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NUMERICAL SOLUTION OF A FREE SURFACE FLOW PROBLEM OVER AN OBSTACLE

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ABSTRACT. We consider a free surface flow problem of an incompressible and inviscid fluid, perturbed by a topography placed on the bottom of a channel. We suppose that the flow is steady, bidimensional and irrotational. We neglect the effects of the superficial tension but we take into account the gravity acceleration g. The main unknown of our problem is the equilibrium free surface of the flow; its determination is based on the Bernoulli equation which is transformed as the forced Korteweg–de Vries equation. The problem is solved numerically via the fourth-order Runge–Kutta method for the subcritical case, and the finite difference method for the supercritical case. The results obtained are illustrated by several figures, where the height h of the obstacle, and the value of the Froude number F of the flow, are varied. Note that different shapes of the obstacle have been considered.

1. Introduction

There have been several studies on two-dimensional surface waves generated over a localized obstacle on the bottom of a channel, with a presence of a uniform stream. They have received much attention over many decades, especially when the effects of surface tension are ignored; in this case, we consider pure gravity waves. Note that the principle parameters describing the generated waves, are the Froude number $F = \frac{U}{\sqrt{gH}}$ and the height h of the obstacle. Here U is the velocity of the uniform stream, H is the undisturbed water depth, and g is the gravity acceleration, we can see Fig. 1

The flow is said to be supercritical when $F > 1$ and it is called subcritical if $F < 1$. Linearized theory gives a good prediction for a small amplitude wave, when F is not close to unity. However the linear model breaks down as $F \to 1$. Here Akylas [**2**] developed a weakly nonlinear theory to study the resonant generation of surface gravity waves. Several authors have studied a 2-D flow over a bump on the bottom of a channel, for example, in [**3**, **4**], the authors solved the problem

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numerically, for a semicircular bump. In [**6**], a numerical study has been done for a trapezoidal obstacle. In [**9**], the authors discussed the accuracy of the stationary forced Korteweg–de Vries equation as a model equation for flow over a bump. In [**1**], the authors have studied a free surface flow problem over an obstacle, when the fluid is viscous and the bottom of the channel is inclined with an angle α with respect to the horizontal. Many other authors have studied the free surface flow problem over an obstacle, we can see [**7**, **8**, **10**]. In [**5**], the authors have considered the same problem when both effects of the gravity and of the superficial tension are taken into account.

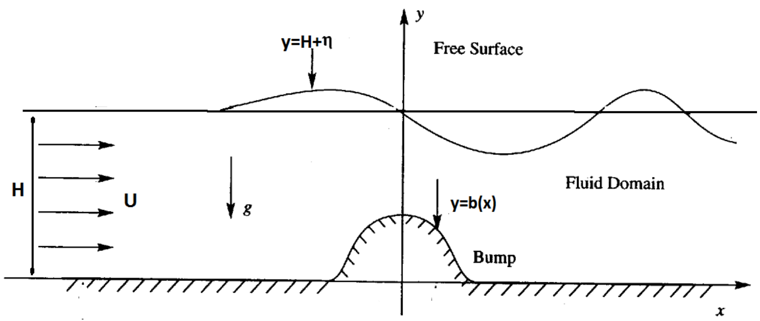

FIGURE 1. Sketch of the flow configuration

In this work, we consider a free surface flow over an obstacle placed on a flat bottom of a channel. The flow is bidimensional and irrotational. The fluid is inviscid and incompressible with constant density ρ. We take into account the gravity g, but we neglect the effects of the superficial tension. Let L be the typical wave length, we are interested in the behaviour of long waves of the fluid motion, for this, we define a small parameter $\varepsilon : 0 < \varepsilon = \left(\frac{H}{L}\right)^2 \ll 1$. We denote by $y = b(x)$ and $y = H + \eta(x,t)$, the profiles of the bottom of the channel and the free surface of the flow, respectively. The functions "b" and "η" are in $C^1(\mathbb{R})$, with compact support included in $\mathbb{R}$. Note that "η" is the principle unknown of our problem, due to the presence of the obstacle. We will see that to find "η", we solve numerically a second order nonlinear ordinary differential equation, which is established from the Euler equations of fluid motion and corresponding boundary conditions.

This paper is organized as follows: In section two, we give the formulation of the problem, where we put the unsteady fully nonlinear equations and we do their nondimensionalization. In section three, we deduce the unsteady forced Korteweg–de Vries equation, which is solved numerically in the fourth section, for the steady case.

We finish this work by a conclusion where we discuss our results obtained in this paper.

2. Formulation of the problem

In this section, we put the governing equations with the potential velocity of the fluid and we do their nondimensionalization.

2.1. Fully nonlinear unsteady equations. In this subsection, we put the governing equations of a free surface flow, perturbed by an obstacle lying on the bottom of a channel. The flow is bidimensional and irrotational, the fluid is perfect and incompressible. We take into account the gravity acceleration g and we neglect the effects of the superficial tension. Here we consider that the flow is not stationary, but in the numerical resolution, the time is neglected; we will see this in the fourth section. Since the flow is irrotational and the fluid is incompressible, the governing equation in the fluid domain is:

$$\Delta\Phi = 0, \quad x \in \mathbb{R}, \quad b(x) < y < H + \eta(x,t), \tag{2.1}$$

where Φ is the velocity potential of the fluid and $\Delta = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}$ is the two-dimensional Laplacian.

The free surface of the flow and the bottom of the channel are streamlines, it means that any fluid particle on the free surface or on the bottom, always stays on this line. So we write the kinematic condition on the free surface and on the bottom, it is given by

$$\frac{D}{Dt}(y - H - \eta) = 0 \quad \text{on} \;\; y = H + \eta(x,t), \tag{2.2}$$

$$\frac{D}{Dt}(y - b) = 0 \quad \text{on} \;\; y = b(x), \tag{2.3}$$

where $\frac{D}{Dt}$ is the material derivative operator.

Since the free surface is not known ($\eta(x,t)$ is an unknown of the problem), so we write a second condition on $y = H+\eta(x,t)$, it is given by the Bernoulli equation:

$$\rho\frac{\partial\Phi}{\partial t} + \frac{\rho}{2}|(\nabla\Phi)^2| + g\rho(H+\eta) = \frac{\rho U^2}{2} + \rho g H \quad \text{on} \quad y = H + \eta(x,t), \tag{2.4}$$

ρ is the density of the fluid and U is the speed of the upstream flow.

Equations (2.2) and (2.3) can be written, respectively, as

$$-\eta_t - \eta_x\Phi_x + \Phi_y = 0 \quad \text{on} \quad y = H + \eta(x,t), \tag{2.5}$$

$$b_x\Phi_x - \Phi_y = 0 \quad \text{on} \quad y = b(x), \tag{2.6}$$

where η_t, Φ_x and Φ_y are the derivatives with respect to t, x and y of η and Φ respectively.

2.2. The governing equations with the perturbation velocity potential. If $b(x) = 0$, then $(\Phi, \eta) = (Ux, 0)$ is a solution of (2.1)–(2.4); so we look for a solution around $(Ux, 0)$. We put $\Phi(x,y,t) = Ux + \varphi(x,y,t)$, then the system (2.1), (2.4)–(2.6) becomes

$$\Delta\varphi = 0, \qquad x \in \mathbb{R}, \quad b(x) < y < H + \eta(x,t), \tag{2.7}$$

$$-\frac{\partial\eta}{\partial t} - \eta_x(U + \varphi_x) + \varphi_y = 0, \qquad x \in \mathbb{R}, \quad y = H + \eta(x,t), \tag{2.8}$$

$$\frac{\partial \varphi}{\partial t} + \frac{1}{2}(\nabla\varphi)^2 + g\eta + U\varphi_x = 0, \qquad x \in \mathbb{R}, \quad y = H + \eta(x,t), \tag{2.9}$$

$$(U + \varphi_x)b_x - \varphi_y = 0, \qquad x \in \mathbb{R}, \quad y = b(x). \tag{2.10}$$

2.3. Nondimensionalization of the governing equations. Choosing H and L (L is the typical wave length), as a caracteristic values, the following dimensionless variables are introduced :

$$x^* = \frac{x}{L}, \qquad y^* = \frac{y}{H}, \qquad t^* = \varepsilon^{3/2}\sqrt{\frac{g}{H}}t, \qquad \eta^* = \frac{\eta}{H},$$

$$\varphi^* = \frac{\varphi}{L\sqrt{gH}}, \qquad b^* = \varepsilon^{-2}\frac{b}{H}, \qquad U^* = \frac{U}{\sqrt{gH}} = F,$$

ε is a small parameter given in the introduction. In this way, we nondimensionalize system (2.7)–(2.10), we neglect the symbol $*$, so we obtain the following dimensionless equations:

$$\varepsilon\varphi_{xx} + \varphi_{yy} = 0, \quad x \in \mathbb{R}, \quad \varepsilon^2 b(x) < y < 1 + \eta(x,t), \tag{2.11}$$

$$\varepsilon\eta_t + (F + \varphi_x)\eta_x = \varepsilon^{-1}\varphi_y, \quad \text{on} \quad y = 1 + \eta(x,t), \tag{2.12}$$

$$\varepsilon\varphi_t + \tfrac{1}{2}(\varphi_x^2 + \varepsilon^{-1}\varphi_y^2) + F\varphi_x + \eta = 0, \quad \text{on} \quad y = 1 + \eta(x,t), \tag{2.13}$$

$$(F + \varphi_x)b_x = \varepsilon^{-3}\varphi_y, \quad \text{on} \quad y = \varepsilon^2 b(x). \tag{2.14}$$

3. Derivation of the forced Kortweg–de Vries equation (fKdV)

In this section, using Taylor expansion about $y = 1$, about $y = 0$ and asymptotic expansion of φ and η, we deduce the forced Korteweg–de Vries equation.

3.1. Formulation of the problem (2.11)–(2.14) **in the fixed domain** $\mathbb{R} \times [\mathbf{0}, \mathbf{1}]$**.** If we consider $\eta(x,t)$ small, then boundary conditions (2.12)–(2.13) on $y = 1 + \eta(x,t)$, can be approximated by their Taylor expansion about $y = 1$. It follows that

$$\varepsilon\eta_t + (F + \varphi_x + \varphi_{xy}\eta)\eta_x = \varepsilon^{-1}\varphi_y + \varepsilon^{-1}\varphi_{yy}\eta, \quad \text{on} \quad y = 1, \tag{3.1}$$

$$\begin{aligned} &\varepsilon\varphi_t + \varepsilon\varphi_{ty}\eta + \tfrac{1}{2}(\varphi_x^2 + \varepsilon^{-1}\varphi_y^2) + \varphi_x\varphi_{xy}\eta \\ &\quad + \varepsilon^{-1}\varphi_y\varphi_{yy}\eta + F\varphi_x + F\varphi_{xy}\eta + \eta = 0, \quad \text{on} \quad y = 1. \end{aligned} \tag{3.2}$$

Also, the Taylor expansion of (2.14) about $y = 0$ results in

$$(F + \varphi_x + \varepsilon^2\varphi_{xy}b)b_x = \varepsilon^{-3}\varphi_y + \varepsilon^{-1}\varphi_{yy}b, \quad \text{on} \quad y = 0. \tag{3.3}$$

We assume that the speed of the upstream flow is near the critical one:

$$F = F_0 + \varepsilon F_1 + O(\varepsilon^2), \tag{3.4}$$

F_0 is the critical speed of the upstream uniform flow which will be determined. F_1 is a measurement of the perturbation of the upstream uniform velocity F, from the critical value F_0.

3.2. Asymptotic expansions. If we know the two functions $\varphi(x,y,t,\varepsilon)$ and $\eta(x,t,\varepsilon)$, then we know everything about the flow and the wave. Note that it is impossible to find exactly these functions φ and η. For this, we proceed to asymptotic approximations. To calculate an asymptotic approximations of these two functions, we suppose that we can write:

$$\varphi = \varepsilon\varphi_1 + \varepsilon^2\varphi_2 + \varepsilon^3\varphi_3 + O(\varepsilon^4), \tag{3.5}$$

$$\eta = \varepsilon\eta_1 + \varepsilon^2\eta_2 + O(\varepsilon^3). \tag{3.6}$$

Substituting (3.4)–(3.6) into (2.11), (3.1)–(3.3) and rearranging the terms with respect to the powers of ε, we obtain a sequence of boundary value problems. The problems of the first three lowest orders are given by:

Lowest order:

$$\begin{aligned} \varphi_{1yy} &= 0 \quad \text{in} \quad 0<y<1,\\ \tfrac{1}{2}\varphi_{1y}^2 + F_0\varphi_{1x} + \eta_1 &= 0 \quad \text{on} \quad y=1,\\ \varphi_{1y} &= 0 \quad \text{on} \quad y=1,\\ \varphi_{1y} &= 0 \quad \text{on} \quad y=0. \end{aligned} \tag{3.7}$$

First order:

$$\varphi_{1xx} + \varphi_{2yy} = 0 \quad \text{in} \quad 0<y<1,$$

$$\begin{aligned} &\varphi_{1t} + \tfrac{1}{2}\varphi_{1x}^2 + \varphi_{1y}\varphi_{2y} + \varphi_{1y}\varphi_{1yy}\eta_1\\ &+ F_0\varphi_{2x} + F_1\varphi_{1x} + F_0\varphi_{1xy}\eta_1 + \eta_2 = 0 \quad \text{on} \quad y=1, \end{aligned} \tag{3.8}$$

$$F_0\eta_{1x} - \varphi_{2y} - \varphi_{1yy}\eta_1 = 0 \quad \text{on} \quad y=1, \tag{3.9}$$

$$\varphi_{2y} = 0 \quad \text{on} \quad y=0.$$

Second order :

$$\varphi_{2xx} + \varphi_{3yy} = 0 \quad \text{in} \quad 0<y<1, \tag{3.10}$$

$$\begin{aligned} &\eta_{1t} + F_0\eta_{2x} + F_1\eta_{1x} + \varphi_{1x}\eta_{1x}\\ &\quad - \varphi_{3y} - \varphi_{1yy}\eta_2 - \eta_1\varphi_{2yy} = 0 \quad \text{on} \quad y=1, \end{aligned} \tag{3.11}$$

$$F_0 b_x - \varphi_{3y} - \varphi_{1yy} b = 0 \quad \text{on} \quad y=0. \tag{3.12}$$

3.3. Deduction of the forced Korteweg–de Vries equation. From integration of lowest order problem (3.7), it is deduced that φ_1 is a function of only x and t, it is independent of y, so: $\varphi_1 = \varphi_1(x,t)$. Also, from (3.7), we have

$$\varphi_{1x} = -\frac{\eta_1(x,t)}{F_0}. \tag{3.13}$$

From the first order problem, we get $\varphi_2 = -\frac{1}{2}\varphi_{1xx}y^2 + G(x,t)$, where $G(x,t)$ is an arbitrary function. By (3.13), we get

$$\varphi_2 = \frac{1}{2F_0}\eta_{1x}y^2 + G(x,t). \tag{3.14}$$

Using (3.8), we obtain:

$$G_x = -\frac{\varphi_{1t}}{F_0} - \frac{1}{2F_0^3}\eta_1^2 - \frac{1}{2F_0}\eta_{1xx} + \frac{F_1}{F_0^2}\eta_1 - \frac{\eta_2}{F_0}, \tag{3.15}$$

then

$$\varphi_{2x} = \frac{1}{2F_0}\eta_{1xx}y^2 - \frac{\varphi_{1t}}{F_0} - \frac{1}{2F_0^3}\eta_1^2 - \frac{1}{2F_0}\eta_{1xx} + \frac{F_1}{F_0^2}\eta_1 - \frac{\eta_2}{F_0}. \tag{3.16}$$

From (3.9) and (3.14), it follows $(F_0^2 - 1)\eta_{1x} = 0$. For a nontrivial solution η_{1x}, we have $\eta_{1x} \neq 0$, so

$$F_0^2 = 1, \tag{3.17}$$

and the critical speed of the upstream flow is one.

Integrating (3.10) with respect to y from $y = 0$ to $y = 1$ and using the boundary conditions (3.11)–(3.12), we obtain

$$\begin{aligned} \eta_{1t} + F_0\eta_{2x} + F_1\eta_{1x} + \varphi_{1x}\eta_{1x} - \eta_1\varphi_{2yy} - F_0 b_x \\ = -\Big[\frac{\eta_{1xxx}}{6F_0} - \frac{\varphi_{1tx}}{F_0} - \frac{(\eta_1^2)_x}{2F_0^3} - \frac{\eta_{1xxx}}{2F_0} + \frac{F_1\eta_{1x}}{F_0^2} - \frac{\eta_{2x}}{F_0}\Big]. \end{aligned} \tag{3.18}$$

Finally, it results

$$\eta_{1t} + F_1\eta_{1x} - \tfrac{3}{2}\eta_1\eta_{1x} - \tfrac{1}{6}\eta_{1xxx} = \tfrac{1}{2}b_x. \tag{3.19}$$

Equation (3.19) is called the forced Korteweg–de Vries equation(fKdV).

In the following of this work, we neglect η_{1t} in (3.19), so we consider the steady equation:

$$F_1\eta_{1x} - \tfrac{3}{2}\eta_1\eta_{1x} - \tfrac{1}{6}\eta_{1xxx} = \tfrac{1}{2}b_x, \quad x \in \mathbb{R}. \tag{3.20}$$

The unknown function $\eta_1(x)$ represents the first order elevation of the free surface of the fluid. Note that when the flow is supercritical, far the obstacle, upstream and downstream, η_1 tends to zero. However, if the flow is subcritical, η_1 and η_{1x} tend to zero, upstream, far the obstacle.

Therefore the solution of equation (3.20), is equivalent to solve the second order nonlinear ordinary differential equation:

$$F_1\eta_1 - \tfrac{3}{4}\eta_1^2 - \tfrac{1}{6}\eta_{1xx} = \tfrac{1}{2}b, \quad x \in \mathbb{R}, \tag{3.21}$$

with

$$\begin{aligned} \lim_{x\to-\infty}\eta_1(x) &= \lim_{x\to+\infty}\eta_1(x) = 0 \quad \text{if} \quad F_1 > 0 \text{ (supercritical flow)}, \\ \lim_{x\to-\infty}\eta_1(x) &= \lim_{x\to-\infty}\eta_{1x}(x) = 0 \quad \text{if} \quad F_1 < 0 \text{ (subcritical flow)}. \end{aligned}$$

For a given bump b and a given upstream near the critical flow speed $F = 1 + \varepsilon F_1$, we can find an asymptotically approximate shape of the free surface $y = 1 + \varepsilon\eta_1(x) + O(\varepsilon^2)$. Since we solve numerically equation (3.21), so we take x in $[-l, l]$, $l \in \mathbb{R}$, then $\eta_1(-l) = \eta_1(l) = 0$ for $F > 1$ and $\eta_1(-l) = \eta_{1x}(-l) = 0$ for $F < 1$.

In the following section, we describe the numerical method which is used for solving equation (3.21), and we present our results in different figures.

4. Numerical procedure and results

In this section, we solve numerically steady equation (3.21), in two situations; first in the supercritical case ($F_1 > 0$ or $F > 1$), with the finite difference method, second in the subcritical case ($F_1 < 0$ or $F < 1$), with the fourth order Runge–Kutta method. We discretize the interval $[-l, l]$ into non-homogeneous subspaces with end points

$$x_j = -l + jp, \qquad j = 0, \ldots, N+1, \quad N \in \mathbb{N}^* \quad \text{and} \quad p = \frac{2l}{N+1}.$$

4.1. Supercritical case. We choose N as a large number, so that x_0 and x_{N+1} are far from the crest of the obstacle. Using the notation $\eta_{1j} \approx \eta_1(x_j)$, we solve equation (3.21) by approximating

$$\eta_{1xx}(x_j) \approx \frac{\eta_{1j+1} - 2\eta_{1j} + \eta_{1j-1}}{p^2},$$

so we obtain an equation for each j, so that there are N equations for determining N unknowns $\eta_{11}, \eta_{12}, \ldots \eta_{1N}$.

As it has been said after (3.20), since the flow is supercritical, two boundary conditions of η_1 are required at $j = 0$ and $j = N+1$. These are set $\eta_{10} = \eta_1(-l) = 0$ and $\eta_{1N+1} = \eta_1(l) = 0$, representing a uniform stream. So for a given F_1 and a fixed obstacle, a system of non-linear equations is constructed, and solved by Newton's method. The free surface of the flow is plotted from the values of (x_j, η_{1j}). In the following figures, we plot the free surface flow obtained from different values of $F(F = 1 + \varepsilon F_1)$, the height h and different types of the obstacle. To simplify notation in figure captions, we put $b(h,x) = \begin{cases} h\cos(\frac{\pi}{2}x) & \text{if } |x| \leqslant 1, \\ 0 & \text{else.} \end{cases}$

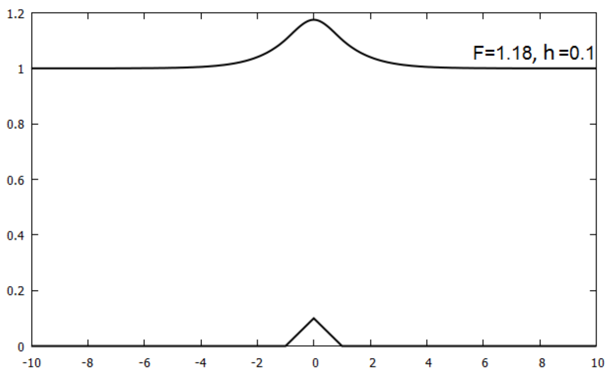

FIGURE 2. Free surface flow profile over a triangular bump in case $F = 1.18$ and $h = 0.1$.

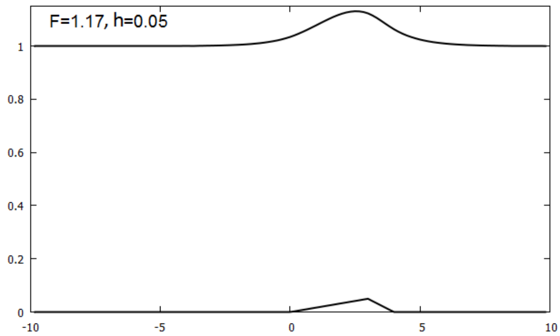

FIGURE 3. Free surface flow profile over a non symmetric triangular bump in case $F = 1.17$ and $h = 0.05$.

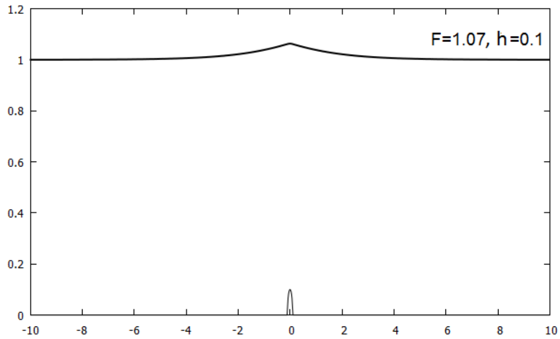

FIGURE 4. Free surface flow profile over a semi-circular bump with radius $h = 0.1$ and $F = 1.07$.

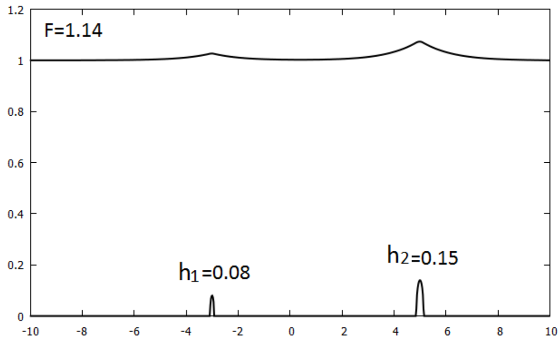

FIGURE 5. Free surface flow profile over two semi-circular bumps with radius $h_1 = 0.08$ and $h_2 = 0.15$ in case $F = 1.14$.

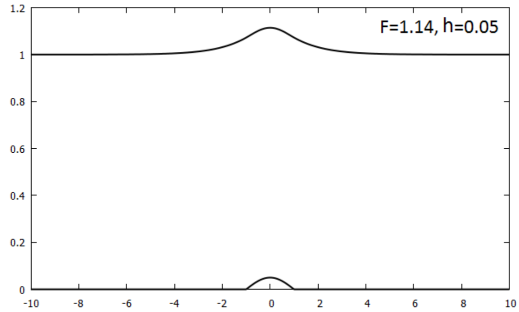

FIGURE 6. Free surface flow profile over a bump $b(h, x)$ for $F = 1.14$ and $h = 0.05$.

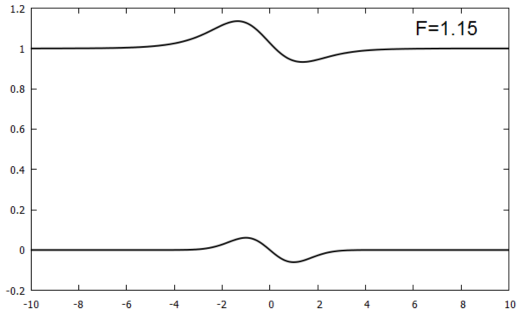

FIGURE 7. The behaviour of the free surface flow profile for $b(x) = e^{-\frac{1}{2}(x+0.05)^2} - e^{-\frac{1}{2}(x-0.05)^2}$ and $F = 1.15$.

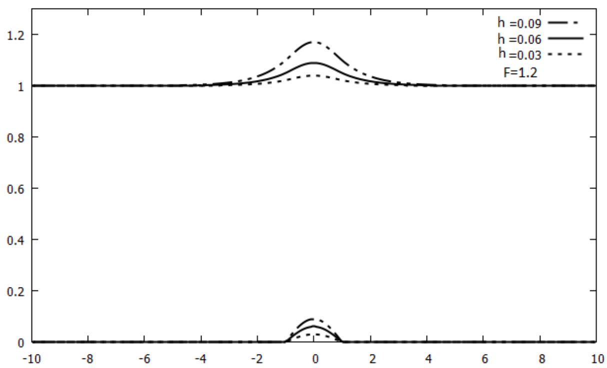

FIGURE 8. Plot of the free surface profile over a bump $b(h, x)$ for different heights h of the obstacle and the Froude number fixed.

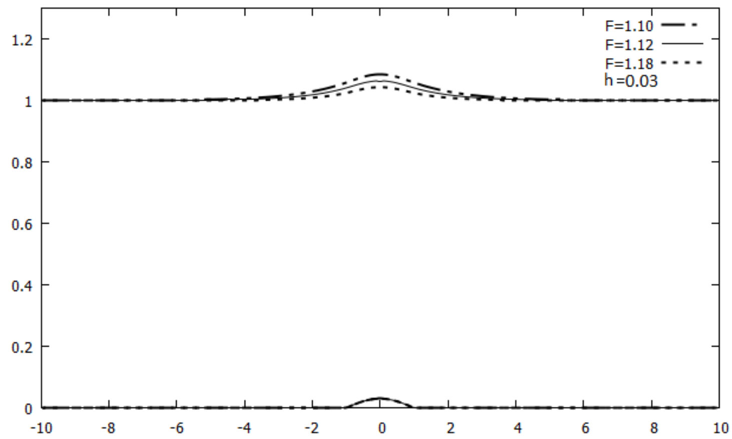

FIGURE 9. Plot of the free surface profile over a bump $b(h, x)$ for different values of the Froude number F and h fixed.

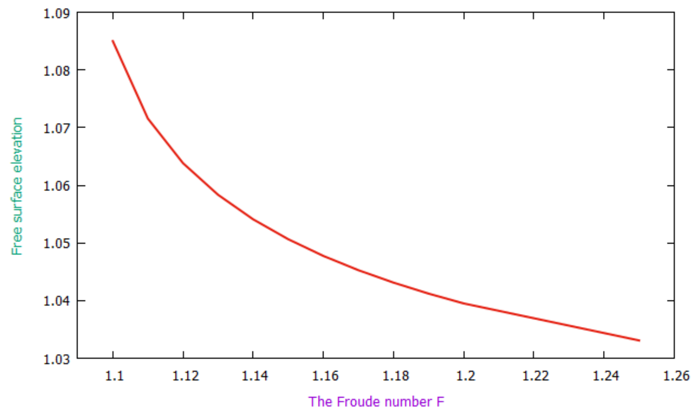

FIGURE 10. Evolution of the free surface elevation, using $b(h, x)$.

4.2. Subcritical case. In this case, η_1 satisfies $\eta_1(-l) = \eta_1'(-l) = 0$. We put $z_1 = \eta_1$ and $z_2 = \eta_1' = \eta_{1x}$; then $z_1(-l) = z_2(-l) = 0$ and it results the first-order system with the unknowns z_1 and z_2:

$$\begin{aligned} z_1' &= \eta_1' = z_2 = f_1(x, z_1, z_2) \\ z_2' &= \eta_1'' = \eta_{1xx} = 6F_1 z_1 - \tfrac{9}{2} z_1^2 - 3b(x) = f_2(x, z_1, z_2). \end{aligned} \tag{4.1}$$

For the expression of z_2', we can see (3.21).

As in the supercritical case, we put $p = \frac{2l}{N+1}$ and consider the discretization of $[-l, l]$: $x_i = -l + ip$, $i = 0, \ldots, N+1$. Using the notation $z_{1,j} \approx z_1(x_j)$, and $z_{2,j} \approx z_2(x_j)$, the fourth order Runge–Kutta method applied to the system (4.1), with the initial condition $z_1(-l) = z_2(-l) = 0$, gives the following equations:

$$\begin{aligned} z_{1,i+1} &= z_{1,i} + \frac{p}{6}(k_{11} + 2k_{21} + 2k_{31} + k_{41}) \\ z_{2,i+1} &= z_{2,i} + \frac{p}{6}(k_{12} + 2k_{22} + 2k_{32} + k_{42}), \end{aligned} \quad i = 0, \dots, N$$

where $z_{1,0} = z_1(-l) = 0$ and $z_{2,0} = z_2(-l) = 0$, with

$$\begin{aligned} k_{11} &= f_1(x_i, z_{1,i}, z_{2,i}), \quad k_{12} = f_2(x_i, z_{1,i}, z_{2,i}), \\ k_{21} &= f_1\Big(x_i + \frac{p}{2}, z_{1,i} + \frac{p}{2}k_{11}, z_{2,i} + \frac{p}{2}k_{12}\Big), \\ k_{22} &= f_2\Big(x_i + \frac{p}{2}, z_{1,i} + \frac{p}{2}k_{11}, z_{2,i} + \frac{p}{2}k_{12}\Big), \\ k_{31} &= f_1\Big(x_i + \frac{p}{2}, z_{1,i} + \frac{p}{2}k_{21}, z_{2,i} + \frac{h}{2}k_{22}\Big), \\ k_{32} &= f_2\Big(x_i + \frac{p}{2}, z_{1,i} + \frac{p}{2}k_{21}, z_{2,i} + \frac{p}{2}k_{22}\Big), \\ k_{41} &= f_1(x_i + p, z_{1,i} + pk_{31}, z_{2,i} + pk_{32}), \\ k_{42} &= f_2(x_i + p, z_{1,i} + pk_{31}, z_{2,i} + pk_{32}). \end{aligned}$$

In the following figures, we plot the free surface flow profile, obtained from different values of F, h, and different types of the obstacle described by $y = b(x)$.

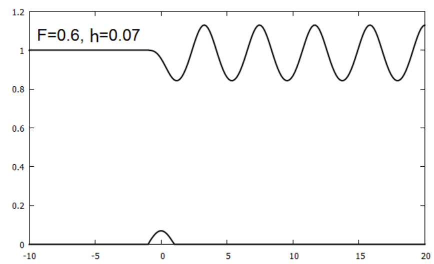

FIGURE 11. The behaviour of the free surface over a bump $b(h, x)$.

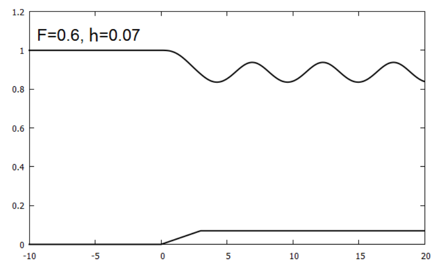

FIGURE 12. Free surface flow profile over a step with small inclination, when $F = 0.6$ and $h = 0.07$.

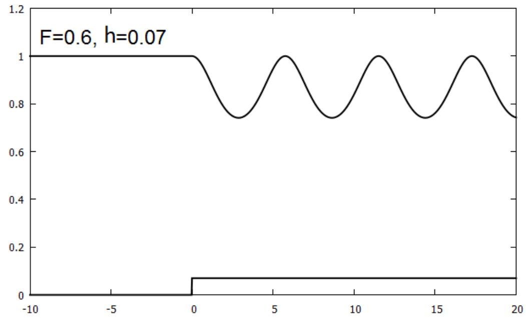

FIGURE 13. Free surface flow profile over a step when $F = 0.6$ and $h = 0.07$.

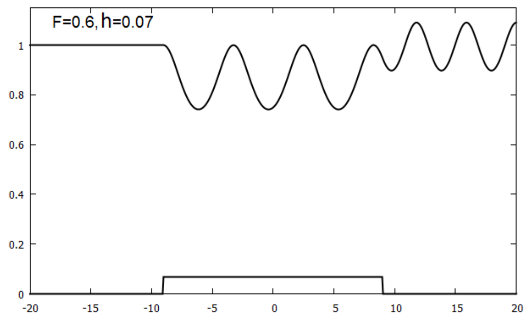

FIGURE 14. Free surface flow profile over a rectangular bump when $F = 0.6$ and $h = 0.07$.

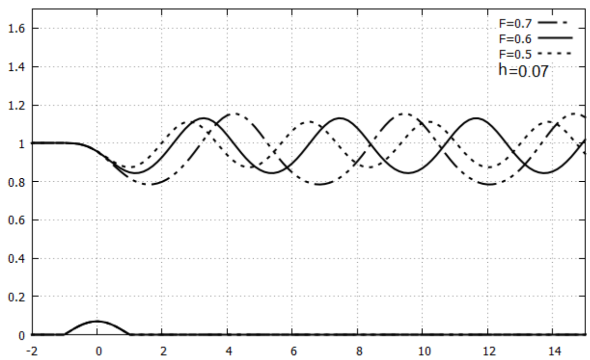

FIGURE 15. Free surface evolution versus F, for h fixed, over a bump $b(h, x)$.

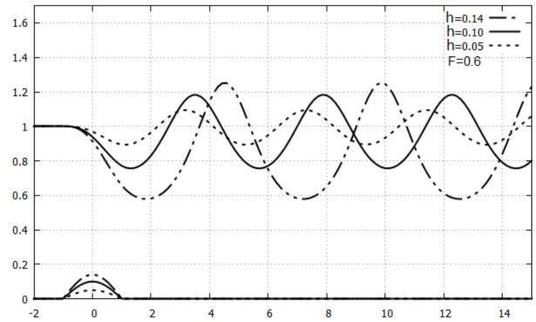

FIGURE 16. Free surface evolution versus h, for F fixed, over a bump $b(h, x)$.

5. Conclusion

In this work, a numerical study of a free surface flow problem, has been done for the determination of the free surface of a bidimensional, steady and irrotational flow, over an obstacle lying on the bottom of an infinite channel. The fluid is perfect and incompressible. The gravity acceleration has been taken into account but the effects of the superficial tension are neglected. The fourth order Runge-Kutta method is used for the subcritical flow and the finite difference method is used when the flow is supercritical. The obtained results show that, when the flow is supercritical, the free surface is horizontal upstream and downstream, far the obstacle, and takes the same shape as the obstacle, above this one. When the flow is subcritical, the free surface is horizontal upstream, far the obstacle, presents a depression over the obstacle, followed by a train of waves. Also, in the supercritical case, when the Froude number increases, the free surface elevation decreases. We note that our results are in a very good agreement with those given by some authors.

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