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PELL FACTORIANGULAR NUMBERS

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Abstract. We show that the only Pell numbers which are factoriangular are $2, 5$ and 12.

1. Introduction

Recall that the *Pell numbers* $\{P_m\}_{m\geqslant 1}$ are given by

$$P_m = \frac{\alpha^m - \beta^m}{\alpha - \beta}, \quad \text{for all} \quad m \geqslant 1,$$

where $\alpha = 1 + \sqrt{2}$ and $\beta = 1 - \sqrt{2}$. The first few Pell numbers are

1, **2**, **5**, **12**, 29, 70, 169, 408, 985, 2378, 5741, 13860, 33461, 80782, 195025, ...

Castillo [**3**], dubbed a number of the form $Ft_n = n! + n(n+1)/2$ for $n \geqslant 1$ a *factoriangular number*. The first few factoriangular numbers are

2, **5**, **12**, 34, 135, 741, 5068, 40356, 362925, ...

Luca and Gómez-Ruiz [**8**], proved that the only Fibonacci factoriangular numbers are 2, 5 and 34. This settled a conjecture of Castillo from [**3**].

In this paper, we prove the following related result.

Theorem 1.1. *The only Pell numbers which are factoriangular are* 2, 5 *and* 12.

Our method is similar to the one from [**8**]. Assuming $P_m = Ft_n$ for positive integers m and n, we use a linear forms in p-adic logarithms to find some bounds on m and n. The resulting bounds are large so we run a calculation to reduce the bounds. This computation is highly nontrivial and relates on reducing the diophantine equation $P_m = Ft_n$ modulo the primes from a carefully selected finite set of suitable primes.

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2. p-adic linear forms in logarithms

Our main tool is an upper bound for a non-zero p-adic linear form in two logarithms of algebraic numbers due to Bugeaud and Laurent [**1**]. Let η be an algebraic number of degree d over $\mathbb{Q}$ with minimal primitive polynomial over the integers

$$f(x) := a_0 \prod_{i=1}^{d}(X - \eta^{(i)}) \in \mathbb{Z}[X],$$

where the leading coefficient a_0 is positive and the $\eta^{(i)}$, $i = 1, \ldots, d$ are the conjugates of η. The *logarithmic height* of η is given by

$$h(\eta) := \frac{1}{d}\bigg(\log a_0 + \sum_{i=1}^{d} \log(\max\{|\eta^{(i)}|, 1\})\bigg).$$

Let $\mathbb{K}$ be an algebraic number field of degree $d_{\mathbb{K}}$. Let $\eta_1, \eta_2 \in \mathbb{K} \setminus \{0, 1\}$ and b_1, b_2 positive integers. We put $\Lambda = \eta_1^{b_1} - \eta_2^{b_1}$. For a prime ideal π of the ring $\mathcal{O}_{\mathbb{K}}$ of algebraic integers in $\mathbb{K}$ and $\eta \in \mathbb{K}$, we denote by $\mathrm{ord}_\pi(\eta)$ the order at which π appears in the prime factorization of the principal fractional ideal $\eta\mathcal{O}_{\mathbb{K}}$ generated by η in $\mathbb{K}$. When η is an algebraic integer, $\eta\mathcal{O}_{\mathbb{K}}$ is an ideal of $\mathcal{O}_{\mathbb{K}}$. When $\mathbb{K} = \mathbb{Q}$, π is just a prime number. Let e_π and f_π be the ramification index and the inertial degree of π, respectively, and let $p \in \mathbb{Z}$ be the only prime number such that $\pi \mid p$. Then

$$p\mathcal{O}_{\mathbb{K}} = \prod_{i=1}^{k} \pi_i^{e_{\pi_i}}, \quad |\mathcal{O}_{\mathbb{K}}/\pi| = p^{f_{\pi_i}}, \quad d_{\mathbb{K}} = \sum_{i=1}^{k} e_{\pi_i} f_{\pi_i},$$

where $\pi_1 := \pi, \ldots, \pi_k$ are prime ideals in $\mathcal{O}_{\mathbb{K}}$. We set $D := d_{\mathbb{K}}/f_\pi$. Let A_1, A_2 be positive real numbers such that

$$\log A_i \geqslant \max\Big\{h(\eta_i, \frac{\log p}{D}\Big\} \qquad (i = 1, 2).$$

Further, let

$$b' := \frac{b_1}{D \log A_2} + \frac{b_2}{D \log A_1}.$$

With the above notation, Bugeaud and Laurent proved the following result (see [**1**, Corollary 1 to Theorem 3]).

THEOREM 2.1. *Assume that η_1, η_2 are algebraic integers which are multiplicatively independent and that π does not divide $\eta_1\eta_2$. Then*

$$\begin{aligned}\mathrm{ord}_\pi(\Lambda) \leqslant\ & \frac{24p(p^{f_\pi} - 1)}{(p-1)(\log p)^4} D^5 (\log A_1)(\log A_2) \\ & \times \Big(\max\Big\{\log b' + \log(\log p) + 0.4, \frac{10 \log p}{D}, 10\Big\}\Big)^2.\end{aligned}$$

In the actual statement of [**1**], there is only a dependence of D^4 in the right-hand side of the above inequality, but there all the valuations are normalized. Since we work with the actual order $\mathrm{ord}_\pi(\Lambda)$, we must multiply the upper bound of [**1**] by another factor of $d_{\mathbb{K}}/f_\pi = D$.

3. Proof of Theorem 1.1

We study the Diophantine equation

$$P_m = Ft_n. \tag{3.1}$$

We may assume that $n \geqslant 10$, since the smaller values can be checked by hand. Before proving our main result, let us prove some preliminary results that are useful for the proof of Theorem 1.1.

LEMMA 3.1. *The following inequalities*

$$\alpha^{n-2} \leqslant P_n \leqslant \alpha^{n-1} \tag{3.2}$$

hold for all $n \geqslant 1$.

PROOF. Follows immediately by induction on n. □

Further (see [**8**]), the inequalities $\left(\frac{n}{e}\right)^n \leqslant Ft_n \leqslant n^n$ hold for all $n \geqslant 3$. Taking logarithms, the above inequalities yield

$$n(\log n - 1) < \log\left(n! + \frac{n(n+1)}{2}\right) < n \log n, \tag{3.3}$$

for all $n \geqslant 3$. Inequalities (3.2) imply $(m-2)\log\alpha \leqslant \log P_m \leqslant (m-1)\log\alpha$. Using (3.1), we get that for positive integers m, n satisfying (3.1) and $n \geqslant 10$, we have

$$(m-2)\log\alpha \leqslant \log\left(n! + \frac{n(n+1)}{2}\right) \leqslant (m-1)\log\alpha.$$

Combining the last inequality above with (3.3), one has

$$n(\log n - 1) < (m-1)\log\alpha \quad \text{and} \quad (m-2)\log\alpha < n\log n.$$

Hence,

$$1 + \frac{n(\log n - 1)}{\log\alpha} < m < 2 + \frac{n\log n}{\log\alpha}. \tag{3.4}$$

If $n \leqslant 100$, the above inequality implies that $m \leqslant 525$. We listed all Pell numbers P_m with $m \leqslant 525$ and all factoriangular numbers Ft_n with $n \leqslant 100$ and intersected these two lists. All solutions in this range of (3.1) are listed in the Theorem 1.1.

We assume from now on that $n > 100$. Rewriting (3.1) as

$$\frac{\alpha^m - \beta^m}{2\sqrt{2}} = n! + \frac{n(n+1)}{2},$$

we get, after some algebraic manipulations using the fact that $\beta = -\alpha^{-1}$, that

$$\left(2\sqrt{2}\right)n! = \alpha^{-m}\left(\alpha^{2m} - n(n+1)\sqrt{2}\alpha^m + \varepsilon_m\right),$$

where $\varepsilon_m := (-1)^{m+1} \in \{\pm 1\}$. We note that

$$\alpha^{2m} - n(n+1)\sqrt{2}\,\alpha^m + \varepsilon_m = (\alpha^m - z_1)(\alpha^m - z_2),$$

where

$$z_{1,2} := \frac{n(n+1)\sqrt{2} \pm \sqrt{2n^2(n+1)^2 - 4\epsilon}}{2}. \tag{3.5}$$

Thus, equation (3.1) is equivalent to

$$(2\sqrt{2})n! = \alpha^{-m}(\alpha^m - z_1)(\alpha^m - z_2).$$

Let $\mathbb{K} = \mathbb{Q}(z_1)$ and let π be a prime ideal lying above 2 in $\mathcal{O}_{\mathbb{K}}$. As α is unit and $\pi \mid 2$, one has

$$\operatorname{ord}_2(n!) \leqslant \operatorname{ord}_\pi(2\sqrt{2}n!) \leqslant \operatorname{ord}_\pi(\alpha^m - z_1) + \operatorname{ord}_\pi(\alpha^m - z_2). \tag{3.6}$$

We use Theorem 2.1 to get an upper bound on $\operatorname{ord}_\pi(\alpha^m - z_i)$, for $i = 1, 2$. We fix $i \in \{1, 2\}$ and put

$$\eta_1 := \alpha, \quad \eta_2 := z_i, \quad b_1 := m, \quad b_2 := 1, \quad \Lambda_i := \alpha^m - z_i.$$

Note that $z_1 z_2 = \epsilon$. Then z_1, z_2 and α are all units so π cannot divide any of them. Further, these three numbers belong to $\mathbb{K}$. Next we prove that α and z_i are multiplicatively independent for $i = 1, 2$. Of course, since $z_2 = \pm z_1^{-1}$, it suffices to show that this is so only for $i = 1$. Well, note first that since $n > 100$, it follows that $\Delta > 0$. Let d be that positive squarefree integer such that for some positive integer u we have $\Delta = 2n^2(n+1)^2 - 4\epsilon = du^2$. Since the left–hand side above is a multiple of 4 and d is squarefree, it follows that $2 \mid u$. Then, using (3.5), we have

$$z_1 = A\sqrt{2} \pm B\sqrt{d},$$

where

$$(A, B) = (n(n+1)/2, u/2) \in \mathbb{Z}^2.$$

Hence, $z_1^2 = C + D\sqrt{2d}$, where C, D are integers. However, since $z_1^2 \in \mathbb{Q}(2\sqrt{d})$ and $\alpha \in \mathbb{Q}(\sqrt{2})$ and they are multiplicatively dependent, it follows that $d \in \{1, 2\}$. The case $d = 2$, leads to

$$u^2 - (n(n+1))^2 = -2\varepsilon,$$

or

$$(u - n(n+1))(u + n(n+1)) = -2\varepsilon,$$

which is impossible since the left–hand side above is an integer multiple of the number $u + n(n+1) > 100^2$. The case $d = 1$ leads to

$$\left(\frac{u}{2}\right)^2 - 2\left(\frac{n(n+1)}{2}\right)^2 = -\varepsilon \in \{\pm 1\}.$$

Hence, $(X, Y) := (u/2, n(n+1)/2)$ is a positive integer solution of the Pell equation

$$X^2 - 2Y^2 = \pm 1.$$

It is known that $Y = P_k$ for some $k \geqslant 1$. Hence, $P_k = n(n+1)/2$ is a triangular number. Luckily all Pell triangular numbers have been found by McDaniel [**9**].

Lemma 3.2 (Theorem [**9**]). *If P_k is triangular then $k = 1$.*

Since for us $n > 100$, it follows that the equation $P_k = n(n+1)/2$ is impossible. This proves that indeed z_1 and α are multiplicatively dependent.

Thus, $\mathbb{K} = \mathbb{Q}(\sqrt{2}, \sqrt{d})$ has $d_{\mathbb{K}} = 4$. Further, since the discriminant of $\mathbb{K}$ is even (because 2 ramifies in $\mathbb{Q}(\sqrt{2}) \subseteq \mathbb{K}$), it follows that for our prime ideal π, we have $f_\pi \geqslant 2$ and so $D = d_{\mathbb{K}}/f_\pi \leqslant 2$.

Next, we calculate upper bounds for the logarithmic heights of α and z_i. The minimal primitive polynomial of α over the integers is x^2-2x-1, so $h(\alpha)=\frac{1}{2}\log\alpha$.

Since $\alpha > 2$, one can take $\log A_1 = \frac{1}{2}\log\alpha$. Next, the minimal primitive polynomial of z_i over the integers is $z^4+(-2n^2(n+1)^2+2\epsilon)z^2+1$. Its roots are either

$$\pm T_n\sqrt{2}+\sqrt{2T_n^2+1} \quad \text{and} \quad \pm T_n\sqrt{2}-\sqrt{2T_n^2+1},$$

or

$$\pm T_n\sqrt{2}+\sqrt{2T_n^2-1} \quad \text{and} \quad \pm T_n\sqrt{2}-\sqrt{2T_n^2-1}<1.$$

Here, we put $T_n=n(n+1)/2$ for the nth triangular number. In both cases, two of the roots are in absolute value larger than 1 and the other two are in absolute value smaller than 1. Since

$$T_n\sqrt{2}+\sqrt{2T_n^2+1}=T_n\sqrt{2}\left(1+\sqrt{1+\frac{2}{n^2(n+1)^2}}\right)<n^{2.1},$$

for $n>100$, we deduce that

$$h(z_i)=\frac{1}{4}\left(\sum_{j=1}^{4}\log\left(\max\{|z_i^{(j)}|,1\}\right)\right)\leqslant\frac{1}{4}(\log n^{2.1}+\log n^{2.1})=1.05\log n,$$

for $i=1,2$. So one can take $\log A_2=1.05\log n$ and therefore

$$b'=\frac{m}{2.1\log n}+\frac{1}{\log\alpha}.$$

From (3.4), one has

$$m<1.135\,n\log n+2<1.15\,n\log n$$

(since $n>100$). We then get

$$b'<\left(\frac{1.15}{2.1}\right)n+1.135<\frac{4n}{7},$$

and so

$$\log b'+\log\log 2+0.4<\log(4n/7)+\log(\log 2)+0.4<\log n.$$

Thus,

$$\max\left\{\log b'+\log\log 2+0.4,\frac{10\log 2}{2},10\right\}$$

equals $\max\{\log n,10\}$ because $5\log 2<\log n$ for $n>100$. From Theorem 2.1, we get

$$\begin{aligned}\text{(3.7)}\quad \operatorname{ord}_\pi(\Lambda_i)&<\frac{24\times 2\times(2^2-1)\times 2^5}{(2-1)(\log 2)^4}(0.5\log\alpha)(1.05\log n)\\ &\quad\times(\max\{\log n,10\})^2<9236.98(\max\{\log n,10\})^3,\quad\text{for } i=1,2.\end{aligned}$$

In order to use inequality (3.6), we need a lower bound to $\operatorname{ord}_2(n!)$. It is well known that

$$\operatorname{ord}_2(n!)=\left\lfloor\frac{n}{2}\right\rfloor+\left\lfloor\frac{n}{4}\right\rfloor+\cdots+\left\lfloor\frac{n}{2^t}\right\rfloor+\cdots$$

Since for $n > 2^k$, we have

$$\Big\lfloor \frac{n}{2^k} \Big\rfloor \geqslant \frac{n}{2^k} - \frac{2^k - 1}{2^k}.$$

we conclude, using the fact that $n > 100 > 2^4$,

$$\operatorname{ord}_2(n!) \geqslant \sum_{k=1}^{4} \Big(\frac{n}{2^k} - \frac{2^k - 1}{2^k} \Big) = \frac{15n - 49}{16} > \frac{7n}{8}. \tag{3.8}$$

Assuming further that $\log n > 10$ (that is, that $n > 22027$) and combining inequalities (3.6), (3.7) and (3.8), we obtain $n < 21114(\log n)^3$, which leads to $n \leqslant 139212946$.

In summary, we proved the following result.

Lemma 3.3. *Let (n, m) be a solution of Diophantine equation* (3.1) *with $n > 100$. Then,*

$$1 + \frac{n(\log n - 1)}{\log \alpha} < m < 2 + \frac{n \log n}{\log \alpha} \quad \textit{and} \quad n \leqslant 1.4 \times 10^8.$$

Let $\lfloor x \rceil$ denote the nearest integer to the real x. It follows that the positive integer solutions (m, n) of the Diophantine equation (3.1) with $n > 100$ are such that (n, m) belongs to

$$[101, 1.4 \times 10^8] \times \Big[\Big\lfloor 1 + \frac{n(\log n - 1)}{\log \alpha} \Big\rceil, \Big\lfloor 2 + \frac{n \log n}{\log \alpha} \Big\rceil \Big].$$

The bounds for n and m are too large to verify our Diophantine equation (3.1) even computationally. To reduce these bounds we use the procedure described in [**8**]. First, (3.1) is equivalent to

$$P_m = n! \Big(1 + \frac{n+1}{2(n-1)!} \Big)$$

and by the arguments in [**8**], if (m, n) is a solution of (3.1) with $n > 100$, then

$$m = \Bigg\lfloor \frac{\big(n + \frac{1}{2}\big) \log n - n + \log(\sqrt{2\pi})}{\log \alpha} \Bigg\rceil + 1.5 + \delta, \tag{3.9}$$

with $\delta \in \{-0.5, 0.5\}$. We consider two cases for $n \in [101, 1.4 \times 10^8]$.

Case 1. $n \in [101, 5.6 \times 10^5]$. For each n in this interval, we generate the list of $P_m \pmod{10^{20}}$ (i.e., we keep only the last 20 digits of the Pell numbers P_m), where m is given by (3.9). So, since $n! \equiv 0 \pmod{10^{20}}$, we explored computationally the congruence $\frac{n(n+1)}{2} \equiv P_m \pmod{10^{20}}$.

A brief calculation in Maple reveals that the above equation has no solutions in this range. Thus, our Diophantine equation (3.1) has no solutions in this range.

Case 2. $n \in [5.6 \times 10^5, 1.4 \times 10^8]$. It is easy to check that for all $m \equiv m'$ (mod 8), one has $P_m \equiv P'_m \pmod 8$. That is, the Pell sequence is periodic modulo 8 with period 8.

We set $A := 2^5 \times 3^2 \times 5^2 \times 7 \times 11$. We found all primes $p \equiv 1 \pmod 8$ such that $p - 1 \mid A$. They are

$$17, 41, 73, 89, 97, 113, 241, 281, 337, 353, 401, 601, 617, 673, 881, 1009,$$
$$1201, 1321, 1801, 2017, 2521, 2801, 3169, 3361, 3697, 4201, 5281, 7393,$$
$$9241, 12601, 15401, 18481, 19801, 55441, 79201, 92401, 110881.$$

For each prime p above, P_m is periodic modulo p and the period of the Pell sequence modulo p divides A. Hence, if (n, m) is a solution of Diophantine equation (3.1) with $n > 5.6 \times 10^5$, then $n! \equiv 0 \pmod p$. Further, $\frac{n(n+1)}{2} \equiv P_m \pmod p$ is equivalent to $8P_m + 1 \equiv (2n+1)^2 \pmod p$. However, a search in Maple shows that for each $m \in [2, A]$, there is a prime p in the above list such that the Legendre symbol

$$\Big(\frac{8P_m + 1}{p}\Big) = -1$$

except for $m = 1$.

We conclude that the only possible values of $n \in [5.6 \times 10^5, 1.4 \times 10^8]$, which can be solutions of the Diophantine equation (3.1) satisfy the conditions

$$\frac{n(n+1)}{2} \equiv 1 \pmod A, \qquad m \equiv 1 \pmod A. \tag{3.10}$$

One generates the set N_1 of residue classes for $n \pmod A$ fulfilling (3.10) obtaining:

$$\begin{aligned} N_1 = \{&1, 16798, 26398, 43198, 66526, 75073, 83326, 91873, 92926, 101473, \\ &109726, 118273, 141601, 158401, 168001, 184798, 184801, 201598, \\ &211198, 227998, 251326, 259873, 268126, 276673, 277726, 286273, \\ &294526, 303073, 326401, 343201, 352801, 369598, 369601, 386398, \\ &395998, 412798, 436126, 444673, 452926, 461473, 462526, 471073, \\ &479326, 487873, 511201, 528001, 537601, 554398\}. \end{aligned}$$

So, we have the following result.

Lemma 3.4. *If $n \in [5.6 \times 10^5, 1.4 \times 10^8]$ and (n, m) is a solution of Diophantine equation (3.1), then $n \equiv n_0 \pmod A$ and $m \equiv 1 \pmod A$ where $A = 2^5 \times 3^2 \times 5^2 \times 7 \times 11$ and $n_0 \in N_1$. Futhermore,*

$$m = \left\lfloor \frac{\big(n + \frac{1}{2}\big)\log n - n + \log(\sqrt{2\pi})}{\log \alpha} \right\rceil + 1.5 + \delta,$$

with $\delta \in \{-0.5, 0.5\}$.

We analyzed computationally equation (3.1) with the restrictions $n = n_0 + A \times t$ with

$$1 \leqslant t \leqslant \left\lfloor \frac{1.4 \times 10^8}{A} \right\rfloor, \quad n_0 \in N_1, \quad m \equiv 1 \pmod A.$$

For this, we first fixed n and checked whether m given by (3.9) satisfies indeed $m \equiv 1 \pmod{A}$. If this doesn't happen, we can discard n. In the very few cases when this actually happened, we checked directly (3.1). An extensive computational search with Maple showed that equation (3.10) has no other solutions than the ones from the statement of Theorem 1.1. This completes the proof of Theorem 1.1.

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