PUBLICATIONS DE L'INSTITUT MATHÉMATIQUE
Nouvelle série, tome 106(120) (2019), 125–133 DOI: https://doi.org/10.2298/PIM1920125Q

# SOME INTEGRAL TRANSFORMS OF THE GENERALIZED $k$-MITTAG-LEFFLER FUNCTION

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Abstract. We generalize the notion "$k$-Mittag-Leffler function", establish some integral transforms of the generalized $k$-Mittag-Leffler function, and derive several special and known conclusions in terms of the generalized Wright function and the generalized $k$-Wright function.

## 1. Preliminaries

Throughout this paper, let $\mathbb{C}$, $\mathbb{R}$, $\mathbb{R}_0^+$, $\mathbb{R}^+$, $\mathbb{Z}_0^-$, and $\mathbb{N}$ denote respectively the sets of complex numbers, real numbers, non-negative numbers, positive numbers, non-positive integers, and positive integers.

The Pochhammer symbol $(\lambda)_\nu$ can be defined for $\lambda, \nu \in \mathbb{C}$ by $(\lambda)_\nu = \frac{\Gamma(\lambda+\nu)}{\Gamma(\lambda)}$, where

$$\Gamma(z) = \lim_{n\to\infty} \frac{n!n^z}{\prod_{k=0}^{n}(z+k)}, \quad z \in \mathbb{C} \setminus \mathbb{Z}_0^-$$

is called the classical gamma function and its reciprocal $\frac{1}{\Gamma}$ is analytic on the whole complex plane $\mathbb{C}$. See [**14**, Chapter 5], [**16**, Section 1], and [**26**, Section 1.1]. In particular, when $\nu \in \{0\} \cup \mathbb{N}$, the quantity

$$(\lambda)_n = \begin{cases} 1, & \nu = 0 \\ \lambda(\lambda+1)\cdots(\lambda+n-1), & n \in \mathbb{N} \end{cases}$$

is called the rising factorial. See [**18**] and closely-related references therein.

The $k$-Pochhammer symbol $(\lambda)_{n,k}$ was defined in [**2**] for $\lambda, \nu \in \mathbb{C}$ and $k \in \mathbb{R}$ by

$$(\lambda)_{\nu,k} = \frac{\Gamma_k(\lambda+\nu k)}{\Gamma_k(\lambda)}, \tag{1.1}$$

where

$$\Gamma_k(z) = k^{z/k-1}\Gamma\Big(\frac{z}{k}\Big) \tag{1.2}$$

2010 *Mathematics Subject Classification:* Primary 33E12; Secondary 33C20; 44A20; 44A30; 65R10.

*Key words and phrases:* integral transform, generalized $k$-Mittag-Leffler function, generalized Wright function, generalized $k$-Wright function, gamma function.

Communicated by Gradimir Milovanović.

is called the $k$-gamma function. In particular,

$$(\lambda)_{n,k} = \begin{cases} 1, & n=0; \\ \lambda(\lambda+k)\cdots(\lambda+(n-1)k), & n\in\mathbb{N}. \end{cases}$$

In 1903, Mittag-Leffler, a Swedish mathematician, introduced and investigated in [**12**, **13**] the so-called Mittag-Leffler function

$$E_\alpha(z) = \sum_{n=0}^{\infty} \frac{z^n}{\Gamma(\alpha n+1)}$$

for $z\in\mathbb{C}$ and $\alpha\in\mathbb{R}_0^+$. In 1905, Wiman [**27**] generalized $E_\alpha(z)$ as

$$E_{\alpha,\beta}(z) = \sum_{n=0}^{\infty} \frac{z^n}{\Gamma(\alpha n+\beta)},$$

where $z\in\mathbb{C}$, $\alpha,\beta\in\mathbb{C}$, and $\operatorname{Re}(\alpha),\operatorname{Re}(\beta)>0$. In 1971, Prabhakar [**15**] introduced the function

$$E_{\alpha,\beta}^{\gamma}(z) = \sum_{n=0}^{\infty} \frac{(\gamma)_n}{\Gamma(\alpha n+\beta)}\frac{z^n}{n!}$$

for $z\in\mathbb{C}$, $\alpha,\beta,\gamma\in\mathbb{C}$, and $\operatorname{Re}(\alpha),\operatorname{Re}(\beta),\operatorname{Re}(\gamma)>0$. In 2012, Dorrego and Cerutti [**3**] introduced the $k$-Mittag-Leffler function

$$E_{k,\alpha,\beta}^{\gamma}(z) = \sum_{n=0}^{\infty} \frac{(\gamma)_{n,k}}{\Gamma_k(\alpha n+\beta)}\frac{z^n}{n!},$$

where $\alpha,\beta,\gamma\in\mathbb{C}$, $k\in\mathbb{R}$, $\operatorname{Re}(\alpha),\operatorname{Re}(\beta),\operatorname{Re}(\gamma)>0$, and $(\gamma)_{n,k}$ is the $k$-Pochhammer symbol. In 2012, a generalization of the $k$-Mittag-Leffler function

$$GE_{k,\alpha,\beta}^{\gamma,q}(z) = \sum_{n=0}^{\infty} \frac{(\gamma)_{nq,k}}{\Gamma_k(\alpha n+\beta)}\frac{z^n}{n!}$$

for $\alpha,\beta,\gamma\in\mathbb{C}$, $k\in\mathbb{R}$, $\operatorname{Re}(\alpha),\operatorname{Re}(\beta),\operatorname{Re}(\gamma)>0$, and $q\in(0,1)\cup\mathbb{N}$ was introduced and studied in [**6**]. For more information on generalizations of the Mittag-Leffler function, please refer to the papers [**1**,**9**,**21**–**23**] and closely-related references therein.

In this paper, we consider a more general generalization

$$E_{k,\alpha,\beta,\delta}^{\gamma,\tau}(z) = \sum_{n=0}^{\infty} \frac{(\gamma)_{n\tau,k}}{\Gamma_k(\alpha n+\beta)}\frac{z^n}{(\delta)_n}, \tag{1.3}$$

where $\alpha,\beta,\gamma,\delta,\tau\in\mathbb{C}$, $k\in\mathbb{R}$, $\operatorname{Re}(\alpha),\operatorname{Re}(\beta)>0$, and $\delta\ne0,-1,-2,\dots$. It is clear that $E_{k,\alpha,\beta,1}^{\gamma,\tau}(z)=GE_{k,\alpha,\beta}^{\gamma,\tau}(z)$ and $E_{k,\alpha,\beta,1}^{\gamma,1}(z)=E_{k,\alpha,\beta}^{\gamma}(z)$.

It is well known [**14**, **19**] that the generalized hypergeometric function can be defined by

$${}_pF_q[(\alpha_p);(\beta_q);z] = \sum_{n=0}^{\infty} \frac{\Pi_{j=1}^{p}(\alpha_j)_n}{\Pi_{j=1}^{q}(\beta_j)_n}\frac{z^n}{n!}$$

for $|z| < 1$ and $p \leqslant q$ with $p = q+1$ and that the generalized Wright hypergeometric function ${}_p\Psi_q(z)$ is given by the series

(1.4)
$$ {}_p\Psi_q(z) = {}_p\Psi_q[(a_i, \alpha_i)_{1,p}; (b_j, \beta_j)_{1,q}; z] = \frac{\prod_{j=1}^{q}\Gamma(\beta_j)}{\prod_{i=1}^{p}\Gamma(\alpha_i)} \sum_{k=0}^{\infty} \frac{\prod_{i=1}^{p}\Gamma(a_i+\alpha_i k)}{\prod_{j=1}^{q}\Gamma(b_j+\beta_j k)} \frac{z^k}{k!} $$

for $a_i, b_j \in \mathbb{C}$ and $\alpha_i, \beta_j \in \mathbb{R}$ with $1 \leqslant i \leqslant p$ and $1 \leqslant j \leqslant q$. Asymptotic behavior of the function ${}_p\Psi_q(z)$ for large values of argument of $z \in \mathbb{C}$ were studied in [**5**, **28**, **29**] under the condition $\sum_{j=1}^{q}\beta_j - \sum_{i=1}^{p}\alpha_i > -1$. If putting $\alpha_1 = \cdots = \alpha_p = \beta_1 = \cdots = \beta_q = 1$ in (1.4), then

$$ \begin{aligned} {}_p\Psi_q(z) &= {}_p\Psi_q[(a_1, 1), \ldots, (a_p, 1); (b_1, 1), \ldots, (b_q, 1); z] \\ &= \frac{\prod_{j=1}^{q}\Gamma(b_j)}{\prod_{i=1}^{p}\Gamma(a_i)} {}_pF_q[a_1, \ldots, a_p; b_1, \ldots, b_q; z]. \end{aligned} $$

The generalized $k$-Wright function was introduced in [**7**] as

$$ {}_p\Psi_q^k(z) = {}_p\Psi_q^k[(a_i, \alpha_i)_{1,p}; (b_j, \beta_j)_{1,q}; z] = \sum_{n=0}^{\infty} \frac{\prod_{i=1}^{p}\Gamma_k(a_i+\alpha_i n)}{\prod_{j=1}^{q}\Gamma_k(b_j+\beta_j n)} \frac{z^n}{n!} $$

for $k \in \mathbb{R}^+$, $z \in \mathbb{C}$, $\alpha_i, \beta_j \in \mathbb{R} \setminus \{0\}$, and $a_i + \alpha_i n, b_j + \beta_j n \in \mathbb{C} \setminus k\mathbb{Z}^-$ with $1 \leqslant p$ and $1 \leqslant j \leqslant q$.

The Euler transform of a function $f(z)$ is defined by

$$ B\{f(z); \alpha, \beta\} = \int_0^1 z^{\alpha-1}(1-z)^{\beta-1} f(z)\,dz \tag{1.5} $$

for $\alpha, \beta \in \mathbb{C}$ and $\mathrm{Re}(\alpha), \mathrm{Re}(\beta) > 0$. The Laplace transform of a function $f(t)$ is defined as $F(s) = L\{f(t); s\} = \int_0^\infty e^{-st} f(t)dt$, $\mathrm{Re}(s) > 0$. The Fourier transform of a function $u = u(t) \in S(\mathbb{R})$ is defined by $\widehat{u} = \mathrm{Im}[u](w) = \int_{\mathbb{R}} u(t)e^{iwt}dt$, where $S(\mathbb{R})$ denotes the Shwartzian space of rapidly decreasing test functions on $\mathbb{R}$. The fractional Fourier transform of order $\alpha$ for $0 \leqslant \alpha \leqslant 1$ was defined in [**10**, **20**] by

$$ \hat{u}_\alpha(w) = \mathrm{Im}_\alpha[u](w) = \int_{\mathbb{R}} e^{iw^{1/\alpha}t} u(t)dt \tag{1.6} $$

for $u \in \Phi(\mathbb{R})$, where $\Phi(\mathbb{R}) = \{\varphi \in S(\mathbb{R}) : \hat{\varphi} \in V(\mathbb{R})\}$ denotes the Lizorkin space of functions and $V(\mathbb{R}) = \{v \in S(\mathbb{R}) : v^{(n)}(0) = 0, n = 0, 1, 2, \ldots\}$. When $\alpha = 1$, the quantity (1.6) reduces to the Fourier transform; when $w > 0$, the transform (1.6) reduces to the fractional Fourier transform.

The aim of this paper is to present the Euler, Laplace, Whittaker, and Fractional Fourier transforms of the generalized $k$-Mittag-Leffler function (1.3). From these conclusions, we can derive some known and new results.

## 2. Main results

Now we are in a position to state and prove our main results.

THEOREM 2.1. *If $k \in \mathbb{R}$, $\alpha, \beta, \gamma, a, b, \sigma \in \mathbb{C}$, $\operatorname{Re}(\alpha), \operatorname{Re}(\beta) > 0$, $\delta \neq 0, -1, -2, \dots$, and $q > 0$, then*

$$\int_0^1 z^{a-1}(1-z)^{b-1} E_{k,\alpha,\beta,\delta}^{\gamma,\tau}(xz^\sigma)dz \tag{2.1}$$
$$= \frac{k^{1-\beta/k}\Gamma(b)\Gamma(\delta)}{\Gamma(\frac{\gamma}{k})} {}_3\Psi_3\Big[\Big(\frac{\gamma}{k}, \tau\Big), (a, \sigma), (1, 1); \Big(\frac{\beta}{k}, \frac{\alpha}{k}\Big), (a+b, \sigma), (\delta, 1); k^{\tau-\alpha/k}x\Big].$$

PROOF. Denote the left-hand side of the equation (2.1) by $\mathcal{I}_1$. By definition of the generalized $k$-Mittag-Leffler function and (1.5), we have

$$\begin{aligned}\mathcal{I}_1 &= \int_0^1 z^{a-1}(1-z)^{b-1} E_{k,\alpha,\beta,\delta}^{\gamma,\tau}(xz^\sigma)dz \\ &= \int_0^1 z^{a-1}(1-z)^{b-1} \sum_{n=0}^{\infty} \frac{(\gamma)_{n\tau,k}}{\Gamma_k(\alpha n+\beta)} \frac{(xz^\sigma)^n}{(\delta)_n} dz.\end{aligned}$$

By interchanging the order of the integration and summation, we obtain

$$\begin{aligned}\mathcal{I}_1 &= \sum_{n=0}^{\infty} \frac{(\gamma)_{n\tau,k}}{\Gamma_k(\alpha n+\beta)} \frac{x^n}{(\delta)_n} \int_0^1 z^{a+\sigma n-1}(1-z)^{b-1}dz \\ &= \sum_{n=0}^{\infty} \frac{(\gamma)_{n\tau,k}}{\Gamma_k(\alpha n+\beta)} \frac{x^n}{(\delta)_n} \frac{\Gamma(a+\sigma n)\Gamma(b)}{\Gamma(a+b+\sigma n)}.\end{aligned}$$

From (1.1) and (1.2), we acquire

$$\mathcal{I}_1 = \frac{k^{1-\beta/k}\Gamma(b)\Gamma(\delta)}{\Gamma(\frac{\gamma}{k})} \sum_{n=0}^{\infty} \frac{\Gamma(\frac{\gamma}{k}+n\tau)\Gamma(a+\sigma n)\Gamma(\delta)}{\Gamma(\frac{\beta}{k}+\frac{\alpha}{k}n)\Gamma(a+b+\sigma n)\Gamma(\delta+n)} \frac{k^{n\tau}}{k^{\alpha n/k}}.$$

In view of (1.4), we arrive at the desired result. □

REMARK 2.1. Taking $\delta = 1$ in Theorem 2.1 gives [**25**, Eq. (24)] which reads that

$$\begin{aligned}&\int_0^1 z^{a-1}(1-z)^{b-1} E_{k,\alpha,\beta}^{\gamma,\tau}(xz^\sigma)dz \\ &\qquad = \frac{k^{1-\beta/k}\Gamma(b)}{\Gamma(\frac{\gamma}{k})} {}_2\Psi_2\Big[\Big(\frac{\gamma}{k}, \tau\Big), (a, \sigma); \Big(\frac{\beta}{k}, \frac{\alpha}{k}\Big), (a+b, \sigma); k^{\tau-\alpha/k}x\Big].\end{aligned}$$

Setting $\delta = 1$ and $\tau = q > 0$ in Theorem 2.1 leads to [**25**, Eq. (25)] which states that

$$\begin{aligned}&\int_0^1 z^{a-1}(1-z)^{b-1} E_{k,\alpha,\beta}^{\gamma,q}(xz^\sigma)dz \\ &\qquad = \frac{k^{1-\beta/k}\Gamma(b)}{\Gamma(\frac{\gamma}{k})} {}_2\Psi_2\Big[\Big(\frac{\gamma}{k}, q\Big), (a, \sigma); \Big(\frac{\beta}{k}, \frac{\alpha}{k}\Big), (a+b, \sigma); k^{q-\alpha/k}x\Big].\end{aligned}$$

Further letting $k=1$ in the above equation derives [**25**, Eq. (26)] which formulates that

$$\int_0^1 z^{a-1}(1-z)^{b-1}E_{\alpha,\beta}^{\gamma,q}(xz^\sigma)dz = \frac{\Gamma(b)}{\Gamma(\gamma)}{}_2\Psi_2[(\gamma,q),(a,\sigma);(\beta,\alpha),(a+b,\sigma);x].$$

THEOREM 2.2. *If $k\in\mathbb{R}$, $\alpha,\beta,\gamma,a,\sigma\in\mathbb{C}$, $\mathrm{Re}(\alpha),\mathrm{Re}(\beta),\mathrm{Re}(s)>0$, $\tau\in\mathbb{C}$, $\left|\frac{x}{s^\sigma}\right|<1$, and $\delta\neq 0,-1,-2,\dots$, then*

$$\begin{aligned}(2.2)\quad &\int_0^\infty z^{a-1}e^{-sz}E_{k,\alpha,\beta,\delta}^{\gamma,\tau}(xz^\sigma)dz\\ &= \frac{k^{1-\beta/k}\Gamma(\delta)}{s^a\Gamma(\frac{\gamma}{k})}{}_3\Psi_2\Big[\Big(\frac{\gamma}{k},\tau\Big),(a,\sigma),(1,1);\Big(\frac{\beta}{k},\frac{\alpha}{k}\Big),(\delta,1);\frac{xk^{\tau-\alpha/k}}{s^\sigma}\Big].\end{aligned}$$

PROOF. Denote the left-hand side of (2.2) by $\mathcal{I}_2$. Applying definition of the generalized $k$-Mittag-Leffler function results in

$$\mathcal{I}_2 = \int_0^\infty z^{a-1}e^{-sz}E_{k,\alpha,\beta}^{\gamma,\tau}(xz^\sigma)dz = \int_0^\infty z^{a-1}e^{-sz}\sum_{n=0}^\infty \frac{(\gamma)_{n\tau,k}}{\Gamma_k(\alpha n+\beta)}\frac{(xz^\sigma)^n}{(\delta)_n}dz.$$

Interchanging the order of the integration and summation leads to

$$\mathcal{I}_2 = \sum_{n=0}^\infty \frac{(\gamma)_{n\tau,k}}{\Gamma_k(\alpha n+\beta)}\frac{x^n}{(\delta)_n}\int_0^\infty z^{a+\sigma n-1}e^{-sz}dz.$$

In view of definition of the Laplace transform, we have

$$\mathcal{I}_2 = \sum_{n=0}^\infty \frac{(\gamma)_{n\tau,k}}{\Gamma_k(\alpha n+\beta)}\frac{x^n}{(\delta)_n}\frac{\Gamma(\sigma n+a)}{s^{\sigma n+a}}.$$

Utilizing (1.1) and (1.2) derives the required result. □

REMARK 2.2. If setting $\delta=1$ in Theorem 2.2, then we deduce [**25**, Eq. (27)] which formulates that

$$\int_0^\infty z^{a-1}e^{-sz}E_{k,\alpha,\beta}^{\gamma,\tau}(xz^\sigma)dz = \frac{k^{1-\beta/k}s^{-a}}{\Gamma(\frac{\gamma}{k})}{}_2\Psi_1\Big[\Big(\frac{\gamma}{k},\tau\Big),(a,\sigma);\Big(\frac{\beta}{k},\frac{\alpha}{k}\Big);\frac{xk^{\tau-\alpha/k}}{s^\sigma}\Big].$$

If taking $\tau=q>0$, $k=1$, and $\delta=1$, then we acquire [**25**, Eq. (29)] which reads that

$$\int_0^\infty z^{a-1}e^{-sz}E_{\alpha,\beta}^{\gamma,q}(xz^\sigma)dz = \frac{s^{-a}}{\Gamma(\gamma)}{}_2\Psi_1\Big[(\gamma,q),(a,\sigma);(\beta,\alpha);\frac{x}{s^\sigma}\Big].$$

If taking $k=q=1$ and $\delta=1$ in the above equation reduces to

$$\int_0^\infty z^{a-1}e^{-sz}E_{\alpha,\beta}^{\gamma}(xz^\sigma)dz = \frac{s^{-a}}{\Gamma(\gamma)}{}_2\Psi_1\Big[(\gamma,1),(a,\sigma);(\beta,\alpha);\frac{x}{s^\sigma}\Big]$$

which is the main result in [**24**].

Recall that

$$\int_0^\infty t^{v-1}e^{-t/2}W_{\lambda,\mu}(t)dt=\frac{\Gamma(\frac{1}{2}+\mu+v)\Gamma(\frac{1}{2}-\mu+v)}{\Gamma(1-\lambda+v)},\quad \operatorname{Re}(v\pm\mu)>-\frac{1}{2}, \tag{2.3}$$

where the Whittaker function

$$W_{\lambda,\mu}(t)=\frac{\Gamma(-2\mu)}{\Gamma(\frac{1}{2}-\mu-\lambda)}M_{\lambda,\mu}(t)+\frac{\Gamma(2\mu)}{\Gamma(\frac{1}{2}+\mu-\lambda)}M_{\lambda,-\mu}(t)$$

and $M_{\lambda,\mu}(t)=z^{\mu+1/2}e^{-t/2}{}_1F_1\big(\frac{1}{2}+\mu+v;2\mu+1;t\big)$ are given in [**4**, **11**].

Theorem 2.3. *If* $k\in\mathbb{R}$, $\alpha,\beta,\gamma,\delta,\tau,\eta\in\mathbb{C}$, $\operatorname{Re}(\alpha),\operatorname{Re}(\beta),\operatorname{Re}(\rho)>0$, $\delta\neq 0,-1,\dots,$ *and* $\operatorname{Re}(\rho\pm\mu)>-\frac{1}{2}$, *then*

$$\int_0^\infty t^{\rho-1}e^{-pt/2}W_{\lambda,\mu}(pt)E_{k,\alpha,\beta,\delta}^{\gamma,\tau}(wt^\eta)dt=\frac{k^{1-\beta/k}p^{-\rho}\Gamma(\delta)}{\Gamma(\frac{\gamma}{k})}{}_3\Psi_3\Big[\Big(\frac{\gamma}{k},\tau\Big),$$
$$\Big(\frac{1}{2}\pm\mu+\rho,\eta\Big),(1,1);\Big(\frac{\beta}{k},\frac{\alpha}{k}\Big),(1-\lambda+\rho,\eta),(\delta,1);\frac{wk^{\tau-\alpha/k}}{p^\eta}\Big].$$

Proof. Letting $pt=v$, interchanging the integration and summation, and using the formula for the Whittaker transform (2.3) yields

$$\begin{aligned}&\int_0^\infty t^{\rho-1}e^{-pt/2}W_{\lambda,\mu}(pt)E_{k,\alpha,\beta,\delta}^{\gamma,\tau}(wt^\eta)dt\\
&=\int_0^\infty e^{-v/2}\Big(\frac{v}{p}\Big)^{\rho-1}W_{\lambda,\mu}(v)\sum_{n=0}^\infty\frac{(\gamma)_{n\tau,k}}{\Gamma_k(\alpha n+\beta)}\frac{w^n}{(\delta)_n}\Big(\frac{v}{p}\Big)^{\delta n}\frac{1}{p}dv\\
&=\sum_{n=0}^\infty\frac{(\gamma)_{n\tau,k}}{\Gamma_k(\alpha n+\beta)}\frac{w^n}{(\delta)_n}\int_0^\infty e^{-v/2}\Big(\frac{v}{p}\Big)^{\rho-1}\Big(\frac{v}{p}\Big)^{\delta n}W_{\lambda,\mu}(v)\frac{1}{p}dv\\
&=\frac{1}{p^\rho}\sum_{n=0}^\infty\frac{(\gamma)_{n\tau,k}}{\Gamma_k(\alpha n+\beta)(\delta)_n}\Big(\frac{w}{p\delta}\Big)^n\int_0^\infty e^{-v/2}v^{\delta n+\rho-1}W_{\lambda,\mu}(v)dv\\
&=\frac{1}{p^\rho}\sum_{n=0}^\infty\frac{(\gamma)_{n\tau,k}\Gamma(\frac{1}{2}+\mu+\delta n+\rho)\Gamma(\frac{1}{2}-\mu+\delta n+\rho)}{\Gamma_k(\alpha n+\beta)(\delta)_n\Gamma(1-\lambda+\delta n+\rho)}\Big(\frac{w}{p\delta}\Big)^n.\end{aligned}$$

In view of (1.1) and (1.2), we find the desired result. □

Remark 2.3. Taking $\delta=1$ in Theorem 2.3 gives [**25**, Eq. (30)] which reads that

$$\begin{aligned}&\int_0^1 t^{\rho-1}e^{-pt/2}W_{\lambda,\mu}(pt)E_{k,\alpha,\beta}^{\gamma,\tau}(wt^\eta)dt\\
&=\frac{k^{1-\beta/k}p^{-\rho}}{\Gamma(\frac{\gamma}{k})}{}_2\Psi_2\Big[\Big(\frac{\gamma}{k},\tau\Big),\Big(\frac{1}{2}\pm\mu+\rho,\eta\Big);\Big(\frac{\beta}{k},\frac{\alpha}{k}\Big),(1-\lambda+\rho,\eta);\frac{wk^{\tau-\alpha/k}}{p^\eta}\Big].\end{aligned}$$

Setting $\tau=q>0$, $k=1$, and $\delta=1$ results in [**25**, Eq. (32)] which states that

$$\int_0^1 t^{\rho-1}e^{-pt/2}W_{\lambda,\mu}(pt)E_{\alpha,\beta}^{\gamma,q}(wt^\eta)dt$$

$$= \frac{1}{p^{\rho}\Gamma(\gamma)}{}_2\Psi_2\Big[(\gamma,\tau),\Big(\frac{1}{2}\pm\mu+\rho,\eta\Big);(\beta,\alpha),(1-\lambda+\rho,\eta);\frac{w}{p^{\eta}}\Big].$$

Letting $q=1$ in the above equation derives a result given in [**24**].

THEOREM 2.4. *If* $k\in\mathbb{R}$, $\alpha,\beta,\gamma,a,b,\sigma\in\mathbb{C}$, $\mathrm{Re}(\alpha),\mathrm{Re}(\beta)>0$, $\tau\in\mathbb{C}$, *and* $\delta\neq 0,-1,-2,\dots$, *then*

$$\mathrm{Im}_{\sigma}[E^{\gamma,\tau}_{k,\alpha,\beta,\delta}(t)](w)=\frac{k^{1-\beta/k}}{\Gamma(\frac{\gamma}{k})}\sum_{n=0}^{\infty}(-1)^n\frac{n!k^{(\tau-\beta/k)n}\Gamma(\frac{\gamma}{k}+n\tau)i^{n-1}w^{-(n+1)/\sigma}}{(\delta)_n\Gamma(\frac{\alpha}{k}n+\frac{\beta}{k})}.$$

PROOF. Using definitions of the generalized $k$-Mittag-Leffler function and the fractional Fourier transform and interchanging the integration and summation give

$$\begin{aligned}\mathrm{Im}_{\sigma}[E^{\gamma,\tau}_{k,\alpha,\beta,\delta}(t)](w)&=\int_0^1\exp(iw^{1/\sigma}t)E^{\gamma,\tau}_{k,\alpha,\beta,\delta}(t)dt\\&=\sum_{n=0}^{\infty}\frac{(\gamma)_{n\tau,k}}{\Gamma_k(\alpha n+\beta)(\delta)_n}\int_R\exp(iw^{1/\sigma}t)t^n dt.\end{aligned}$$

Letting $iw^{1/\sigma}t=-\eta$ reduces to

$$\begin{aligned}\mathrm{Im}_{\sigma}[E^{\gamma,\tau}_{k,\alpha,\beta,\delta}(t)](w)&=\sum_{n=0}^{\infty}\frac{(\gamma)_{n\tau,k}}{\Gamma_k(\alpha n+\beta)(\delta)_n}\int_{-\infty}^{0}\exp(-\eta)\Big(\frac{-\eta}{iw^{1/\sigma}}\Big)^n\Big(\frac{-d\eta}{iw^{1/\sigma}}\Big)\\&=\sum_{n=0}^{\infty}\frac{(\gamma)_{n\tau,k}}{\Gamma_k(\alpha n+\beta)(\delta)_n i^{n+1}w^{(n+1)/\sigma}(-1)^n}\int_0^{\infty}e^{-\eta}\eta^n d\eta\\&=\sum_{n=0}^{\infty}\frac{(\gamma)_{n\tau,k}i^{-n-1}w^{-(n+1)/\sigma}(-1)^n n!}{\Gamma_k(\alpha n+\beta)(\delta)_n}\end{aligned}$$

Further using formulas (1.1) and (1.2) arrives at the required result. □

REMARK 2.4. If taking $\delta=1$ in Theorem 2.4, then the equation

$$\mathrm{Im}_{\sigma}[E^{\gamma,\tau}_{k,\alpha,\beta}(t)](w)=\frac{k^{1-\beta/k}}{\Gamma(\frac{\gamma}{k})}\sum_{n=0}^{\infty}(-1)^n\frac{k^{(\tau-\beta/k)n}\Gamma(\frac{\gamma}{k}+n\tau)i^{n-1}w^{-(n+1)/\sigma}}{\Gamma(\frac{\alpha}{k}n+\frac{\beta}{k})}.$$

in [**25**, Eq. (33)] follows readily.

If setting $\delta=1$ and $\tau=q$ in Theorem 2.4, then the equation

$$\mathrm{Im}_{\sigma}[E^{\gamma,\tau}_{k,\alpha,\beta}(t)](w)=\frac{k^{1-\beta/k}}{\Gamma(\frac{\gamma}{k})}\sum_{n=0}^{\infty}(-1)^n\frac{k^{(q-\beta/k)n}\Gamma(\frac{\gamma}{k}+nq)i^{n-1}w^{-(n+1)/\sigma}}{\Gamma(\frac{\alpha}{k}n+\frac{\beta}{k})}$$

in [**25**, Eq. (34)] can be derived immediately.

If letting $\delta=k=1$ and $\tau=q$ in Theorem 2.4, then

$$\mathrm{Im}_{\sigma}[E^{\gamma,q}_{\alpha,\beta}(t)](w)=\frac{1}{\Gamma(\gamma)}\sum_{n=0}^{\infty}(-1)^n\frac{\Gamma(\gamma+n\tau)i^{n-1}w^{-(n+1)/\sigma}}{\Gamma(\alpha n+\beta)}$$

in [**25**, Eq. (34)] can be deduced straightforwardly.

REMARK 2.5. In [**8**, Section 3], the quantity $i^k$ for $k\in\mathbb{N}$ was computed generally by three approaches.

## 3. Concluding remarks

Some integral transforms of the generalized $k$-Mittag-Leffler function are established and the results are expressed in terms of the generalized Wright function. By taking $\delta = 1$ and using formulas (1.1) and (1.2), we express Theorems 2.1 to 2.3 in terms of the generalized $k$-Wright function as follows.

It is noted that, using the appropriate formulas mentioned in Section 1, one can easily express the Euler integral in terms of the $k$-Wright function as

$$\int_0^1 z^{a-1}(1-z)^{b-1}E^{\gamma,\tau}_{k,\alpha,\beta}(xz^\sigma)dz \\ = \frac{\Gamma(b)k^b}{\Gamma_k(\gamma)}{}_2\Psi^k_2[(\gamma,\tau k),(ak,\sigma k);(\beta,\alpha),((a+b)k,\sigma k);x].$$

By applying suitable formula for the $k$-gamma function, Theorem 2.2 can be expressed in terms of the $k$-Wright function as

$$\int_0^1 z^{a-1}e^{-sz}E^{\gamma,\tau}_{k,\alpha,\beta}(xz^\sigma)dz = \frac{k^{2-\gamma/k}}{(sk)^a\Gamma_k(\gamma)}{}_2\Psi^k_1\Big[(\gamma,\tau k),(ak,\sigma k);(\beta,\alpha);\frac{x}{(ks)^\sigma}\Big].$$

Theorem 2.3 can be expressed in terms of the $k$-Wright function as

$$\int_0^1 t^{\rho-1}e^{-pt/2}W_{\lambda,\mu}(pt)E^{\gamma,\tau}_{k,\alpha,\beta}(wt^\eta)dt = \frac{p^{-\rho}k^{1-\rho-\lambda}}{\Gamma_k(\gamma)} \\ \times {}_2\Psi^k_2\Big[(\gamma,\tau k),\Big(\Big(\frac{1}{2}\pm\mu+\rho\Big)k,\eta k\Big);(\beta,\alpha),((1-\lambda+\rho)k,\eta k);\frac{xk^{\tau-\eta-\alpha/k}}{p^\eta}\Big].$$

Remark 3.1. This paper is a slightly revised version of the preprint [**17**].

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(Received 23 02 2017)

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