, ZD-module.
Communicated by Zoran Petrović.

generated R-module and $M/IM \notin S$. This common length is called the S-depth of I on M, denoted by S-depth(I, M).

In Section 2, we generalize the concept of S-depth to ZD-modules. Let S satisfy the condition C_I, M be a ZD-module and I contain a maximal S-sequence on M. In Theorem 2.1, we show that all maximal S-sequences on M in I, have the same length, equal to $\inf\{i : \mathrm{Ext}^i_R(R/I, M) \notin S\}$. Also it is proved in Proposition 2.1 that if $M/IM \notin S$, then I contains maximal S-sequences on M.

In Section 3, we get some formulas and inequalities for S-depth of I on M. Let S satisfy the condition C_I, and M be a ZD-module. In Lemma 3.1, we show that S-depth$(I, M) = \inf\{i : H^i_I(M) \notin S\}$. Also it is proved that S-depth$(I, M) = \inf\{\operatorname{depth} M_{\mathfrak{p}} : \mathfrak{p} \in V(I) \text{ and } R/\mathfrak{p} \notin S\} = \inf\{S\text{-depth}(\mathfrak{p}, M) : \mathfrak{p} \in V(I) \text{ and } R/\mathfrak{p} \notin S\}$ whenever S is closed under taking injective hulls; see Theorem 3.1 and Corollary 3.4.

2. Regular sequences and depth of an ideal on ZD-modules

Recall that R is a Noetherian ring, I is an ideal of R and M is an R-module.

Definition 2.1. A full subcategory of the category of R-modules is said to be Serre subcategory, if it is closed under taking submodules, quotients and extensions. A Serre subcategory S is said to satisfy the condition C_I if for any I-torsion R-module M, $0 :_M I \in S$ implies that $M \in S$.

It is well known that if a Serre subcategory is closed under taking injective hulls, then it satisfies the condition C_I; see [**1**, Lemma 2.2].

Example 2.1. [**1**, Example 2.4] The following classes of modules are Serre subcategories closed under taking injective hulls and hence satisfy the condition C_I.

(a) The class of zero modules.
(b) The class of Artinian R-modules.
(c) The class of R-modules with finite support.
(d) The class of all R-modules M with $\dim_R M \leqslant k$, where k is a nonnegative integer. When $k = 0$, we get the class of semiartinian modules, i.e., the modules M with $\mathrm{Supp}_R M \subseteq \mathrm{Max}(R)$.
(e) Let $Z \subseteq \mathrm{Spec}(R)$ be a closed set under specialization, that is if $\mathfrak{q} \supseteq \mathfrak{p} \in Z$, then $\mathfrak{q} \in Z$. The class of all R-modules M with $\mathrm{Ass}_R M \subseteq Z$ (equivalently $\mathrm{Supp}_R M \subseteq Z$). For example Z could be a closed set $Z = V(\mathfrak{c})$, for a given ideal $\mathfrak{c}$ of R. If we take $Z = \{\mathfrak{p} \in \mathrm{Spec}(R) : \dim R/\mathfrak{p} \leqslant k\}$ where k is a nonnegative integer, then we recover (d).

Example 2.2. [**1**, Example 2.5] The class of I-cofinite Artinian modules is a Serre subcategory of the category of R-modules satisfying the condition C_I, but is not closed under taking injective hulls.

In the rest of the paper, S denotes a Serre subcategory of the category of R-modules. Aghapournahr and Melkersson [**1**] introduced the notion of S-sequences on M as a generalization of regular sequences.

Definition 2.2. An element a of R is called S-regular on M, if $0:_M a \in S$. A sequence $a_1, \ldots, a_t$ is an S-sequence on M, if a_i is S-regular on $M/(a_1, \ldots, a_{i-1})M$ for $i = 1, \ldots, t$.

Example 2.3. [**1**, Example 2.8] Let S be as in Example 2.1. S-sequences on M are just:

(a) Poor M-sequences, [**5**, Definition 6.2.1].
(b) Filter-regular sequences which have been defined in the local case for finitely generated modules in [**14**].
(c) Generalized regular sequences which have been defined in the local case for finitely generated modules in [**12**, Definition 2.1].
(d) M-sequences in dimension $> k$ which have been defined for finitely generated modules in [**4**, Definition 2.1].
(e) When $Z = V(\mathfrak{c})$ for a given ideal $\mathfrak{c}$ of R, they are called $\mathfrak{c}$-filter regular sequences. For a local ring $(R, \mathfrak{m})$ and $\mathfrak{c} = \mathfrak{m}$, we recover (b) in the local case when M is finitely generated.

As a generalization of finitely generated modules, Evans [**8**] introduced zero-divisor modules (ZD-modules). Let $Z_R(M)$ denote the set of zero-divisors of M.

Definition 2.3. An R-module M is said to be zero-divisor module if for any submodule N of M, the set $Z_R(M/N)$ is a finite union of prime ideals in $\operatorname{Ass}_R M/N$.

According to [**7**, Example 2.2], the class of ZD-modules contains finitely generated, Laskerian, weakly Laskerian, linearly compact and Matlis reflexive modules. Also it contains modules whose quotients have finite Goldie dimension, and modules with finite support, in particular Artinian modules. Therefore the class of ZD-modules is much larger than that of finitely generated modules.

The following result yields a characterization of local cohomology modules of ZD-modules that belong to a Serre subcategory of the category of R-modules satisfying the condition C_I.

Lemma 2.1. *Let S satisfy the condition C_I and M be a ZD-module. Then the following conditions are equivalent:*

(i) *There is an S-sequence on M in I of length t.*
(ii) *$H^i_I(M) \in S$ for all $i < t$.*

Proof. The proof, which we include for the reader's convenience, proceeds like that used in the proof of equivalence of conditions (i) and (vii) in [**1**, Theorem 2.9]. We use induction on t. Let $t = 1$. Let $a \in I$ be an S-sequence on M. Then $0:_{\Gamma_I(M)} a = 0:_M a \in S$, and so $\Gamma_I(M) \in S$ by [**1**, Theorem 2.3]. Now suppose that $\Gamma_I(M) \in S$. Since $\Gamma_I(M/\Gamma_I(M)) = 0$, it follows by [**7**, Lemma 2.4] that I contains an element a which is a non zero-divisor on $M/\Gamma_I(M)$. The exact sequence $0 \longrightarrow \Gamma_I(M) \longrightarrow M \longrightarrow M/\Gamma_I(M) \longrightarrow 0$ induces the following exact sequence

$$\Gamma_{(a)}(\Gamma_I(M)) \longrightarrow \Gamma_{(a)}(M) \longrightarrow \Gamma_{(a)}(M/\Gamma_I(M)).$$

Because $\Gamma_{(a)}(\Gamma_I(M)) = \Gamma_I(M)$ and $\Gamma_{(a)}(M/\Gamma_I(M)) = 0$, it follows by the above exact sequence that $\Gamma_{(a)}(M) \in S$. Therefore $0:_M a \in S$, and $a \in I$ is an S-sequence on M.

Now assume inductively that $t > 1$ and the result has been proved for smaller values of t. Let $a_1, \dots, a_t$ be an S-sequence on M in I. Since $0 :_M a_1 \in S$, it follows by [**3**, Lemma 2.1] that $\operatorname{Ext}^i_R(N, 0 :_M a_1) \in S$ for all $i \in \mathbb{N}_0$ and for any finitely generated R-module N. Therefore $H^i_I(0 :_M a_1) \in S$ for all $i < t$, by [**1**, Theorem 2.9]. Since $a_2, \dots, a_t$ is an S-sequence on M/a_1M, it follows by inductive hypothesis that $H^i_I(M/a_1M) \in S$ for all $i < t-1$. Consider the R-homomorphism $f : M \xrightarrow{a_1} M$. Since $H^i_I(\operatorname{Ker} f) \in S$ and $H^{i-1}_I(\operatorname{Coker} f) \in S$ for all $i < t$, it follows by [**11**, Lemma 3.1] that $(0 :_{H^i_I(M)} a_1) = \operatorname{Ker} H^i_I(f) \in S$ for all $i < t$. Therefore $H^i_I(M) \in S$ for all $i < t$.

Suppose that $H^i_I(M) \in S$ for all $i < t$. Take a_1 as in the proof of the case $t = 1$. Then a_1 is a non zero-divisor on $M/\Gamma_I(M)$, and we may replace M with $M/\Gamma_I(M)$ and assume that a_1 is M-regular. Now the exact sequence $0 \longrightarrow M \xrightarrow{a_1} M \longrightarrow M/a_1M \longrightarrow 0$ induces the following exact sequence

$$\cdots \longrightarrow H^{i-1}_I(M/a_1M) \longrightarrow H^i_I(M) \xrightarrow{a_1} H^i_I(M) \longrightarrow H^i_I(M/a_1M) \longrightarrow \cdots .$$

Hence $H^i_I(M/a_1M) \in S$ for all $i < t-1$. Now it follows by inductive hypothesis that $a_2, \dots, a_t$ is an S-sequence on M/a_1M. Therefore $a_1, \dots, a_t$ is an S-sequence on M. □

The following result, which generalizes [**8**, Theorem 15] and [**1**, Lemma 2.14(b)], is one of the main results of this paper. Note that an S-sequence $a_1, \dots, a_t$ (contained in ideal I) is maximal (in I), if $a_1, \dots, a_t, b$ is not an S-sequence for any $b \in R$ ($b \in I$).

THEOREM 2.1. *Let S satisfy the condition C_I and M be a ZD-module. Let I contain a maximal S-sequence on M. Then all maximal S-sequences on M in I have the same length, and this length is equal to* $\inf\{i : \operatorname{Ext}^i_R(R/I, M) \notin S\}$.

PROOF. Let $a_1, \dots, a_t$ be a maximal S-sequence on M in I. Then $H^i_I(M) \in S$ for all $i < t$, by Lemma 2.1. Now it follows by [**1**, Theorem 2.9] that $\operatorname{Ext}^i_R(R/I, M) \in S$ for all $i < t$. Therefore it is enough to show that $\operatorname{Ext}^t_R(R/I, M) \notin S$. First we reduce to the case that a_1 is a non zero-divisor on M. Since $0 :_{\Gamma_{(a_1)}(M)} a_1 = 0 :_M a_1 \in S$, thus $\Gamma_{(a_1)}(M) \in S$. Now it follows by [**1**, Theorem 2.7(b)] that $a_1, \dots, a_t$ is an S-sequence on $M' = M/\Gamma_{(a_1)}(M)$ in I, and a_1 is a regular element on M'. The exact sequence $0 \longrightarrow \Gamma_{(a_1)}(M) \longrightarrow M \longrightarrow M' \longrightarrow 0$ induces the following exact sequence

$$\begin{aligned} \operatorname{Ext}^t_R(R/I, \Gamma_{(a_1)}(M)) \longrightarrow \operatorname{Ext}^t_R(R/I, M) \longrightarrow \operatorname{Ext}^t_R(R/I, M') \\ \longrightarrow \operatorname{Ext}^{t+1}_R(R/I, \Gamma_{(a_1)}(M)). \end{aligned}$$

Since $\Gamma_{(a_1)}(M) \in S$, thus $\operatorname{Ext}^i_R(R/I, \Gamma_{(a_1)}(M)) \in S$ for all $i \in \mathbb{N}_0$. Therefore the above exact sequence implies that $\operatorname{Ext}^t_R(R/I, M) \in S$ if and only if $\operatorname{Ext}^t_R(R/I, M') \in S$. Now we may replace M with M' and assume that a_1 is a non zero-divisor on M.

We use induction on t. Let $t = 1$. We suppose that $\operatorname{Ext}^1_R(R/I, M) \in S$ and look for a contradiction. The exact sequence $0 \longrightarrow M \xrightarrow{a_1} M \longrightarrow M/a_1M \longrightarrow 0$

induces the following exact sequence

$$\begin{aligned}0 &\longrightarrow \operatorname{Hom}_R(R/I, M/a_1M) \longrightarrow \operatorname{Ext}^1_R(R/I,M) \xrightarrow{a_1} \operatorname{Ext}^1_R(R/I,M) \longrightarrow \cdots \\ &\longrightarrow \operatorname{Ext}^{t-1}_R(R/I,M) \longrightarrow \operatorname{Ext}^{t-1}_R(R/I,M/a_1M) \longrightarrow \operatorname{Ext}^t_R(R/I,M) \longrightarrow \cdots .\end{aligned}$$

Thus $\operatorname{Hom}_R(R/I, \Gamma_I(M/a_1M)) = \operatorname{Hom}_R(R/I, M/a_1M) \in S$ and so $\Gamma_I(M/a_1M) \in S$. Since a_1 is a maximal S-sequence on M in I, thus $0 :_{M/a_1M} a \notin S$ for all $a \in I$. Hence for any $a \in I$, there is $x_a \in M$ such that $x_a + a_1M \notin S$ and $a(x_a + a_1M) = 0$. Set $X = \{x_a + a_1M : a \in I\}$ and $N = (X)$. Then $N \notin S$ and $I \subseteq Z_R(N) = \bigcup_{\mathfrak{p} \in \operatorname{Ass}_R N} \mathfrak{p}$. So there is $\mathfrak{q} \in \operatorname{Ass}_R N$ such that $I \subseteq \mathfrak{q}$. Therefore there is $a' \in I$ such that $x_{a'} + a_1M \notin S$ and $I(x_{a'} + a_1M) = 0$. Hence $x_{a'} + a_1M \in \Gamma_I(M/a_1M)$ and so $\Gamma_I(M/a_1M) \notin S$ which is a contradiction.

Now assume inductively that $t > 1$ and the result has been proved for smaller values of t. Since $a_2, \ldots, a_t$ is a maximal S-sequence on M/a_1M in I, it follows by inductive hypothesis that $\operatorname{Ext}^{t-1}_R(R/I, M/a_1M) \notin S$. Now by the above exact sequence, we have $\operatorname{Ext}^t_R(R/I,M) \notin S$. □

Definition 2.4. Let S satisfy the condition C_I, M be a ZD-module and I contain a maximal S-sequence on M. The common length of all maximal S-sequences on M in I is called the S-depth of I on M, denoted by $S\text{-depth}(I,M)$.

Remark 2.1. Let S satisfy the condition C_I and M be a ZD-module. We complement the above definition by setting $S\text{-depth}(I,M) = \infty$, whenever there is no maximal S-sequence on M in I. This is consistent with Theorem 2.1:

$$S\text{-depth}(I,M) = \infty \iff \operatorname{Ext}^i_R(R/I,M) \in S \text{ for all } i.$$

For, if there is no maximal S-sequence on M in I, then there are S-sequences on M in I of arbitrary length, and therefore $H^i_I(M) \in S$ for all i, by Theorem 2.1. Now it follows by [**1**, Theorem 2.9] that $\operatorname{Ext}^i_R(R/I,M) \in S$ for all i. The converse is seen similarly.

The following result provides an important case that I contains a maximal S-sequence on M.

Proposition 2.1. *If $M/IM \notin S$, then every S-sequence on M in I can be extended to a maximal one.*

Proof. Let $a_1, a_2, a_3, \ldots$ be a sequence of elements of I such that for any positive integer t, the finite sequence $a_1, \ldots, a_t$ is an S-sequence on M, and look for a contradiction. Therefore, for any positive integer t, we must have $(a_1, \ldots, a_t) \subset (a_1, \ldots, a_t, a_{t+1})$; otherwise we would have $a_{t+1} \in (a_1, \ldots, a_t)$, so $M/(a_1, \ldots, a_t)M = 0 :_{M/(a_1,\ldots,a_t)M} a_{t+1} \in S$, and hence $M/IM = (M/(a_1, \ldots, a_t)M) \otimes_R R/I \in S$ which is a contradiction. Thus

$$(a_1) \subset (a_1, a_2) \subset \cdots \subset (a_1, \ldots, a_t) \subset (a_1, \ldots, a_t, a_{t+1}) \subset \cdots$$

is an infinite strictly ascending chain of ideals of R, contrary to the fact that R is Noetherian. □

Now a natural question arises.

Question. Let S satisfy the condition C_I, and M be a ZD-module. If $M/IM \in S$, are there S-sequences on M in I of arbitrary length?

The answer to the question is 'yes', whenever M is a finitely generated R-module; see [**1**, Lemma 2.14].

EXAMPLE 2.4. [**1**, Example 2.16] Let M be a ZD-module. The following are some examples of S-depth(I, M) with S as in Example 2.1.

(a) It is the same as ordinary depth(I, M).
(b) It is the same as f-depth(I, M) (filter-depth) which has been defined in the local case in [**10**] and [**9**, Definition 3.3].
(c) It is the same as g-depth(I, M) (generalized depth) which has been defined in the local case in [**12**, Definition 4.2].

3. Some properties of S-depth of I on ZD-modules

In this section, we investigate some properties of S-depth of I on a ZD-module M. In the first, we get the following result, which is a generalization of [**1**, Theorem 2.18(a)], and yields a relation between S-depth of I on M and local cohomology modules of M with respect to I.

LEMMA 3.1. *Let S satisfy the condition C_I, and M be a ZD-module. Then*

$$S\text{-depth}(I, M) = \inf\{i : \operatorname{Ext}^i_R(R/I, M) \notin S\} = \inf\{i : H^i_I(M) \notin S\}.$$

PROOF. The first equality follows by Theorem 2.1 and our discussion in Remark 2.1. The second follows by the first one and [**1**, Theorem 2.9]. □

COROLLARY 3.1. *Let $\mathfrak{p}$ be a prime ideal of R, and let M be a ZD-module. Then* depth $M_{\mathfrak{p}} = \inf\{i : \mu^i(\mathfrak{p}, M) \neq 0\}$*, where $\mu^i(\mathfrak{p}, M)$ denotes the i-th Bass number of M with respect to $\mathfrak{p}$.*

PROOF. We note that $\mu^i(\mathfrak{p}, M) = \dim_{R_{\mathfrak{p}}} \operatorname{Ext}^i_{R_{\mathfrak{p}}}(R_{\mathfrak{p}}/\mathfrak{p}R_{\mathfrak{p}}, M_{\mathfrak{p}})$ for any integer i. Now the claim follows by Lemma 3.1, whenever S is the class of zero modules. □

The next proposition collects some formulas which are useful in the computation of S-depth of I on M.

PROPOSITION 3.1. *Let S satisfy the condition C_I, M be a ZD-module, and I' be an ideal of R. Then*

(i) *If $I \subseteq I'$, then* S-depth$(I, M) \leqslant S$-depth(I', M).
(ii) S-depth$(I, M) = S$-depth$(\sqrt{I}, M)$.
(iii) S-depth$(II', M) = S$-depth$(I \cap I', M)$.
(iv) *If $\underline{a} = a_1, \ldots, a_t$ is an S-sequence on M in I, then* S-depth$\left(\frac{I}{(\underline{a})}, \frac{M}{(\underline{a})M}\right) = S$-depth$\left(I, \frac{M}{(\underline{a})M}\right) = S$-depth$(I, M) - t$.

PROOF. (i) The claim follows easily from the definition.

(ii) The claim follows by Lemma 3.1. We note that $H^i_I(M) \cong H^i_{\sqrt{I}}(M)$ for any i.

(iii) The claim follows by (ii).

(iv) It is clear that I contains an S-sequence on $\frac{M}{(\underline{a})M}$, if and only if, $\frac{I}{(\underline{a})}$ contains an S-sequence on $\frac{M}{(\underline{a})M}$. This proves the first equation.

To prove the second equation, let $S\text{-depth}(I, \frac{M}{(\underline{a})M}) = s$ and $b_1, \dots, b_s$ be a maximal S-sequence on $\frac{M}{(\underline{a})M}$ in I. It follows by [**1**, Proposition 2.7(a)] that $a_1, \dots, a_t, b_1, \dots, b_s$ is a maximal S-sequence on M, and hence $S\text{-depth}(I, M) = t + s$. Now let $S\text{-depth}\left(I, \frac{M}{(\underline{a})M}\right) = \infty$. Then I contains S-sequences on $\frac{M}{(\underline{a})M}$ of arbitrary length. It follows by [**1**, Proposition 2.7(a)] that I contains S-sequences on M of arbitrary length, and so $S\text{-depth}(I, M) = \infty$. □

We now study the behaviour of $S\text{-depth}(I, M)$ along exact sequences.

PROPOSITION 3.2. *Let S satisfy the condition C_I, and $0 \longrightarrow U \longrightarrow M \longrightarrow N \longrightarrow 0$ be an exact sequence of ZD-modules. Then*

(i) $S\text{-depth}(I, M) \geqslant \min\{S\text{-depth}(I, U), S\text{-depth}(I, N)\}$.

(ii) $S\text{-depth}(I, U) \geqslant \min\{S\text{-depth}(I, M), S\text{-depth}(I, N) + 1\}$.

(iii) $S\text{-depth}(I, N) \geqslant \min\{S\text{-depth}(I, U) - 1, S\text{-depth}(I, M)\}$.

PROOF. The given exact sequence induces a long exact sequence

$$\cdots \longrightarrow \operatorname{Ext}^{i-1}_R(R/I, N) \longrightarrow \operatorname{Ext}^i_R(R/I, U) \longrightarrow \operatorname{Ext}^i_R(R/I, M) \longrightarrow \operatorname{Ext}^i_R(R/I, N) \longrightarrow \operatorname{Ext}^{i+1}_R(R/I, U) \longrightarrow \cdots .$$

One observes that $\operatorname{Ext}^i_R(R/I, M) \in S$ if $\operatorname{Ext}^i_R(R/I, U) \in S$ and $\operatorname{Ext}^i_R(R/I, N) \in S$. Therefore (i) follows by Lemma 3.1. Completely analogous arguments prove (ii) and (iii). □

The following lemma has a main role in the next results on this section and is a generalization of [**11**, Lemma 5.4].

LEMMA 3.2. *Let S be a Serre subcategory closed under taking injective hulls, and $0 \longrightarrow X^0 \longrightarrow X^1 \longrightarrow X^2 \longrightarrow \cdots$ be a cochain complex with cocycles Z^i, coboundaries B^i and cohomology $H^i = Z^i/B^i$. Let X^i be injective hull of Z^i for all i. The following conditions are equivalent:*

(i) $H^i \in S$ *for all* $i < t$. (ii) $X^i \in S$ *for all* $i < t$.

PROOF. The only nontrivial part is (i)⇒(ii). We prove the claim by induction on t. Let $t = 1$. Then $Z^0 \in S$ and hence $X^0 \in S$. So assume, inductively, that $t > 1$ and the result has been proved for smaller values of t. It follows by inductive hypothesis that $X^i \in S$ for all $i < t - 1$, and it is enough to show that $X^{t-1} \in S$. Since $X^{t-2} \in S$, thus $B^{t-1} \in S$. Now the exact sequence $0 \longrightarrow B^{t-1} \longrightarrow Z^{t-1} \longrightarrow H^{t-1} \longrightarrow 0$ implies that $Z^{t-1} \in S$, and hence $X^{t-1} \in S$. □

Corollary 3.2. *Let S be a Serre subcategory closed under taking injective hulls, and* $0 \longrightarrow M \longrightarrow E^0 \xrightarrow{d^0} E^1 \longrightarrow \cdots \longrightarrow E^i \xrightarrow{d^i} E^{i+1} \longrightarrow \cdots$ *be a minimal injective resolution of M. The following conditions are equivalent:*

(i) $H_I^i(M) \in S$ *for all* $i < t$. (ii) $\Gamma_I(E^i) \in S$ *for all* $i < t$.

Proof. Let $E^i \cong \bigoplus_{\mathfrak{p}\in \mathrm{Spec}(R)} \mu^i(\mathfrak{p}, M)E_R(R/\mathfrak{p})$ be a decomposition of E^i as the direct sum of indecomposable injective R-modules, where $E_R(R/\mathfrak{p})$ denotes the injective hull of $R/\mathfrak{p}$. Then $\Gamma_I(E^i) \cong \bigoplus_{\mathfrak{p}\in V(I)} \mu^i(\mathfrak{p}, M)E_R(R/\mathfrak{p})$, and hence $\Gamma_I(E^i)$ is injective. Also it is easy to see that $\mathrm{Ker}\,\Gamma_I(d^i) = \mathrm{Ker}\, d^i \cap \Gamma_I(E^i)$ for all i. Therefore $\Gamma_I(E^i)$ is injective hull of $\mathrm{Ker}\,\Gamma_I(d^i)$. Now the claim follows by Lemma 3.2. □

The following result generalizes [**6**, Proposition 1.2.10(a)].

Theorem 3.1. *Let S be a Serre subcategory closed under taking injective hulls, and M be a ZD-module. Then*

$$S\text{-depth}(I, M) = \inf\{\mathrm{depth}\, M_{\mathfrak{p}} : \mathfrak{p} \in V(I) \text{ and } R/\mathfrak{p} \notin S\}.$$

Proof. Let $0 \longrightarrow M \longrightarrow E^0 \longrightarrow E^1 \longrightarrow \cdots$ be a minimal injective resolution of M, and $E^i \cong \bigoplus_{\mathfrak{p}\in \mathrm{Spec}(R)} \mu^i(\mathfrak{p}, M)E_R(R/\mathfrak{p})$ be a decomposition of E^i as the direct sum of indecomposable injective R-modules. Let $t = \inf\{\mathrm{depth}\, M_{\mathfrak{p}} : \mathfrak{p} \in V(I) \text{ and } R/\mathfrak{p} \notin S\}$. It follows by Corollary 3.1 that if $\mathfrak{p} \in V(I)$ and $R/\mathfrak{p} \notin S$, then $\mu^i(\mathfrak{p}, M) = 0$ for all $i < t$. Therefore $\Gamma_I(E^i) \in S$ for all $i < t$, and so $H_I^i(M) \in S$ for all $i < t$ by Corollary 3.2. Hence $t \leqslant \inf\{i : H_I^i(M) \notin S\}$. Now it follows by Lemma 3.1 that $t \leqslant S\text{-depth}(I, M)$ and it is enough to show that $H_I^t(M) \notin S$. By assumption, there is $\mathfrak{q} \in V(I)$ with $R/\mathfrak{q} \notin S$ such that $t = \mathrm{depth}\, M_{\mathfrak{q}}$. It follows by Corollary 3.1 that $\mu^t(\mathfrak{q}, M) \neq 0$. Therefore $\Gamma_I(E^t) \notin S$, and hence $H_I^t(M) \notin S$ by Corollary 3.2. □

Corollary 3.3. *Let S be a Serre subcategory closed under taking injective hulls, M be a ZD-module, and I' be an ideal of R. Then*

$$S\text{-depth}(I \cap I', M) = \min\{S\text{-depth}(I, M), S\text{-depth}(I', M)\}.$$

The following result is a generalization of [**1**, Theorem 2.18(e)].

Corollary 3.4. *Let S be a Serre subcategory closed under taking injective hulls, and M be a ZD-module. Then*

$$S\text{-depth}(I, M) = \inf\{S\text{-depth}(\mathfrak{p}, M) : \mathfrak{p} \in V(I) \text{ and } R/\mathfrak{p} \notin S\}.$$

Proof. By 3.1(i), we have $S\text{-depth}(I, M) \leqslant S\text{-depth}(\mathfrak{p}, M)$ for all $\mathfrak{p} \in V(I)$. Thus $S\text{-depth}(I, M) \leqslant \inf\{S\text{-depth}(\mathfrak{p}, M) : \mathfrak{p} \in V(I) \text{ and } R/\mathfrak{p} \notin S\}$. To prove the converse inequality, let $\mathfrak{p} \in V(I)$ and $R/\mathfrak{p} \notin S$. It follows by Theorem 3.1 that

$$S\text{-depth}(\mathfrak{p}, M) = \inf\{\mathrm{depth}\, M_{\mathfrak{q}} : \mathfrak{q} \in V(\mathfrak{p}) \text{ and } R/\mathfrak{q} \notin S\} \leqslant \mathrm{depth}\, M_{\mathfrak{p}}.$$

Therefore

$$\begin{aligned}&\inf\{S\text{-depth}(\mathfrak{p}, M) : \mathfrak{p} \in V(I) \text{ and } R/\mathfrak{p} \notin S\}\\ &\quad \leqslant \inf\{\mathrm{depth}\, M_{\mathfrak{p}} : \mathfrak{p} \in V(I) \text{ and } R/\mathfrak{p} \notin S\}\end{aligned}$$

and the claim follows by Theorem 3.1. □

References

1. M. Aghapournahr, L. Melkersson, *Local cohomology and Serre subcategories*, J. Algebra **320** (2008), 1275–1287.
2. M. Aghapournahr, Kh. Ahmadi-Amoli, M. Y. Sadeghi, *Cofiniteness and Artinianness of certain local cohomology modules*, Ric. Mat. **65** (2016), 21–36.
3. M. Asgharzadeh, M. Tousi, *A unified approach to local cohomology modules using Serre classes*, Canad. Math. Bull. **53** (2010), 577–586.
4. M. Brodmann, L. T. Nhan, *A finiteness result for associated primes of certain Ext-modules*, Comm. Algebra **36** (2008), 1527–1536.
5. M. P. Brodmann, R. Y. Sharp, *Local Cohomology: An Algebraic Introduction with Geometric Applications*, Cambridge University Press, Cambridge, 1998.
6. W. Bruns, J. Herzog, *Cohen-Macaulay Rings*, 2nd ed., Cambridge University Press, Cambridge, 1998.
7. K. Divaani-Aazar, M. A. Esmkhani, *Artinianness of local cohomology modules of ZD-modules*, Comm. Algebra **33** (2005), 2857–2863.
8. E. G. Evans, *Zero divisors in Noetherian-like rings*, Trans. Amer. Math. Soc. **155** (1971), 505–512.
9. R. Lü, Z. Tang, *The* $f-\mathrm{depth}$ *of an ideal on a module*, Proc. Amer. Math. Soc. **130** (2002), 1905–1912.
10. L. Melkersson, *Some applications of a criterion for Artinianness of a module*, J. Pure Appl. Algebra **101** (1995), 291–303.
11. ______, *Modules cofinite with respect to an ideal*, J. Algebra **285** (2005), 649–668.
12. L. T. Nhan, *On generalized regular sequences and the finiteness for associated primes of local cohomology modules*, Comm. Algebra **33** (2005), 793–806.
13. Sh. Payrovi, M. Lotfi Parsa, *Finiteness of local cohomology modules defined by a pair of ideals*, Comm. Algebra **41** (2013), 627–637.
14. P. Schenzel, N. V. Trung, N. T. Cuong, *Verallgemeinerte Cohen-Macaulay moduln*, Math. Nachr. **85** (1978), 57–73.

Sayyed Jamaleddin Asadabadi University
Asadabad, Iran
lotfi.parsa@yahoo.com

(Received 31 03 2016)