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SUBCLASSES OF HARMONIC UNIVALENT FUNCTIONS ASSOCIATED WITH GENERALIZED RUSCHEWEYH OPERATOR

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Abstract. We introduce a new subclass of functions defined by multiplier differential operator and give coefficient bounds for these subclasses. Also, we obtain necessary and sufficient convolution conditions, distortion bounds and extreme points for these subclasses of functions.

1. Introduction

A real-valued function u is said to be harmonic in a domain $D \subset \mathbb{C}$ if it has continuous second order partial derivatives in D, which satisfy the Laplace equation $\Delta u := \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$. A harmonic mapping f of the simply connected domain D is a complex-valued function of the form $f = h + \overline{g}$, where h and g are analytic in D. We call h and g analytic and co-analytic part of f, respectively (see [**4**]). The Jacobian of f is given by $J_{f(z)} = |f_z(z)|^2 - |f_{\overline{z}}(z)|^2 = |h'(z)|^2 - |g'(z)|^2$. A result of Lewy [**16**] states that f is locally univalent if and only if its Jacobian is never zero, and is sense-preserving if the Jacobian is positive.

Let $\mathcal{H}$ indicate the class of harmonic functions in the unit disc $\mathbb{U}$. By $\mathcal{S_H}$ we indicate the class of function $f \in \mathcal{H}$ of the form

$$f(z) = z + \sum_{n=2}^{\infty} \left(a_n z^n + \overline{b_n z^n}\right) \quad (z \in \mathbb{U}), \tag{1.1}$$

which are univalent and sense-preserving in $\mathbb{U}$ and $h(0) = h'(0) - 1 = 0$, $g(0) = 0$. Also note that $\mathcal{H}$ reduces to the class $\mathcal{A}$ of analytic functions in $\mathbb{U}$ if co-analytic part of f is identically zero.

We say that a function $f \in \mathcal{S_H}$ is harmonic starlike in $\mathbb{U}(r)$ if $\frac{\partial}{\partial t}(\arg f(re^{it})) > 0$ $(0 \leqslant t \leqslant 2\pi)$ i.e., f maps the circle $\partial\mathbb{U}(r)$ onto a closed curve that is starlike with

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respect to the origin. It is easy to verify, that the above condition is equivalent to

$$\operatorname{Re}\frac{D_{\mathcal{H}}f(z)}{f(z)}>0\ \ (|z|=r),\quad D_{\mathcal{H}}f(z):=zh'(z)-\overline{zg'(z)}.$$

Ruscheweyh [20] introduced an operator $\mathcal{R}^m:\mathcal{A}\to\mathcal{A}$, defined by

$$\mathcal{R}^m f(z)=\frac{z(z^{m-1}f(z))^{(m)}}{m!}\quad (m\in\mathbb{N}_0=\mathbb{N}\cup\{0\},\ z\in\mathbb{U}).$$

The Ruscheweyh derivative $\mathcal{R}^m$ was extended in [18] (see also [7, 9, 11, 23]) on the class of harmonic functions. Let $D_{\mathcal{H}}^{m,\lambda}:\mathcal{H}\to\mathcal{H}$ represent the linear operator defined for a function $f=h+\overline{g}\in\mathcal{H}$ by

$$\begin{aligned} D_{\mathcal{H}}^{m,\lambda}f&:=\lambda D_{\mathcal{H}}^{m+1}f+(1-\lambda)D_{\mathcal{H}}^{m}f\quad (0\leqslant\lambda\leqslant 1),\\ D_{\mathcal{H}}^{m}f&:=\mathcal{R}^m h+(-1)^m\overline{\mathcal{R}^m g}. \end{aligned}\tag{1.2}$$

We say that a function $f\in\mathcal{H}$ is *subordinate* to a function $F\in\mathcal{H}$, and write $f(z)\prec F(z)$ (or simply $f\prec F$) if there exists a complex-valued function ω which maps $\mathbb{U}$ into oneself with $\omega(0)=0$, such that $f(z)=F(\omega(z))$ $(z\in\mathbb{U})$.

Let A,B be real parameters, $-B\leqslant A<B\leqslant 1$. We represent by $\mathcal{S}_{\mathcal{H}}^{m,\lambda}(A,B)$, the class of functions $f\in\mathcal{S}_{\mathcal{H}}$ such that

$$\frac{D_{\mathcal{H}}\big(D_{\mathcal{H}}^{m,\lambda}f\big)(z)}{D_{\mathcal{H}}^{m,\lambda}f(z)}\prec\frac{1+Az}{1+Bz}.\tag{1.3}$$

The class $\mathcal{S}_{\mathcal{H}}^{m}(A,B):=\mathcal{S}_{\mathcal{H}}^{m,0}(A,B)$ was investigated in [6]. In particular, the class $\mathcal{S}_{\mathcal{H}}^{m}(\alpha):=\mathcal{S}_{\mathcal{H}}^{m}(2\alpha-1,1)$ $(0\leqslant\alpha<1)$ is related to the class of Sălăgean-type harmonic functions studied by Yalçin [22]. The classes $\mathcal{S}_{\mathcal{H}}(A,B):=\mathcal{S}_{\mathcal{H}}^{0}(A,B)$, and $\mathcal{K}_{\mathcal{H}}(A,B):=\mathcal{S}_{\mathcal{H}}^{1}(A,B)$ were defined in [7] (see also [8]).

Making use of the techniques and methodology used by Dziok [7], we will give necessary and sufficient conditions, distortion bounds, compactness and extreme points for the classes defined above. Some applications of the main results are also considered.

2. Analytic criteria

For functions $f_1,f_2\in\mathcal{H}$ of the form

$$f_l(z)=\sum_{k=0}^{\infty}\big(a_{l,k}z^k+\overline{b_{l,k}z^k}\big)\quad (z\in\mathbb{U},\ l\in\{1,2\})\tag{2.1}$$

we define the *Hadamard product or convolution* of f_1 and f_2 by

$$(f_1*f_2)(z)=\sum_{k=0}^{\infty}\big(a_{1,k}a_{2,k}z^k+\overline{b_{1,k}b_{2,k}z^k}\big)\quad (z\in\mathbb{U}).$$

In our first theorem, we obtain the necessary and sufficient conditions for harmonic functions in $\mathcal{S}_{\mathcal{H}}^{m,\lambda}(A,B)$.

THEOREM 2.1. *Let $f \in \mathcal{S}_{\mathcal{H}}$. Then $f \in \mathcal{S}_{\mathcal{H}}^{m,\lambda}(A,B)$ if and only if*

$$f(z) * D_{\mathcal{H}}^{m,\lambda}\psi_{\xi}(z) \neq 0 \quad (z \in \mathbb{U}_0 = \mathbb{U} \setminus \{0\},\ |\xi| = 1),$$

where

$$\psi_{\xi}(z) = z\frac{1 + B\xi - (1 + A\xi)(1 - z)}{(1 - z)^2} - \overline{z}\frac{1 + B\xi + (1 + A\xi)(1 - \overline{z})}{(1 - \overline{z})^2} \quad (z \in \mathbb{U}_0).$$

PROOF. Let $f \in \mathcal{S}_{\mathcal{H}}$ be of the form (1.1). Then $f \in \mathcal{S}_{\mathcal{H}}^{m,\lambda}(A,B)$ if and only if it satisfies (1.3), or equivalently

$$\frac{D_{\mathcal{H}}\big(D_{\mathcal{H}}^{m,\lambda}f\big)(z)}{D_{\mathcal{H}}^{m,\lambda}f(z)} \neq \frac{1 + A\xi}{1 + B\xi} \quad (z \in \mathbb{U}_0,\ |\xi| = 1).$$

Since

$$D_{\mathcal{H}}\big(D_{\mathcal{H}}^{m,\lambda}h\big)(z) = D_{\mathcal{H}}^{m,\lambda}h(z) * \frac{z}{(1 - z)^2}, \quad D_{\mathcal{H}}^{m,\lambda}h(z) = D_{\mathcal{H}}^{m,\lambda}h(z) * \frac{z}{1 - z},$$

the above inequalities yield

$$\begin{aligned}
&\big(1 + B\xi\big)D_{\mathcal{H}}\big(D_{\mathcal{H}}^{m,\lambda}f\big)(z) - \big(1 + A\xi\big)D_{\mathcal{H}}^{m,\lambda}f(z)\\
&\qquad = \big(1 + B\xi\big)D_{\mathcal{H}}\big(D_{\mathcal{H}}^{m,\lambda}h\big)(z) - \big(1 + A\xi\big)D_{\mathcal{H}}^{m,\lambda}h(z)\\
&\qquad\quad - (-1)^m\Big[\big(1 + B\xi\big)\overline{D_{\mathcal{H}}\big(D_{\mathcal{H}}^{m,\lambda}g\big)(z)} + \big(1 + A\xi\big)\overline{D_{\mathcal{H}}^{m,\lambda}g(z)}\Big]\\
&\qquad = D_{\mathcal{H}}^{m,\lambda}h(z) * \left(\frac{(1 + B\xi)z}{(1 - z)^2} - \frac{(1 + A\xi)z}{1 - z}\right)\\
&\qquad\quad - (-1)^m\overline{D_{\mathcal{H}}^{m,\lambda}g(z)} * \left(\frac{(1 + B\xi)\overline{z}}{(1 - \overline{z})^2} + \frac{(1 + A\xi)\overline{z}}{1 - \overline{z}}\right)\\
&\qquad = f(z) * D_{\mathcal{H}}^{m,\lambda}\psi_{\xi}(z) \neq 0 \quad (z \in \mathbb{U}_0,\ |\xi| = 1).
\end{aligned}$$

Thus, $f \in \mathcal{S}_{\mathcal{H}}^{m,\lambda}(A,B)$ if and only if $f(z) * D_{\mathcal{H}}^{m,\lambda}\psi_{\xi}(z) \neq 0$ for $z \in \mathbb{U}_0$, $|\xi| = 1$. □

Let $f \in \mathcal{H}$ be of the form (1.1). Thus, by (1.2) we have

$$D_{\mathcal{H}}^{m,\lambda}f(z) = z + \sum_{n=2}^{\infty}\lambda_n a_n z^n + (-1)^m\sum_{n=2}^{\infty}\lambda_n\overline{b_n}\overline{z}^n \quad (z \in \mathbb{U}),$$

where $\lambda_n := (\lambda(n-1) + m + 1)\frac{n\cdot\ldots\cdot(m+n)}{(m+n)(m+1)!}$.

THEOREM 2.2. *If a function $f \in \mathcal{H}$ of the form* (1.1) *satisfies the condition*

$$\sum_{n=2}^{\infty}(\alpha_n|a_n| + \beta_n|b_n|) \leqslant B - A, \tag{2.2}$$

where

$$\alpha_n = \lambda_n\{n(1 + B) - (1 + A)\}, \quad \beta_n = \lambda_n\{n(1 + B) + (1 + A)\}, \tag{2.3}$$

then $f \in \mathcal{S}_{\mathcal{H}}^{m,\lambda}(A,B)$.

PROOF. It is clear that the theorem is true for the function $f(z) \equiv z$. Let $f \in \mathcal{H}$ be a function of the form (1.1) and let there exist $n \in \mathbb{N}_2$ such that $a_n \neq 0$ or $b_n \neq 0$. Since $\lambda_n \geqslant \lambda_2 \geqslant 1$, we have

$$\frac{\alpha_n}{B-A} \geqslant n,\ \frac{\beta_n}{B-A} \geqslant n, \quad n \in \mathbb{N}_2.$$

Thus, by (2.2) we get

$$\sum_{n=2}^{\infty}(n|a_n| + n|b_n|) \leqslant 1 \tag{2.4}$$

and

$$\begin{aligned}|h'(z)| - |g'(z)| &\geqslant 1 - \sum_{n=2}^{\infty} n|a_n||z|^n - \sum_{n=2}^{\infty} n|b_n||z|^n \geqslant 1 - |z|\sum_{n=2}^{\infty}(n|a_n| + n|b_n|)\\ &\geqslant 1 - \frac{|z|}{B-A}\sum_{n=2}^{\infty}(\alpha_n|a_n| + \beta_n|b_n|) \geqslant 1 - |z| > 0 \quad (z \in \mathbb{U}).\end{aligned}$$

Therefore, by (1.1) the function f is locally univalent and sense-preserving in $\mathbb{U}$. Moreover, if $z_1, z_2 \in \mathbb{U}$, $z_1 \neq z_2$, then

$$\left|\frac{z_1^n - z_2^n}{z_1 - z_2}\right| = \left|\sum_{l=1}^{n} z_1^{l-1} z_2^{n-l}\right| \leqslant \sum_{l=1}^{n} |z_1|^{l-1}|z_2|^{n-l} < n \quad (n \in \mathbb{N}_2).$$

Hence, by (2.4) we have

$$\begin{aligned}|f(z_1) - f(z_2)| &\geqslant |h(z_1) - h(z_2)| - |g(z_1) - g(z_2)|\\ &= \left|z_1 - z_2 - \sum_{n=2}^{\infty} a_n(z_1^n - z_2^n)\right| - \left|\sum_{n=2}^{\infty} \overline{b_n(z_1^n - z_2^n)}\right|\\ &\geqslant |z_1 - z_2| - \sum_{n=2}^{\infty} |a_n||z_1^n - z_2^n| - \sum_{n=2}^{\infty} |b_n||z_1^n - z_2^n|\\ &= |z_1 - z_2|\left(1 - \sum_{n=2}^{\infty} |a_n|\left|\frac{z_1^n - z_2^n}{z_1 - z_2}\right| - \sum_{n=2}^{\infty} |b_n|\left|\frac{z_1^n - z_2^n}{z_1 - z_2}\right|\right)\\ &> |z_1 - z_2|\left(1 - \sum_{n=2}^{\infty} n|a_n| - \sum_{n=2}^{\infty} n|b_n|\right) \geqslant 0.\end{aligned}$$

This leads to the univalence of f i.e., $f \in \mathcal{S}_{\mathcal{H}}$. Therefore, $f \in \mathcal{S}^{m,\lambda}(A, B)$ if and only if there exists a complex-valued function ω, $\omega(0) = 0$, $|\omega(z)| < 1$ $(z \in \mathbb{U})$ such that

$$\frac{D_{\mathcal{H}}\big(D_{\mathcal{H}}^{m,\lambda} f\big)(z)}{D_{\mathcal{H}}^{m,\lambda} f(z)} = \frac{1 + A\omega(z)}{1 + B\omega(z)} \quad (z \in \mathbb{U}),$$

or equivalently

$$\left|\frac{D_{\mathcal{H}}\big(D_{\mathcal{H}}^{m,\lambda} f\big)(z) - D_{\mathcal{H}}^{m,\lambda} f(z)}{BD_{\mathcal{H}}\big(D_{\mathcal{H}}^{m,\lambda} f\big)(z) - AD_{\mathcal{H}}^{m} f(z)}\right| < 1 \quad (z \in \mathbb{U}). \tag{2.5}$$

Thus, it is suffice to prove that

$$\left|D_{\mathcal{H}}\big(D_{\mathcal{H}}^{m,\lambda}f\big)(z)-D_{\mathcal{H}}^{m,\lambda}f(z)\right|-\left|BD_{\mathcal{H}}\big(D_{\mathcal{H}}^{m,\lambda}f\big)(z)-AD_{\mathcal{H}}^{m}f(z)\right|<0\ \ (z\in\mathbb{U}_0).$$

Indeed, letting $|z|=r$ $(0<r<1)$ we have

$$\begin{aligned}
&\left|D_{\mathcal{H}}\big(D_{\mathcal{H}}^{m,\lambda}f\big)(z)-D_{\mathcal{H}}^{m,\lambda}f(z)\right|-\left|BD_{\mathcal{H}}\big(D_{\mathcal{H}}^{m,\lambda}f\big)(z)-AD_{\mathcal{H}}^{m}f(z)\right|\\
&=\left|\sum_{n=2}^{\infty}(n-1)\lambda_n a_n z^n-(-1)^m\sum_{n=2}^{\infty}(n+1)\lambda_n\overline{b_n}\overline{z}^n\right|\\
&\quad-\left|(B-A)z+\sum_{n=2}^{\infty}(Bn-A)\lambda_n a_n z^n+(-1)^m\sum_{n=2}^{\infty}(Bn+A)\lambda_n\overline{b_n}\overline{z}^n\right|\\
&\leqslant\sum_{n=2}^{\infty}(n-1)\lambda_n|a_n|r^n+\sum_{n=2}^{\infty}(n+1)\lambda_n|b_n|r^n-(B-A)r\\
&\quad+\sum_{n=2}^{\infty}(Bn-A)\lambda_n|a_n|r^n+\sum_{n=2}^{\infty}(Bn+A)\lambda_n|b_n|r^n\\
&\leqslant r\left\{\sum_{n=2}^{\infty}\big(\alpha_n|a_n|+\beta_n|b_n|\big)r^{n-1}-(B-A)\right\}<0,
\end{aligned}$$

whence $f\in\mathcal{S}_{\mathcal{H}}^{m,\lambda}(A,B)$. □

Motivated by Silverman [**21**] we denote by $\mathcal{T}^m$ $(m\in\{0,1\})$ the class of functions $f\in\mathcal{H}$ of the form (1.1) such that $a_n=-|a_n|,\ \ b_n=(-1)^m|b_n|\quad(n=2,3,\dots)$ i.e.,

$$f=h+\overline{g},\ \ h(z)=z-\sum_{n=2}^{\infty}|a_n|z^n,\ \ g(z)=(-1)^m\sum_{n=2}^{\infty}|b_n|\overline{z}^n\quad(z\in\mathbb{U}).\tag{2.6}$$

These functions were intensively investigated by many authors (for example, see [**5**, **7**–**10**, **12**, **14**, **25**]).

Moreover, let us define

$$\mathcal{S}_{\mathcal{T}}^{m,\lambda}(A,B):=\mathcal{T}^m\cap\mathcal{S}_{\mathcal{H}}^{m,\lambda}(A,B)$$

where A,B are real parameters with $B>\max\{0,A\}$.

The next theorem shows that condition (2.2) is also the sufficient condition for functions $f\in\mathcal{T}^m$ to be in the class $\mathcal{S}_{\mathcal{T}}^{m,\lambda}(A,B)$.

THEOREM 2.3. *Let $f\in\mathcal{T}^m$ be a function of the form* (2.6). *Then $f\in\mathcal{S}_{\mathcal{T}}^{m,\lambda}(A,B)$ if and only if condition* (2.2) *holds true.*

PROOF. In view of Theorem 2.2 we need only to show that each function $f\in\mathcal{S}_{\mathcal{T}}^{m,\lambda}(A,B)$ satisfies coefficient inequality (2.2). If $f\in\mathcal{S}_{\mathcal{T}}^{m,\lambda}(A,B)$, then it is of the form (2.6) and it satisfies (2.5) or equivalently

$$\left|\frac{-\sum_{n=2}^{\infty}(n-1)\lambda_n|a_n|z^n-(-1)^{2m}\sum_{n=2}^{\infty}(n+1)\lambda_n|b_n|\overline{z}^n}{(B-A)z-\sum_{n=2}^{\infty}(Bn-A)\lambda_n|a_n|z^n-(-1)^{2m}\sum_{n=2}^{\infty}(Bn+A)\lambda_n|b_n|\overline{z}^n}\right|<1.$$

Therefore, putting $z=r$ $(0\leqslant r<1)$, we obtain

$$\frac{\sum_{n=2}^{\infty}\left[(n-1)\lambda_n|a_n|+(n+1)\lambda_n|b_n|\right]r^{n-1}}{(B-A)-\sum_{n=2}^{\infty}\left\{(Bn-A)\lambda_n|a_n|+(Bn+A)\lambda_n|b_n|\right\}r^{n-1}}<1. \tag{2.7}$$

It is clear that the denominator of the left-hand side cannot vanish for $r \in (0,1)$. Moreover, it is positive for $r = 0$, and in consequence for $r \in (0,1)$. Thus, by (2.7) we have

$$\sum_{n=2}^{\infty}\left(\alpha_n|a_n|+\beta_n|b_n|\right)r^{n-1}<B-A \quad (0\leqslant r<1). \tag{2.8}$$

The sequence of partial sums $\{S_n\}$ associated with the series $\sum_{n=2}^{\infty}\left(\alpha_n|a_n|+\beta_n|b_n|\right)$ is a nondecreasing sequence. Moreover, by (2.8) it is bounded by $B-A$. Hence, the sequence $\{S_n\}$ is convergent and $\sum_{n=2}^{\infty}\left(\alpha_n|a_n|+\beta_n|b_n|\right)=\lim_{n\to\infty}S_n\leqslant B-A$, which yields assertion (2.2). □

3. Topological properties

We consider the usual topology on $\mathcal{H}$ defined by a metric in which a sequence $\{f_n\}$ in $\mathcal{H}$ converges to f if and only if it converges to f uniformly on each compact subset of $\mathbb{U}$. It follows from the theorems of Weierstrass and Montel that this topological space is complete.

Let $\mathcal{F}$ be a subclass of the class $\mathcal{H}$. A function $f\in\mathcal{F}$ is called *an extreme point of* $\mathcal{F}$ if the condition $f=\gamma f_1+(1-\gamma)f_2$ $(f_1,f_2\in\mathcal{F},\ 0<\gamma<1)$ implies $f_1=f_2=f$. We shall use the notation $E\mathcal{F}$ to denote the set of all extreme points of $\mathcal{F}$. It is clear that $E\mathcal{F}\subset\mathcal{F}$.

We say that $\mathcal{F}$ is *locally uniformly bounded* if for each r, $0<r<1$, there is a real constant $M=M(r)$ so that $|f(z)|\leqslant M$ $(f\in\mathcal{F},\ |z|\leqslant r)$.

We say that a class $\mathcal{F}$ is *convex* if $\gamma f+(1-\gamma)g\in\mathcal{F}$ $(f,g\in\mathcal{F},\ 0\leqslant\gamma\leqslant1)$. Moreover, we define *the closed convex* hull of $\mathcal{F}$ as the intersection of all closed convex subsets of $\mathcal{H}$ that contain $\mathcal{F}$. We denote the closed convex hull of $\mathcal{F}$ by $\overline{\mathrm{co}}\,\mathcal{F}$.

A real-valued functional $\mathcal{J}:\mathcal{H}\to\mathbb{R}$ is called *convex* on a convex class $\mathcal{F}\subset\mathcal{H}$ if

$$\mathcal{J}\big(\gamma f+(1-\gamma)g\big)\leqslant\gamma\mathcal{J}(f)+(1-\gamma)\mathcal{J}(g)\quad(f,g\in\mathcal{F},\ 0\leqslant\gamma\leqslant1).$$

The Krein–Milman theorem (see [**15**]) is fundamental in the theory of extreme points. In particular, it implies the following lemma.

Lemma 3.1. [**7**, p. 45] *Let $\mathcal{F}$ be a nonempty compact convex subclass of the class $\mathcal{H}$ and $\mathcal{J}:\mathcal{H}\to\mathbb{R}$ be a real-valued, continuous and convex functional on $\mathcal{F}$. Then* $\max\{\mathcal{J}(f):f\in\mathcal{F}\}=\max\{\mathcal{J}(f):f\in E\mathcal{F}\}$.

Since $\mathcal{H}$ is a complete metric space, Montel's theorem (see [**17**]) implies the following lemma.

Lemma 3.2. *A class $\mathcal{F}\subset\mathcal{H}$ is compact if and only if $\mathcal{F}$ is closed and locally uniformly bounded.*

Theorem 3.1. *The class $\mathcal{S}_{\mathcal{T}}^{m,\lambda}(A,B)$ is a convex and compact subset of $\mathcal{H}$.*

Proof. Let $f_1, f_2 \in \mathcal{S}_{\mathcal{T}}^{m,\lambda}(A,B)$ be functions of the form (2.1), $0 \leqslant \gamma \leqslant 1$. Since

$$\gamma f_1(z) + (1-\gamma) f_2(z) = z - \sum_{n=2}^{\infty} \big\{ \big(\gamma |a_{1,n}| + (1-\gamma)|a_{2,n}|\big) z^n - (-1)^m \big(\gamma |b_{1,n}| + (1-\gamma)|b_{2,n}|\big) \overline{z}^n \big\},$$

and by Theorem 2.3, we have

$$\begin{aligned}\sum_{n=2}^{\infty} \big\{ \alpha_n \big(\gamma |a_{1,n}| + (1-\gamma)|a_{2,n}|\big) + \beta_n \big(\gamma |b_{1,n}| + (1-\gamma)|b_{2,n}|\big) \big\} \\ = \gamma \sum_{n=2}^{\infty} \big\{ \alpha_n |a_{1,n}| + \beta_n |b_{1,n}| \big\} + (1-\gamma) \sum_{n=2}^{\infty} \big\{ \alpha_n |a_{2,n}| + \beta_n |b_{2,n}| \big\} \\ \leqslant \gamma (B-A) + (1-\gamma)(B-A) = B - A,\end{aligned}$$

the function $\phi = \gamma f_1 + (1-\gamma) f_2$ belongs to the class $\mathcal{S}_{\mathcal{T}}^{m,\lambda}(A,B)$. Hence, the class is convex. Furthermore, for $f \in \mathcal{S}_{\mathcal{T}}^{m,\lambda}(A,B)$, $|z| \leqslant r$, $0 < r < 1$, we have

$$|f(z)| \leqslant r + \sum_{n=2}^{\infty} \big(|a_n| + |b_n|\big) r^n < r + \sum_{n=2}^{\infty} \big(\alpha_n |a_n| + \beta_n |b_n|\big) \leqslant r + (B-A).$$

Thus, we conclude that the class $\mathcal{S}_{\mathcal{T}}^{m,\lambda}(A,B)$ is locally uniformly bounded. By Lemma 3.2, we only need to show that it is closed i.e., if $f_l \in \mathcal{S}_{\mathcal{T}}^{m,\lambda}(A,B)$ $(l \in \mathbb{N})$ and $f_l \to f$, then $f \in \mathcal{S}_{\mathcal{T}}^{m,\lambda}(A,B)$. Let f_l and f be given by (2.1) and (1.1), respectively. Using Theorem 2.3 we have

$$\sum_{n=2}^{\infty} \big(|\alpha_n a_{l,n}| + |\beta_n b_{l,n}|\big) \leqslant B - A \quad (l \in \mathbb{N}). \tag{3.1}$$

Since $f_l \to f$, we conclude that $|a_{l,n}| \to |a_n|$ and $|b_{l,n}| \to |b_n|$ as $l \to \infty$ $(n \in \mathbb{N})$. The sequence of partial sums $\{S_n\}$ associated with the series $\sum_{n=2}^{\infty} (\alpha_n |a_n| + \beta_n |b_n|)$ is a nondecreasing sequence. Moreover, by (3.1) it is bounded by $B - A$. Therefore, the sequence $\{S_n\}$ is convergent and $\sum_{n=2}^{\infty} \big(\alpha_n |a_n| + \beta_n |b_n|\big) = \lim_{n\to\infty} S_n \leqslant B - A$. This gives condition (2.2), and, in consequence, $f \in \mathcal{S}_{\mathcal{T}}^{m,\lambda}(A,B)$. □

Theorem 3.2. *We have* $E\mathcal{S}_{\mathcal{T}}^{m,\lambda}(A,B) = \{h_n :\ n \in \mathbb{N}\} \cup \{g_n :\ n \in \mathbb{N}_2\}$ *where*

$$h_1(z) = z,\ h_n(z) = z - \frac{B-A}{\alpha_n} z^n,\ g_n(z) = z + (-1)^m \frac{B-A}{\beta_n} \overline{z}^n \ \ (z \in \mathbb{U}). \tag{3.2}$$

Proof. Suppose that $0 < \gamma < 1$ and $g_n = \gamma f_1 + (1-\gamma) f_2$, where $f_1, f_2 \in \mathcal{S}_{\mathcal{T}}^{m,\lambda}(A,B)$ are functions of the form (2.1). Then, by (2.2) we have $|b_{1,n}| = |b_{2,n}| = \frac{B-A}{\beta_n}$, and, in consequence, $a_{1,l} = a_{2,l} = 0$ for $l \in \mathbb{N}_2$ and $b_{1,l} = b_{2,l} = 0$ for $l \in \mathbb{N}_2 \setminus \{n\}$. It follows that $g_n = f_1 = f_2$, and consequently $g_n \in E\mathcal{S}_{\mathcal{T}}^{*}(A,B)$. Similarly, we verify that the functions h_n of the form (3.2) are the extreme points of the class $\mathcal{S}_{\mathcal{T}}^{m,\lambda}(A,B)$. Now, suppose that a function f belongs to the set $E\mathcal{S}_{\mathcal{T}}^{m,\lambda}(A,B)$ and

f is not of the form (3.2). Then there exists $l \in \mathbb{N}_2$ such that $0 < |a_l| < \frac{B-A}{\alpha_l}$ or $0 < |b_l| < \frac{B-A}{\beta_l}$. If $0 < |a_l| < \frac{B-A}{\alpha_l}$, then putting

$$\gamma = \frac{\alpha_l |a_l|}{B-A}, \ \varphi = \frac{1}{1-\gamma}(f - \gamma h_l),$$

we have that $0 < \gamma < 1$, $h_l \neq \varphi$ and $f = \gamma h_l + (1-\gamma)\varphi$. Thus, $f \notin E\mathcal{S}_{\mathcal{T}}^{m,\lambda}(A,B)$. Similarly, if $0 < |b_l| < \frac{B-A}{\beta_l}$, then putting

$$\gamma = \frac{\beta_l |b_l|}{B-A}, \ \phi = \frac{1}{1-\gamma}(f - \gamma g_l),$$

we have that $0 < \gamma < 1$, $g_l \neq \phi$ and $f = \gamma g_l + (1-\gamma)\phi$. It follows that $f \notin E\mathcal{S}_{\mathcal{T}}^{m,\lambda}(A,B)$. □

4. Applications

It is clear that if the class $\mathcal{F} = \{f_n \in \mathcal{H} : n \in \mathbb{N}\}$ is locally uniformly bounded, then

$$\overline{\text{co}}\,\mathcal{F} = \left\{ \sum_{n=1}^{\infty} \gamma_n f_n : \sum_{n=1}^{\infty} \gamma_n = 1, \ \gamma_n \geqslant 0 \ (n \in \mathbb{N}) \right\}.$$

Thus, by Theorem 1.3 we have the following corollary.

Corollary 4.1.

$$\mathcal{S}_{\mathcal{T}}^{m,\lambda}(A,B) = \left\{ \sum_{n=1}^{\infty} (\gamma_n h_n + \delta_n g_n) : \sum_{n=1}^{\infty} (\gamma_n + \delta_n) = 1 \ (\delta_1 = 0, \ \gamma_n, \delta_n \geqslant 0) \right\},$$

where h_n, g_n *are defined by* (3.2).

For each fixed value of $n \in \mathbb{N}_2$, $z \in \mathbb{U}$, the following real-valued functions are continuous and convex on $\mathcal{H}$:

$$\mathcal{J}(f) = |a_n|, \ \mathcal{J}(f) = |b_n|, \ \mathcal{J}(f) = |f(z)|, \ \mathcal{J}(f) = |D_{\mathcal{H}} f(z)| \quad (f \in \mathcal{H}).$$

Moreover, for $\gamma \geqslant 1$, $0 < r < 1$, the real-valued functional

$$\mathcal{J}(f) = \left(\frac{1}{2\pi} \int_0^{2\pi} \left| f\left(re^{i\theta}\right)\right|^{\gamma} d\theta \right)^{1/\gamma} \quad (f \in \mathcal{H})$$

is also continuous and convex on $\mathcal{H}$.

Therefore, by Lemma 3.1 and Theorem 1.3 we have the following corollaries.

Corollary 4.2. *Let* $f \in \mathcal{S}_{\mathcal{T}}^{m,\lambda}(A,B)$, $|z| = r < 1$. *Then*

$$r - \frac{B-A}{(\lambda+m+1)(1+2B-A)} r^2 \leqslant |f(z)| \leqslant r + \frac{B-A}{(\lambda+m+1)(1+2B-A)} r^2,$$

$$r - \frac{B-A}{(1+2B-A)} r^2 \leqslant \left| D_{\mathcal{H}}^{m,\lambda} f(z) \right| \leqslant r + \frac{B-A}{(1+2B-A)} r^2,$$

The result is sharp. The function h_2 *of the form* (3.2) *is the extremal function.*

Corollary 4.3. *Let* $f \in \mathcal{S}_{\mathcal{T}}^{m,\lambda}(A,B)$ *be a function of the form* (2.6). *Then* $|a_n| \leqslant \frac{B-A}{\alpha_n}$, $|b_n| \leqslant \frac{B-A}{\beta_n}$ $(n \in \mathbb{N}_2)$, *where* α_n, β_n *are defined by* (2.3). *The result is sharp. The functions* h_n, g_n *of the form* (3.2) *are the extremal functions.*

Corollary 4.4. *Let* $0 < r < 1$, $\gamma \geqslant 1$. *If* $f \in \mathcal{S}_{\mathcal{T}}^{m,\lambda}(A,B)$, *then*

$$\frac{1}{2\pi}\int_0^{2\pi} \left|f(re^{i\theta})\right|^{\gamma} d\theta \leqslant \frac{1}{2\pi}\int_0^{2\pi} \left|h_2(re^{i\theta})\right|^{\gamma} d\theta,$$
$$\frac{1}{2\pi}\int_0^{2\pi} \left|D_{\mathcal{H}}f(z)\right|^{\gamma} d\theta \leqslant \frac{1}{2\pi}\int_0^{2\pi} \left|D_{\mathcal{H}}h_2(re^{i\theta})\right|^{\gamma} d\theta,$$

where h_2 *is the function defined by* (3.2).

The following covering result follows from Corollary 4.2.

Corollary 4.5. *If* $f \in \mathcal{S}_{\mathcal{T}}^{m,\lambda}(A,B)$, *then* $\mathbb{U}(r) \subset f(\mathbb{U})$, *where*

$$r = 1 - \frac{B-A}{(\lambda+m+1)(1+2B-A)}.$$

The class $\mathcal{S}_{\mathcal{H}}^{m}(A,B)$ is related to harmonic starlike functions, harmonic convex functions and harmonic Janowski functions.

The classes $\mathcal{S}_{\mathcal{H}}(\alpha) := \mathcal{S}_{\mathcal{H}}^{0}(2; 2\alpha-1, 1)$ and $\mathcal{K}_{\mathcal{H}}(\alpha) := \mathcal{K}_{\mathcal{H}}(2; 2\alpha-1, 1)$ were investigated by Jahangiri [**10**] (see also [**2**, **19**]). They are the classes of starlike and convex functions of order α, respectively. Finally, the classes $\mathcal{S}_{\mathcal{H}} := \mathcal{S}_{\mathcal{H}}(0)$ and $\mathcal{K}_{\mathcal{H}} := \mathcal{K}_{\mathcal{H}}(0)$ are the classes of functions which are starlike and convex in $\mathbb{U}(r)$, for all $r \in (0,1\rangle$. We should notice, that the classes $\mathcal{S}(A,B) := \mathcal{S}_{\mathcal{H}}(A,B) \cap \mathcal{A}$ and $\mathcal{K}(A,B) := \mathcal{K}_{\mathcal{H}}(A,B) \cap \mathcal{A}$ are introduced by Janowski [**13**].

The class $\mathcal{S}_{\mathcal{H}}^{m}(A,B)$ generalize also classes of starlike functions of complex order. The class $\mathcal{CS}_{\mathcal{H}}(\gamma) := \mathcal{S}_{\mathcal{H}}(1-2\gamma, 1)$, $\gamma \in \mathbb{C} \setminus \{0\}$, was defined by Yalçin and Öztürk [**24**]. In particular, if we put $\gamma := \frac{1-\alpha}{1+e^{i\eta}}$, $\eta \in \mathbb{R}$, then we obtain the class $\mathcal{RS}_{\mathcal{H}}(\alpha,\eta) := \mathcal{S}_{\mathcal{H}}\big(\frac{2\alpha-1+e^{i\eta}}{1+e^{i\eta}}, 1\big)$ studied by Yalçin et al. [**25**]. It is the class of functions $f \in \mathcal{H}_0$ such that $\operatorname{Re}\big\{(1+e^{i\eta})\frac{D_{\mathcal{H}}f(z)}{f(z)} - e^{i\eta}\big\} > \alpha \quad (z \in \mathbb{U})$.

Applying the obtained results to the classes defined above, we can obtain new and also well-known results (see for example [**1**–**3**, **5**–**14**, **19**, **21**–**25**].

Remark 4.1. The results obtained in classes of harmonic functions can be transferred to corresponding classes of analytic functions.

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References

1. O. P. Ahuja, *Connections between various subclasses of planar harmonic mappings involving hypergeometric functions*, Appl. Math. Comput. **198**(2008), 305–316.
2. O. P. Ahuja, J. M. Jahangiri, *Certain multipliers of univalent harmonic functions*, Appl. Math. Lett. **18** (2005), 1319–1324.
3. H. A. Al-Kharsani, R. A. Al-Khal, *Univalent harmonic functions*, JIPAM. J. Inequal. Pure Appl. Math. **8**(2) (2007), Article 59, 8 pp.

4. J. Clunie, T. Sheil Small, *Harmonic univalent functions*, Ann. Acad. Sci. Fenn. Ser. A I Math. **9** (1984), 3–25.
5. M. Darus, K. Al-Shaqsi, *K. On certain subclass of harmonic univalent functions*, J. Anal. Appl. **6** (2008), 17–28.
6. J. Dziok, *Classes of harmonic functions associated with Ruscheweyh derivatives*, Rev. R. Acad. Cienc. Exactas Fís. Nat. Ser. A Mat. RACSAM **113**(2) (2019), 1315–1329.
7. ———, *On Janowski harmonic functions*, J. Appl. Anal. **21**(2) (2015), 99–107.
8. J. Dziok, J. M. Jahangiri, H. Silverman, *Harmonic functions with varying coefficients*, J. Inequal. Appl. **139** (2016), pages.
9. J. Dziok, M. Darus, J. Sokół, T. Bulboaca, *Generalizations of starlike harmonic functions*, C. R. Math. Acad. Sci. Paris **354** (2016), 13–18.
10. J. M. Jahangiri, *Harmonic functions starlike in the unit disk*, J. Math. Anal. Appl. **235** (1999), 470–477.
11. J. M. Jahangiri, G. Murugusundaramoorthy, K. Vijaya, *Starlikeness of harmonic functions defined by Ruscheweyh derivatives*, J. Indian Acad. Math. **26** (2004), 191–200.
12. J. M. Jahangiri, H. Silverman, *Harmonic univalent functions with varying arguments*, Int. J. Appl. Math. **8**, 267–275 (2002).
13. W. Janowski, *Some extremal problems for certain families of analytic functions I*, Ann. Polon. Math. **28** (1973), 297–326.
14. S. Y. Karpuzoğulları, M. Öztürk, M. Yamankaradeniz, *A subclass of harmonic univalent functions with negative coefficients*, Appl. Math. Comput. **142**(2-3) (2003), 469–476.
15. M. Krein, D. Milman, *On the extreme points of regularly convex sets*, Studia Mathematica **9** (1940), 133–138.
16. H. Lewy, *On the non-vanishing of the Jacobian in certain one-to-one mappings*, Bull. Amer. Math. Soc. **42** (1936), 689–692.
17. P. Montel, *Sur les families de functions analytiques qui admettent des valeurs exceptionelles dans un domaine*, Ann. Sci. Ecole Norm. Sup. **23** (1912), 487–535.
18. G. Murugusundaramoorthy, K. Vijaya, *On certain classes of harmonic functions involving Ruscheweyh derivatives*, Bull. Calcutta Math. Soc. **96** (2004), 99–108.
19. M. Öztürk, S. Yalçın, M. Yamankaradeniz, *Convex subclass of harmonic starlike functions*, Appl. Math. Comput. **154** (2004), 449–459.
20. S. Ruscheweyh, *New criteria for univalent functions*, Proc. Amer. Math Soc. **49** (1975), 109–115.
21. H. Silverman, *Harmonic univalent functions with negative coefficients*, J. Math. Anal. Appl. **220** (1998), 283-289.
22. S. Yalçın, *A new class of Sălăgean-type harmonic univalent functions*, Appl. Math. Lett. **18** (2005), 191–198.
23. S. Yalçın, S. B. Joshi, E. Yaşar, *On certain subclass of harmonic univalent functions defined by a generalized Ruscheweyh derivatives operator*, Appl. Math. Sci. (Ruse) **4** (2010), 327–336.
24. S. Yalçın, M. Öztürk, *Harmonic functions starlike of the complex order*, Mat. Vesnik **58** (2006), 7–11.
25. S. Yalçın, M. Öztürk, M. Yamankaradeniz, *On the subclass of Sălăgean-type harmonic univalent functions*, JIPAM. J. Inequal. Pure Appl. Math. **8** (2007), Article 54, 9 pp.

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