

IDENTITIES AND RELATIONS FOR FUBINI TYPE NUMBERS AND POLYNOMIALS VIA GENERATING FUNCTIONS AND p -ADIC INTEGRAL APPROACH

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ABSTRACT. The Fubini type polynomials have many application not only especially in combinatorial analysis, but also other branches of mathematics, in engineering and related areas. Therefore, by using the p -adic integrals method and functional equation of the generating functions for Fubini type polynomials and numbers, we derive various different new identities, relations and formulas including well-known numbers and polynomials such as the Bernoulli numbers and polynomials, the Euler numbers and polynomials, the Stirling numbers of the second kind, the λ -array polynomials and the Lah numbers.

1. Introduction

Recently these special numbers and polynomials and their applications have occupied many mathematicians and researchers of other scholars (see [1–24] and the references therein). Thus by applying the p -adic integrals technique to generating functions and their functional equations we give many new formulas and relations including combinatorial sums, and special numbers and polynomials. In order to give these formulas and relations we do not need only the following notations, but also generating functions for the specials numbers and polynomials.

Let $\mathbb{N} = \{1, 2, 3, \dots\}$, $\mathbb{N}_0 = \{0, 1, 2, 3, \dots\} = \mathbb{N} \cup \{0\}$ and let \mathbb{Z} denote the set of integers, \mathbb{R}^+ denote the set of positive real numbers, \mathbb{C} denote the set of complex numbers and \mathbb{Z}_p denote the set of p -adic integers.

Now let us introduce generating functions for some well known special numbers and polynomials.

The Bernoulli polynomials $B_n(x)$ are defined by means of the following generating function:

$$(1.1) \quad \frac{t}{e^t - 1} e^{xt} = \sum_{n=0}^{\infty} B_n(x) \frac{t^n}{n!},$$

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where $|t| < 2\pi$ (cf. [1–24]). By (1.1), one can easily deduce the following formula:

$$B_n(x) = \sum_{j=0}^n \binom{n}{j} x^{n-j} B_j,$$

where B_j denote the Bernoulli numbers, which are defined by the following recurrence relation:

$$B_0 = 1 \text{ and } \sum_{j=0}^{n-1} \binom{n}{j} B_j = 0 \quad (n \in \mathbb{N} \setminus \{1\}).$$

We also see that $B_n = B_n(0)$ ($n \in \mathbb{N}_0$) (cf. [1–24] and the references therein).

The Euler polynomials of the first kind $E_n(x)$ are defined by means of the following generating function:

$$(1.2) \quad \frac{2}{e^t + 1} e^{xt} = \sum_{n=0}^{\infty} E_n(x) \frac{t^n}{n!},$$

where $|t| < \pi$ (cf. [3, 7, 16, 24] and the references therein).

By (1.2), one can easily deduce the following formula:

$$E_n(x) = \sum_{j=0}^n \binom{n}{j} x^{n-j} E_j,$$

where E_j denote the Euler numbers, which are defined by the following recurrence relation:

$$E_0 = 1 \text{ and } E_n = -\frac{1}{2} \sum_{j=0}^{n-1} \binom{n}{j} E_j \quad (n \in \mathbb{N}).$$

We also see that $E_n = E_n(0)$ ($n \in \mathbb{N}_0$) (cf. [3, 7, 16, 24] and the references therein).

The Stirling numbers of the second kind are defined by means of the following generating function:

$$\frac{(e^t - 1)^m}{m!} = \sum_{n=0}^{\infty} S_2(n, m) \frac{t^n}{n!}.$$

By using this generating function, we have

$$S_2(n, m) = \frac{1}{m!} \sum_{j=0}^m (-1)^j \binom{m}{j} (m-j)^n,$$

$$S_2(n, n) = 1 \quad (n \in \mathbb{N}_0), \quad S_2(n, 0) = 0 \quad (n \in \mathbb{N}) \quad S_2(n, m) = 0, \quad (m > n)$$

(cf. [1–24] and the references therein).

Let $m \in \mathbb{N}_0$ and $\lambda \in \mathbb{C}$. The λ -array polynomials $S_m^n(x; \lambda)$ are defined by means of the following generating function:

$$\frac{(\lambda e^t - 1)^m}{m!} e^{xt} = \sum_{n=0}^{\infty} S_m^n(x; \lambda) \frac{t^n}{n!}$$

(cf. [1,4,16] and the references therein). Substituting $\lambda = 1$ into the above equation, we have

$$S_m^n(x) = \frac{1}{m!} \sum_{j=0}^m (-1)^{m-j} \binom{m}{j} (x+j)^n$$

with $S_0^0(x) = S_n^n(x) = 1$, $S_0^n(x) = x^n$. If $m > n$, then $S_m^n(x) = 0$ (cf. [1,4,16,18] and the references therein).

The Lah numbers $L(n, k)$ are defined by means of the following generating function:

$$\sum_{n=0}^{\infty} L(n, k) \frac{t^n}{n!} = \frac{1}{k!} \left(\frac{t}{1-t} \right)^k,$$

where $k \in \mathbb{N}$ (cf. [5,6,14]). The Lah numbers are given by the explicit formula:

$$L(n, k) = (-1)^n \frac{n!}{k!} \binom{n-1}{k-1}, \quad n \geq k \geq 1.$$

The falling factorial is defined by $(x)_n = x(x-1)(x-2)\dots(x-n+1)$, $(x)_0 = 1$, (cf. [5,6,24] and the references therein).

Riordan [14, p. 43] gave the relation between falling factorial and Lah numbers as the following definition:

$$(1.3) \quad (-x)_n = \sum_{k=0}^n L(n, k)(x)_k$$

so that $(x)_n = \sum_{k=0}^n L(n, k)(-x)_k$.

The Fubini numbers $w_g(n)$ are defined by means of the following generating function [8]:

$$Fw_g(t) = \frac{1}{2 - e^t} = \sum_{n=0}^{\infty} w_g(n) \frac{t^n}{n!},$$

where $w_g(0) = 1$ and $|t| < \ln 2$.

In [9], we defined a new family of polynomials $a_n^{(l)}(x)$ by means of the following generating function:

$$(1.4) \quad F_a(t, x; l) = \frac{2^l}{(2 - e^t)^{2l}} e^{xt} = \sum_{n=0}^{\infty} a_n^{(l)}(x) \frac{t^n}{n!},$$

where $l \in \mathbb{N}_0$ and $|t| < 2\pi$.

By (1.4), we have $a_n^{(l)}(0) = a_n^{(l)}$ which denotes Fubini type numbers of order l [9]. By (1.4), one can easily deduce the following formula:

$$(1.5) \quad a_n^{(l)}(x) = \sum_{j=0}^n \binom{n}{j} x^{n-j} a_j^{(l)}.$$

1.1. p -adic integral and related formulas. It is well known that in recent years p -adic integrals and applications have been widely used in fields such as physics and engineering besides mathematics. Some special numbers and polynomials can also be computed with the help of p -adic integrals besides the construction of their generating functions. In order to give some combinatorial sums including the Bernoulli numbers and the Euler numbers in Section 3, we need some basic properties of p -adic integrals including the bosonic or Volkenborn integral and the fermionic p -adic integral.

Let \mathbb{K} be a field with a complete valuation and let $C^1(\mathbb{Z}_p \rightarrow \mathbb{K})$ be a set of continuous derivative functions:

$$\left\{ f: \mathbb{Z}_p \rightarrow \mathbb{K}: f(x) \text{ is differentiable and } \frac{d}{dx}f(x) \text{ is continuous} \right\}.$$

Let $f \in (\mathbb{Z}_p \rightarrow \mathbb{K})$. The Volkenborn integral (bosonic p -adic integral) of function f is given by

$$(1.6) \quad \int_{\mathbb{Z}_p} f(x) d\mu_1(x) = \lim_{n \rightarrow \infty} \frac{1}{p^N} \sum_{x=0}^{p^N-1} f(x),$$

where $\mu_1(x) = \mu_1(x + p^N \mathbb{Z}_p) = \frac{1}{p^N}$ (cf. [11, 12, 15] and the references therein).

By using (1.6), the Witt formula for the Bernoulli numbers B_n is given by

$$(1.7) \quad \int_{\mathbb{Z}_p} x^n d\mu_1(x) = B_n$$

(cf. [11, 12, 15] and the references therein).

Let $f \in (\mathbb{Z}_p \rightarrow \mathbb{K})$. The fermionic p -adic integral of function f is given by

$$(1.8) \quad \int_{\mathbb{Z}_p} f(x) d\mu_{-1}(x) = \lim_{n \rightarrow \infty} \sum_{x=0}^{p^N-1} (-1)^x f(x),$$

where $\mu_{-1}(x) = \mu_{-1}(x + p^N \mathbb{Z}_p) = \frac{(-1)^x}{p^N}$ (cf. [11, 12, 15] and the references therein).

By using (1.8), Witt formula for the Euler numbers E_n is given by

$$(1.9) \quad \int_{\mathbb{Z}_p} x^n d\mu_{-1}(x) = E_n$$

(cf. [11, 12, 15] and the references therein).

Let us briefly summarize the rest of this article. Section 2, by using functional equations for the generating functions for the special numbers and polynomials, we derive many identities and formulas including combinatorial sums, the Fubini numbers and polynomials, the Bernoulli numbers and polynomials, the Euler numbers and polynomials, the Stirling numbers, the Lah numbers and the array polynomials. In section 3, by applying p -adic integrals methods to identities and formulas, we give some new relations including special numbers and polynomials and combinatorial sums.

2. Identities and relations related to functional equations of the generating functions

In this section, we give some relationships among the Fubini type numbers, the Bernoulli numbers, the Euler numbers, the array polynomials and the Lah numbers.

THEOREM 2.1. *The following identity holds true:*

$$(2.1) \quad 2^l x^n = \sum_{j=0}^{2l} \binom{2l}{j} 2^j (-1)^{2l-j} \sum_{m=0}^n \binom{n}{m} a_m^{(l)}(x) (2l-j)^{n-m}.$$

PROOF. By using (1.4), we get

$$2^l \sum_{n=0}^{\infty} x^n \frac{t^n}{n!} = (2 - e^t)^{2l} \sum_{n=0}^{\infty} a_n^{(l)}(x) \frac{t^n}{n!}.$$

After some elementary calculation in the above equation, we get

$$\sum_{n=0}^{\infty} 2^l x^n \frac{t^n}{n!} = \sum_{j=0}^{2l} \binom{2l}{j} 2^j (-1)^{2l-j} \sum_{n=0}^{\infty} (2l-j)^n \frac{t^n}{n!} \sum_{n=0}^{\infty} a_n^{(l)}(x) \frac{t^n}{n!}.$$

Therefore,

$$\sum_{n=0}^{\infty} 2^l x^n \frac{t^n}{n!} = \sum_{n=0}^{\infty} \left(\sum_{m=0}^n \binom{n}{m} a_m^{(l)}(x) (2l-j)^{n-m} \sum_{j=0}^{2l} \binom{2l}{j} 2^j (-1)^{2l-j} \right) \frac{t^n}{n!}.$$

Comparing the coefficients of $\frac{t^n}{n!}$ on both sides of the above equation, we arrive at the desired result. \square

By integrating both sides of (2.1) with x , from 0 to 1, we get with help of (34) in [9, p.1619], we get the following theorem:

THEOREM 2.2. *The following identity holds true:*

$$(2.2) \quad \frac{2^l}{n+1} = \sum_{j=0}^{2l} \binom{2l}{j} 2^j (-1)^{2l-j} \sum_{m=0}^n \binom{n}{m} (2l-j)^{n-m} \sum_{k=0}^m \binom{m}{k} \frac{a_k^{(l)}}{m-k+1}.$$

Combining (2.2) with Theorem 4.13 in [9, p.1619], we get the following combinatorial sum:

THEOREM 2.3. *The following identity holds true:*

$$\begin{aligned} \sum_{j=0}^{2l} \sum_{m=0}^n \sum_{k=0}^m \binom{m}{k} \binom{n}{m} \binom{2l}{j} 2^j (-1)^{2l-j} (2l-j)^{n-m} \frac{a_k^{(l)}}{m-k+1} \\ = \sum_{j=0}^{2l} \sum_{m=0}^n \sum_{k=0}^{n-m} (-1)^j \binom{n-m}{k} \binom{2l}{j} \binom{n}{m} j! \frac{S_2(m, j) a_k^{(l)}}{n+1-k-m}. \end{aligned}$$

THEOREM 2.4. *The following identity holds true:*

$$(2.3) \quad 2^l \sum_{m=0}^n (-2l)_m (-1)^m S_m^n(x) = a_n^{(l)}(x).$$

PROOF. By using (1.4), we get $2^l e^{xt}(2 - e^t)^{-2l} = \sum_{n=0}^{\infty} a_n^{(l)}(x) \frac{t^n}{n!}$. After some elementary calculation in the above equation, we get

$$2^l \sum_{m=0}^{\infty} (-2l)_m (-1)^m \frac{(e^t - 1)^m}{m!} e^{xt} = \sum_{n=0}^{\infty} a_n^{(l)}(x) \frac{t^n}{n!}.$$

Therefore,

$$2^l \sum_{m=0}^{\infty} (-2l)_m (-1)^m \sum_{n=0}^{\infty} S_m^n(x) \frac{t^n}{n!} = \sum_{n=0}^{\infty} a_n^{(l)}(x) \frac{t^n}{n!}.$$

Since $n < m$, $S_m^n(x) = 0$, we deduce the above equation

$$2^l \sum_{n=0}^{\infty} \left(\sum_{m=0}^n (-2l)_m (-1)^m S_m^n(x) \right) \frac{t^n}{n!} = \sum_{n=0}^{\infty} a_n^{(l)}(x) \frac{t^n}{n!}.$$

Comparing the coefficients of $\frac{t^n}{n!}$ on both sides of the above equation, we arrive at the desired result. \square

THEOREM 2.5. *The following identity holds true:*

$$2^l \sum_{m=0}^n \sum_{k=0}^m L(m, k) (2l)_k (-1)^m S_m^n(x) = a_n^{(l)}(x).$$

PROOF. By combining (1.3) with (2.3), we arrive at the desired result. \square

THEOREM 2.6. *We have*

$$\sum_{j=0}^n \binom{n}{j} a_j^{(-l)} \frac{1}{n-j+1} = (2l)! (-1)^l \sum_{j=0}^n \binom{n}{j} y_2 \left(j, l; -\frac{1}{2} \right) \left(\frac{(1+l)^{n-j+1} - l^{n-j+1}}{n-j+1} \right)$$

where [22]

$$y_2(n, k; \lambda) = \frac{1}{(2k)!} \sum_{j=0}^k \binom{k}{j} 2^{k-j} \sum_{l=0}^j \binom{j}{l} (2l-j)^n \lambda^{2n-j}.$$

PROOF. By using equation (1.5) in which l is replaced by $-l$, we obtain

$$(2.4) \quad a_n^{(-l)}(x) = \sum_{j=0}^n \binom{n}{j} a_j^{(-l)} x^{n-j}.$$

Integrating both sides of (2.4) from 0 to 1 with respect to x , we get

$$(2.5) \quad \int_0^1 a_n^{(-l)}(x) dx = \sum_{j=0}^n \binom{n}{j} a_j^{(-l)} \frac{1}{n-j+1}.$$

And also by integrating both sides of (10) in [10] from 0 to 1 with respect to x , we get

$$\begin{aligned}
 (2.6) \quad \int_0^1 a_n^{(-l)}(x)dx &= (2l)!(-1)^l \sum_{j=0}^n \binom{n}{j} y_2\left(j, l; -\frac{1}{2}\right) \int_0^1 (x+l)^{n-j} dx \\
 &= (2l)!(-1)^l \sum_{j=0}^n \binom{n}{j} y_2\left(j, l; -\frac{1}{2}\right) \left(\frac{(1+l)^{n-j+1} - l^{n-j+1}}{n-j+1}\right).
 \end{aligned}$$

By combining (2.5) with (2.6), we arrive at the desired result. □

THEOREM 2.7. *The following identity holds true:*

$$a_{n+1}^{(l)}(x) = xa_n^{(l)}(x) + 2la_n^{(l+1)}(x+1) - la_n^{(l+1)}(x+2).$$

PROOF. By applying $\frac{\partial}{\partial t}$ derivative operator to equation (1.4), we get the following functional equation:

$$(2.7) \quad \frac{\partial}{\partial t} F_a(t, x; l) = xF_a(t, x; l) + 2lF_a(t, x+1; l+1) - lF_a(t, x+2; l+1).$$

From this equation, we get

$$\sum_{n=0}^{\infty} a_{n+1}^{(l)}(x) \frac{t^n}{n!} = x \sum_{n=0}^{\infty} a_n^{(l)}(x) \frac{t^n}{n!} + 2l \sum_{n=0}^{\infty} a_n^{(l+1)}(x+1) \frac{t^n}{n!} - l \sum_{n=0}^{\infty} a_n^{(l+1)}(x+2) \frac{t^n}{n!}.$$

Comparing the coefficients of $\frac{t^n}{n!}$ on both sides of the above equation, we arrive at the desired result. □

THEOREM 2.8. *The following identity holds true:*

$$a_{n+1}^{(l)}(x) = xa_n^{(l)}(x) + 2l \sum_{j=0}^n \binom{n}{j} w_g(j) a_{n-j}^{(l)}(x+1).$$

PROOF. By applying $\frac{\partial}{\partial t}$ derivative operator to equation (1.4), we get the following functional equations:

$$(2.8) \quad \frac{\partial}{\partial t} F_a(t, x; l) = xF_a(t, x; l) + 2lF_a(t, x+1; l)F_{w_g}(t).$$

From this equation, we obtain recurrence relations for the Fubini type polynomials

$$\frac{\partial}{\partial t} \sum_{n=0}^{\infty} a_n^{(l)}(x) \frac{t^n}{n!} = x \sum_{n=0}^{\infty} a_n^{(l)}(x) \frac{t^n}{n!} + 2l \sum_{n=0}^{\infty} a_n^{(l)}(x+1) \frac{t^n}{n!} \sum_{n=0}^{\infty} w_g(n) \frac{t^n}{n!}.$$

Therefore,

$$\sum_{n=0}^{\infty} a_{n+1}^{(l)}(x) \frac{t^n}{n!} = x \sum_{n=0}^{\infty} a_n^{(l)}(x) \frac{t^n}{n!} + 2l \sum_{n=0}^{\infty} \left(\sum_{j=0}^n \binom{n}{j} w_g(j) a_{n-j}^{(l)}(x+1) \right) \frac{t^n}{n!}.$$

Comparing the coefficients of $\frac{t^n}{n!}$ on both sides of the above equation, we arrive at the desired result. □

THEOREM 2.9. *The following identity holds true:*

$$\frac{\partial}{\partial x} a_{n+1}^{(l)}(x) = a_n^{(l)}(x) + xna_{n-1}^{(l)}(x) + 2nla_{n-1}^{(l+1)}(x+1) - nla_{n-1}^{(l+1)}(x+2).$$

PROOF. By applying the $\frac{\partial}{\partial x}$ derivative operator to equation (1.4),

$$(2.9) \quad \frac{\partial}{\partial x} F_a(t, x; l) = tF_a(t, x; l).$$

By applying the $\frac{\partial}{\partial t}$ derivative operator to equation (2.9),

$$(2.10) \quad \frac{\partial^2}{\partial x \partial t} F_a(t, x; l) = F_a(t, x; l) + t \frac{\partial}{\partial t} F_a(t, x; l).$$

By using this function with (2.7), we get the following functional equation:

$$\frac{\partial^2}{\partial x \partial t} F_a(t, x; l) = F_a(t, x; l) + t(xF_a(t, x; l) + 2lF_a(t, x+1; l+1) - lF_a(t, x+2; l+1)).$$

From this equation, we obtain another recurrence relation for the Fubini type polynomials

$$\begin{aligned} \frac{\partial}{\partial x} \sum_{n=0}^{\infty} a_{n+1}^{(l)}(x) \frac{t^n}{n!} &= \sum_{n=0}^{\infty} a_n^{(l)}(x) \frac{t^n}{n!} + xt \sum_{n=0}^{\infty} a_n^{(l)}(x) \frac{t^n}{n!} \\ &\quad + 2lt \sum_{n=0}^{\infty} a_n^{(l+1)}(x+1) \frac{t^n}{n!} - lt \sum_{n=0}^{\infty} a_n^{(l+1)}(x+2) \frac{t^n}{n!}. \end{aligned}$$

After some elementary calculation in the above equation, we get

$$\begin{aligned} \frac{\partial}{\partial x} \sum_{n=0}^{\infty} a_{n+1}^{(l)}(x) \frac{t^n}{n!} &= \sum_{n=0}^{\infty} a_n^{(l)}(x) \frac{t^n}{n!} + x \sum_{n=0}^{\infty} na_{n-1}^{(l)}(x) \frac{t^n}{n!} \\ &\quad + 2l \sum_{n=0}^{\infty} na_{n-1}^{(l+1)}(x+1) \frac{t^n}{n!} - l \sum_{n=0}^{\infty} na_{n-1}^{(l+1)}(x+2) \frac{t^n}{n!}. \end{aligned}$$

Comparing the coefficients of $\frac{t^n}{n!}$ on both sides of the above equations, we arrive at the desired result. \square

THEOREM 2.10. *The following identity holds true:*

$$\frac{\partial}{\partial x} a_{n+1}^{(l)}(x) = a_n^{(l)}(x) + xna_{n-1}^{(l)}(x) + 2nl \sum_{j=0}^{n-1} \binom{n-1}{j} w_g(j) a_{n-j-1}^{(l)}(x+1).$$

PROOF. By using (2.8) and (2.10), we get the following functional equation:

$$\frac{\partial^2}{\partial x \partial t} F_a(t, x; l) = F_a(t, x; l) + t(xF_a(t, x; l) + 2lF_a(t, x+1; l)F_{w_g}(t)).$$

From this equation, we obtain another recurrence relation for the Fubini type polynomials

$$\begin{aligned} \frac{\partial}{\partial x} \sum_{n=0}^{\infty} a_{n+1}^{(l)}(x) \frac{t^n}{n!} &= \sum_{n=0}^{\infty} a_n^{(l)}(x) \frac{t^n}{n!} + xt \sum_{n=0}^{\infty} a_n^{(l)}(x) \frac{t^n}{n!} \\ &\quad + 2lt \sum_{n=0}^{\infty} a_n^{(l)}(x+1) \frac{t^n}{n!} \sum_{n=0}^{\infty} w_g(n) \frac{t^n}{n!}. \end{aligned}$$

After some elementary calculation in the above equation, we get

$$\begin{aligned} \frac{\partial}{\partial x} \sum_{n=0}^{\infty} a_{n+1}^{(l)}(x) \frac{t^n}{n!} &= \sum_{n=0}^{\infty} a_n^{(l)}(x) \frac{t^n}{n!} + x \sum_{n=0}^{\infty} na_{n-1}^{(l)}(x) \frac{t^n}{n!} \\ &\quad + 2l \sum_{n=0}^{\infty} \left(n \sum_{j=0}^{n-1} \binom{n-1}{j} w_g(j) a_{n-j-1}^{(l)}(x+1) \right) \frac{t^n}{n!}. \end{aligned}$$

Comparing the coefficients of $\frac{t^n}{n!}$ on both sides of the above equation, we arrive at the desired result. \square

3. Identities and relations using p -adic integrals

In this section, by applying p -adic integrals to the identities and formulas in Section 2, we derive combinatorial sums, some new relations and identities including special numbers and polynomials and combinatorial sums. These identities are related to the Fubini numbers and polynomials, the Bernoulli numbers and polynomials, the Euler numbers and polynomials and the Stirling numbers.

It is time to give some applications of p -adic integrals in order to give combinatorial sums including not only the Bernoulli numbers and polynomials, the Euler numbers and polynomials, but also the Fubini numbers and polynomials and the Stirling numbers.

By applying the Volkenborn integral to equation (2.1) and combining with (1.7), we arrive at the following theorem:

THEOREM 3.1.

$$(3.1) \quad B_n = 2^{-l} \sum_{j=0}^{2l} \binom{2l}{j} 2^j (-1)^{2l-j} \sum_{m=0}^n \binom{n}{m} (2l-j)^{n-m} \sum_{i=0}^m \binom{m}{i} B_{m-i} a_i^{(l)}.$$

By applying the fermionic integral to equation (2.1) and combining with (1.9), we arrive at the following theorem:

THEOREM 3.2.

$$(3.2) \quad E_n = 2^{-l} \sum_{j=0}^{2l} \binom{2l}{j} 2^j (-1)^{2l-j} \sum_{m=0}^n \binom{n}{m} (2l-j)^{n-m} \sum_{i=0}^m \binom{m}{i} E_{m-i} a_i^{(l)}.$$

By applying the Volkenborn integral to equation (31) in [9, Theorem 4.8, p. 1616] and combining with (1.7), we arrive at the following theorem:

THEOREM 3.3.

$$(3.3) \quad B_n = 2^{-l} \sum_{m=0}^n \sum_{j=0}^{2l} (-1)^j \binom{2l}{j} \binom{n}{m} j! S_2(m, j) \sum_{i=0}^{n-m} \binom{n-m}{i} B_{n-m-i} a_i^{(l)}.$$

By applying the fermionic integral to equation (31) in [9, Theorem 4.8, p. 1616] and combining with (1.9), we arrive at the following theorem:

THEOREM 3.4.

$$(3.4) \quad E_n = 2^{-l} \sum_{m=0}^n \sum_{j=0}^{2l} (-1)^j \binom{2l}{j} \binom{n}{m} j! S_2(m, j) \sum_{i=0}^{n-m} \binom{n-m}{i} E_{n-m-i} a_i^{(l)}.$$

Combining (3.1) with (3.3), we arrive at the following theorem:

THEOREM 3.5. *The following identity holds true:*

$$\begin{aligned} \sum_{m=0}^n \sum_{j=0}^{2l} (-1)^j \binom{2l}{j} \binom{n}{m} j! S_2(m, j) \sum_{i=0}^{n-m} \binom{n-m}{i} B_{n-m-i} a_i^{(l)} \\ = \sum_{j=0}^{2l} \binom{2l}{j} 2^j (-1)^{2l-j} \sum_{m=0}^n \binom{n}{m} (2l-j)^{n-m} \sum_{i=0}^m \binom{m}{i} B_{m-i} a_i^{(l)}. \end{aligned}$$

Combining (3.2) with (3.4), we arrive at the following theorem:

THEOREM 3.6. *The following identity holds true:*

$$\begin{aligned} \sum_{m=0}^n \sum_{j=0}^{2l} (-1)^j \binom{2l}{j} \binom{n}{m} j! S_2(m, j) \sum_{i=0}^{n-m} \binom{n-m}{i} E_{n-m-i} a_i^{(l)} \\ = \sum_{j=0}^{2l} \binom{2l}{j} 2^j (-1)^{2l-j} \sum_{m=0}^n \binom{n}{m} (2l-j)^{n-m} \sum_{i=0}^m \binom{m}{i} E_{m-i} a_i^{(l)}. \end{aligned}$$

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