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THE BOOLE POLYNOMIALS ASSOCIATED WITH THE p-ADIC GAMMA FUNCTION

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Abstract. We set some correlations between Boole polynomials and p-adic gamma function in conjunction with p-adic Euler contant. We develop diverse formulas for p-adic gamma function by means of their Mahler expansion and fermionic p-adic integral on $\mathbb{Z}_p$. Also, we acquire two fermionic p-adic integrals of p-adic gamma function in terms of Boole numbers and polynomials. We then provide fermionic p-adic integral of the derivative of p-adic gamma function and a representation for the p-adic Euler constant by means of the Boole polynomials. Furthermore, we investigate an explicit representation for the aforesaid constant covering Stirling numbers of the first kind.

1. Introduction

Let $\mathbb{N} := \{1, 2, 3, \cdots\}$ and $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$. Throughout this paper, $\mathbb{Z}$ denotes the set of integers, $\mathbb{R}$ denotes the set of real numbers and $\mathbb{C}$ denotes the set of complex numbers. Let p be chosen as an odd fixed prime number. The symbols $\mathbb{Z}_p$, $\mathbb{Q}_p$ and $\mathbb{C}_p$ denote the ring of p-adic integers, the field of p-adic numbers and the completion of an algebraic closure of $\mathbb{Q}_p$, respectively. The normalized absolute value according to the theory of p-adic analysis is given by $|p|_p = p^{-1}$ (for details [**1**–**12**]; see also the related references cited therein).

The fermionic p-adic integral on $\mathbb{Z}_p$ of a function

$$f \in C(\mathbb{Z}_p) = \{f \mid f\colon \mathbb{Z}_p \to \mathbb{Z}_p \text{ be a continuous function}\}$$

is defined [**5**, **12**] as follows:

$$\int_{\mathbb{Z}_p} f(x)\, d\mu_{-1}(x) = \lim_{N\to\infty} \frac{1}{p^N} \sum_{k=0}^{p^N-1} (-1)^k f(k). \tag{1.1}$$

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By (1.1), the following integral equation holds true, see [**1**, **2**, **5**–**7**]:

$$\int_{\mathbb{Z}_p} f(x+1)\, d\mu_{-1}(x) + \int_{\mathbb{Z}_p} f(x)\, d\mu_{-1}(x) = 2f(0), \tag{1.2}$$

which intensely holds usability in introducing assorted generalizations of many special polynomials such as Euler, Genocchi, Frobenius–Euler and Changhee polynomials, see [**1**, **2**, **4**–**7**, **12**].

The familiar Boole polynomials $\mathrm{Bl}_n(x)$ of the first kind are defined by means of the following generating function [**7**]):

$$\sum_{n=0}^{\infty} \mathrm{Bl}_n(x \mid \omega) \frac{t^n}{n!} = \frac{1}{1+(1+t)^{\omega}}(1+t)^x = \int_{\mathbb{Z}_p} (1+t)^{x+\omega y} d\mu_{-1}(y). \tag{1.3}$$

When $\omega = 1$, we have $\mathrm{Bl}_n(x \mid 1) := 2^{-1}\,\mathrm{Ch}_n(x)$ which are the Changhee polynomials given by the following generating function to be [**6**]

$$\sum_{n=0}^{\infty} \mathrm{Ch}_n(x) \frac{t^n}{n!} = \frac{2}{2+t}(1+t)^x. \tag{1.4}$$

In the case $x = 0$ in the (1.4), one can get $\mathrm{Ch}_n(0) := \mathrm{Ch}_n$ standing for n-th Changhee number [**3**, **8**].

The Boole polynomials of the first kind can be represented by

$$\mathrm{Bl}_n(x \mid \omega) = 2^{-1} \int_{\mathbb{Z}_p} (x+\omega y)_n\, d\mu_{-1}(y), \tag{1.5}$$

where $(x)_n$ is a falling factorial given by [**1**–**3**, **8**, **9**]

$$(x)_n = x(x-1)(x-2)\cdots(x-n+1). \tag{1.6}$$

In the special case, $\mathrm{Bl}_n(0 \mid \omega) := \mathrm{Bl}_n(\omega)$ is called n-th Boole number.

The Boole polynomials of the second kind are defined by means of the following fermionic p-adic integral, see [**6**]:

$$\sum_{n=0}^{\infty} \widehat{\mathrm{Bl}}_n(x \mid \omega) \frac{t^n}{n!} = \frac{1}{2} \int_{\mathbb{Z}_p} (1+t)^{x-\omega y} d\mu_{-1}(y) = \frac{(1+t)^{\omega}}{1+(1+t)^{\omega}}(1+t)^x. \tag{1.7}$$

which also means

$$\widehat{\mathrm{Bl}}_n(x \mid \omega) = 2^{-1} \int_{\mathbb{Z}_p} (x-\omega y)_n d\mu_{-1}(y). \tag{1.8}$$

When $x = 0$, we have $\widehat{\mathrm{Bl}}_n(0 \mid \omega) := \widehat{\mathrm{Bl}}_n(\omega)$ which is called the Boole numbers of the second kind [**6**].

In recent years, the Boole and the Changhee polynomials with their several generalizations studied and developed by a lot of mathematicians possess various applications in p-adic analysis, see [**2**, **4**, **6**, **7**] and also references cited therein.

Formula (1.6) satisfies the following identity:

$$(x)_n = \sum_{k=0}^{n} S_1(n,k) x^k. \tag{1.9}$$

where $S_1(n,k)$ denotes the Stirling numbers of the first kind [**1**, **2**, **4**, **6**, **7**].

The following relation holds true for $n \geqslant 0$:

$$\int_{\mathbb{Z}_p} \binom{x+\omega y}{n} d\mu_{-1}(y) = \sum_{m=0}^{n} \omega^m S_1(n,m) E_m\Big(\frac{x}{\omega}\Big),$$

where $E_m(x/\omega)$ denotes m-th Euler polynomials with the value x/ω defined by [**6**]

$$\sum_{n=0}^{\infty} E_n(y)\frac{t^n}{n!} = \int_{\mathbb{Z}_p} (x+y)^n d\mu_{-1}(x) = \frac{2}{e^t+1}e^{yt}.$$

Note that when $y=0$, we have $E_n(0) := E_n$ called n-th Euler number (see [**6**]).

In this paper, we investigate several relations for p-adic gamma function by means of their Mahler expansion and fermionic p-adic integral on $\mathbb{Z}_p$. We also derived two fermionic p-adic integrals of p-adic gamma function in terms of Boole polynomials and numbers. Moreover, we discover fermionic p-adic integral of the derivative of p-adic gamma function. We acquire a representation for the p-adic Euler constant by means of the Boole polynomials. We finally develop a novel, explicit and interesting representation for the p-adic Euler constant covering Stirling numbers of the first kind.

2. The Boole polynomials related to p-adic gamma function

Throughout this paper, we suppose that $t \in \mathbb{C}_p$ with $|t|_p < p^{1/1-p}$. In this part, we perform to derive some relationships among the two types of Boole polynomials, p-adic gamma function and p-adic Euler constant by making use of the Mahler expansion of the p-adic gamma function.

The p-adic gamma function (see [**3**, **4**, **8**–**11**]) is given by

$$\Gamma_p(x) = \lim_{n\to x} (-1)^n \prod_{\substack{j<n \\ (p,j)=1}} j \quad (x \in \mathbb{Z}_p),$$

where n approaches x through positive integers.

The p-adic Euler constant γ_p is given by

$$\gamma_p := -\frac{\Gamma_p'(1)}{\Gamma_p(0)} = \Gamma_p'(1) = -\Gamma_p'(0). \tag{2.1}$$

The p-adic gamma function in conjunction with its various generalizations and p-adic Euler constant have been investigated and studied by many mathematicians, [**3**, **4**, **8**–**11**]; see also the references cited in each of these earlier works.

For $x \in \mathbb{Z}_p$, the symbol $\binom{x}{n}$ is given by

$$\binom{x}{0} = 1 \text{ and } \binom{x}{n} = \frac{x(x-1)\cdots(x-n+1)}{n!} \quad (n \in \mathbb{N}).$$

Let $x \in \mathbb{Z}_p$ and $n \in \mathbb{N}$. The functions $x \to \binom{x}{n}$ form an orthonormal base of the space $C(\mathbb{Z}_p \to \mathbb{C}_p)$ with respect to the Euclidean norm $|\cdot|_\infty$. The mentioned orthonormal base satisfy the formula

$$\binom{x}{n}' = \sum_{j=0}^{n-1} \frac{(-1)^{n-j-1}}{n-j} \binom{x}{j} \qquad \text{(see [9] and [11])}. \tag{2.2}$$

Kurt Mahler, German mathematician, provided an extension for continuous maps of a p-adic variable using the special polynomials as binomial coefficient polynomial [**9**] in 1958 as follows.

THEOREM 2.1. [**9**] *Every continuous function $f\colon \mathbb{Z}_p \to \mathbb{C}_p$ can be written in the form*

$$f(x) = \sum_{n=0}^{\infty} a_n \binom{x}{n}$$

for all $x \in \mathbb{Z}_p$, where $a_n \in \mathbb{C}_p$ and $a_n \to 0$ as $n \to \infty$.

The base $\{\binom{*}{n} : n \in \mathbb{N}\}$ is termed as Mahler base of the space $C(\mathbb{Z}_p \to \mathbb{C}_p)$, and the components $\{a_n : n \in \mathbb{N}\}$ in $f(x) = \sum_{n=0}^{\infty} a_n \binom{x}{n}$ are called Mahler coefficients of $f \in C(\mathbb{Z}_p \to \mathbb{C}_p)$. The Mahler expansion of the p-adic gamma function Γ_p and its Mahler coefficients are given in [**11**] as follows.

PROPOSITION 2.1. *For $x \in \mathbb{Z}_p$, let $\Gamma_p(x+1) = \sum_{n=0}^{\infty} a_n \binom{x}{n}$ be Mahler series of Γ_p. Then its coefficients satisfy the following expression:*

$$\sum_{n\geqslant 0} (-1)^{n+1} a_n \frac{x^n}{n!} = \frac{1-x^p}{1-x} \exp\left(x + \frac{x^p}{p}\right).$$

The fermionic p-adic integral on $\mathbb{Z}_p$ of the p-adic gamma function via Eq. (1.5) and Proposition 2.1 is as follows.

THEOREM 2.2. *The following identity holds true for $n \in \mathbb{N}$:*

$$\int_{\mathbb{Z}_p} \Gamma_p(\omega x + 1)\, d\mu_{-1}(x) = 2 \sum_{n=0}^{\infty} \frac{a_n}{n!} \operatorname{Bl}_n(\omega),$$

where a_n is given by Proposition 2.1.

PROOF. For $x, \omega \in \mathbb{Z}_p$, by Proposition 2.1, we get

$$\int_{\mathbb{Z}_p} \Gamma_p(\omega x + 1)\, d\mu_{-1}(x) = \sum_{n=0}^{\infty} a_n \int_{\mathbb{Z}_p} \binom{\omega x}{n} d\mu_{-1}(x)$$

and using (1.5), we acquire

$$\int_{\mathbb{Z}_p} \Gamma_p(\omega x + 1)\, d\mu_{-1}(x) = \sum_{n=0}^{\infty} \frac{2a_n}{n!} \operatorname{Bl}_{n,1}(\omega),$$

which gives the asserted result. □

We here present another fermionic p-adic integral of the p-adic gamma function related to the Boole polynomials as follows.

THEOREM 2.3. *Let $x, y, \omega \in \mathbb{Z}_p$. We have*

$$\int_{\mathbb{Z}_p} \Gamma_p(x + \omega y + 1)\, d\mu_{-1}(y) = 2 \sum_{n=0}^{\infty} \frac{a_n}{n!} \operatorname{Bl}_n(x \mid \omega), \tag{2.3}$$

where a_n is given by Proposition 2.1.

PROOF. For $x, y, \omega \in \mathbb{Z}_p$, by the relation $\binom{x+\omega y}{n} = \frac{(x+\omega y)_n}{n!}$ and Proposition 2.1, we get

$$\begin{aligned}\int_{\mathbb{Z}_p} \Gamma_p(x + \omega y + 1)\, d\mu_{-1}(y) &= \int_{\mathbb{Z}_p} \sum_{n=0}^{\infty} a_n \frac{(x + \omega y)_n}{n!} d\mu_{-1}(y) \\ &= \sum_{n=0}^{\infty} a_n \frac{1}{n!} \int_{\mathbb{Z}_p} (x + \omega y)_n d\mu_{-1}(y),\end{aligned}$$

which is the desired result (2.3) via (1.3). □

A relation between $\Gamma_p(x)$ and $\widehat{\operatorname{Bl}}_n(x \mid \omega)$ is stated by the following theorem.

THEOREM 2.4. *For $x, y, \omega \in \mathbb{Z}_p$, we have*

$$\int_{\mathbb{Z}_p} \Gamma_p(x - \omega y + 1)\, d\mu_{-1}(y) = 2 \sum_{n=0}^{\infty} a_n \frac{\widehat{\operatorname{Bl}}_n(x \mid \omega)}{n!},$$

where a_n is given by Proposition 2.1.

PROOF. For $x, y, \omega \in \mathbb{Z}_p$, by the relation $\binom{x-\omega y}{n} = \frac{(x-\omega y)_n}{n!}$ and Proposition 2.1, we get

$$\begin{aligned}\int_{\mathbb{Z}_p} \Gamma_p(x - \omega y + 1)\, d\mu_{-1}(y) &= \int_{\mathbb{Z}_p} \sum_{n=0}^{\infty} a_n \frac{(x - \omega y)_n}{n!} d\mu_{-1}(y) \\ &= \sum_{n=0}^{\infty} a_n \frac{1}{n!} \int_{\mathbb{Z}_p} (x - \omega y)_n d\mu_{-1}(y),\end{aligned}$$

which is the desired result thanks to (1.8). □

A consequence of Theorem 2.4 is given by the following corollary.

COROLLARY 2.1. *Upon setting $x = 0$ in Theorem* 2.4 *gives the following relation for Γ_p and $\widehat{\operatorname{Bl}}_n(\omega)$:*

$$\int_{\mathbb{Z}_p} \Gamma_p(-\omega y + 1)\, d\mu_{-1}(y) = 2 \sum_{n=0}^{\infty} a_n \frac{\widehat{\operatorname{Bl}}_n(\omega)}{n!},$$

where a_n is given by Proposition 2.1.

Here is the fermionic p-adic integral of the derivative of the p-adic gamma function.

THEOREM 2.5. *For $x, y, \omega \in \mathbb{Z}_p$, we have*

$$\int_{\mathbb{Z}_p} \Gamma_p'(x+\omega y+1)\, d\mu_{-1}(y) = 2\sum_{n=0}^{\infty}\sum_{j=0}^{n-1} a_n \frac{(-1)^{n-j-1}\operatorname{Bl}_j(x \mid \omega)}{(n-j)j!}.$$

PROOF. In view of Proposition 2.1, we obtain

$$\begin{aligned}\int_{\mathbb{Z}_p} \Gamma_p'(x+\omega y+1)\, d\mu_{-1}(y) &= \int_{\mathbb{Z}_p}\sum_{n=0}^{\infty} a_n \binom{x+\omega y}{n}' d\mu_{-1}(y)\\ &= \sum_{n=0}^{\infty} a_n \int_{\mathbb{Z}_p}\binom{x+\omega y}{n}' d\mu_{-1}(y)\end{aligned}$$

and using (2.2), we derive

$$\begin{aligned}\int_{\mathbb{Z}_p} \Gamma_p'(x+\omega y+1)\, d\mu_{-1}(y) &= \sum_{n=0}^{\infty}\sum_{j=0}^{n-1} a_n \frac{(-1)^{n-j-1}}{n-j}\int_{\mathbb{Z}_p}\binom{x+\omega y}{j} d\mu_{-1}(y)\\ &= 2\sum_{n=0}^{\infty}\sum_{j=0}^{n-1} a_n \frac{(-1)^{n-j-1}}{n-j}\frac{\operatorname{Bl}_j(x \mid \omega)}{j!}.\end{aligned}$$ □

The immediate result of Theorem 2.5 is given as follows.

COROLLARY 2.2. *For $y \in \mathbb{Z}_p$, we have*

$$\int_{\mathbb{Z}_p} \Gamma_p'(\omega y+1)\, d\mu_{-1}(y) = 2\sum_{n=0}^{\infty}\sum_{j=0}^{n-1} a_n \frac{(-1)^{n-j-1}\operatorname{Bl}_j(\omega)}{(n-j)j!}. \tag{2.4}$$

We now provide a new and interesting representation of the p-adic Euler constant by means of Boole polynomials of the second kind.

THEOREM 2.6. *We have*

$$\gamma_p = \sum_{n=0}^{\infty}\sum_{j=0}^{n-1} a_n (-1)^{n-j}\frac{\operatorname{Bl}_j(\omega-1 \mid \omega)-\operatorname{Bl}_j(-1 \mid \omega)}{(n-j)j!}. \tag{2.5}$$

PROOF. Taking $f(y) = \Gamma_p'(\omega y)$ in (1.2) yields the following result

$$\int_{\mathbb{Z}_p} \Gamma_p'(\omega y+\omega-1+1)\, d\mu_{-1}(y) + \int_{\mathbb{Z}_p} \Gamma_p'(\omega y)\, d\mu_{-1}(y) = 2\Gamma_p'(0).$$

Using (2.1), (2.4) and Theorem 2.5 along with some basic calculations, we have

$$2\sum_{n=0}^{\infty}\sum_{j=0}^{n-1} a_n \frac{(-1)^{n-j-1}\operatorname{Bl}_j(\omega-1 \mid \omega)}{(n-j)j!} + 2\sum_{n=0}^{\infty}\sum_{j=0}^{n-1} a_n \frac{(-1)^{n-j-1}\operatorname{Bl}_j(-1 \mid \omega)}{(n-j)j!} = -2\gamma_p,$$

which implies the asserted result. □

We give the following explicit formula for the p-adic Euler constant.

THEOREM 2.7. *The following explicit formula is valid:*

$$\gamma_p = \sum_{n=0}^{\infty}\sum_{j=0}^{n-1}\frac{a_n}{(n-j)j!}\sum_{m=0}^{\infty}(-1)^{m+n-j} \cdot \sum_{k=0}^{n} S_1(n,k)((-1-\omega m)^k - (-1-\omega-\omega m)^k).$$

PROOF. By (1.7), we get

$$\sum_{n=0}^{\infty}\widehat{\mathrm{Bl}}_n(x\mid\omega)\frac{t^n}{n!} = \frac{1}{1+(1+t)^{\omega}}(1+t)^{x+\omega} = \sum_{m=0}^{\infty}(-1)^m(1+t)^{x+\omega+\omega m}$$
$$= \sum_{m=0}^{\infty}(-1)^m(1+t)^{x+\omega+\omega m} = \sum_{m=0}^{\infty}(-1)^m\sum_{n=0}^{\infty}\binom{x+\omega+\omega m}{n}t^n$$
$$= \sum_{n=0}^{\infty}\left(\sum_{m=0}^{\infty}(-1)^m(x+\omega+\omega m)_n\right)\frac{t^n}{n!},$$

which gives, from (1.9), that

$$\widehat{\mathrm{Bl}}_n(x\mid\omega) = \sum_{m=0}^{\infty}(-1)^m\sum_{k=0}^{n}S_1(n,k)(x+\omega+\omega m)^k.$$

In view of (1.5) and (1.8), we easily obtain that

$$\widehat{\mathrm{Bl}}_n(x\mid\omega) = \mathrm{Bl}_n(x\mid-\omega).$$

So, we derive that

$$\mathrm{Bl}_n(x\mid\omega) = \sum_{m=0}^{\infty}(-1)^m\sum_{k=0}^{n}S_1(n,k)(x-\omega-\omega m)^k.$$

Thus, we have

$$\mathrm{Bl}_n(-1\mid\omega) = \sum_{m=0}^{\infty}(-1)^m\sum_{k=0}^{n}S_1(n,k)(-1-\omega-\omega m)^k \tag{2.6}$$

and

$$\mathrm{Bl}_n(\omega-1\mid\omega) = \sum_{m=0}^{\infty}(-1)^m\sum_{k=0}^{n}S_1(n,k)(-1-\omega m)^k. \tag{2.7}$$

By combining (2.5), (2.6) and (2.7), we arrive at the desired result. □

3. Conclusions and Observations

In this work, we first have handled some multifarious relations for the p-adic gamma function and the Boole polynomials of both sides. We also have acquired the fermionic p-adic integral of the derivative of p-adic gamma function. We then have obtained a new representation for the p-adic Euler constant via the Boole polynomials of both kinds. Lastly, we have investigated an interesting identity for the mentioned constant.

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