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REVERSE HILBERT INEQUALITIES INVOLVING SERIES

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Abstract. Some reverse Hilbert's type inequalities involving series of non-negative terms are established by the use of the technique of real analysis, which provides new estimates on inequalities of these type.

1. Introduction

The well-known Hilbert's inequality can be stated as below [**6**, p.226].

Theorem 1.1. *Let* $a_m, b_n \geqslant 0$, $0 < \sum_1^\infty a_m^p \leqslant \infty$ *and* $0 < \sum_1^\infty b_n^q \leqslant \infty$. *If* $p > 1$ *and* $q = p/(p-1)$, *then*

$$\sum_1^\infty \sum_1^\infty \frac{a_m b_n}{m+n} \leqslant \frac{\pi}{\sin(\pi/p)} \left(\sum_1^\infty a_m^p\right)^{1/p} \left(\sum_1^\infty b_n^q\right)^{1/q}. \tag{1.1}$$

Hilbert's inequality and its integral form were studied extensively and numerous variants, generalizations, and extensions appeared in the literatures [**2**–**5**, **7**–**13**, **16**, **17**] and the references cited therein. Some researches on reverse Hilbert inequalities were published in [**15**, **18**, **19**] and et al. In 1998, Pachpatte [**14**] proved some new inequalities similar to Hilbert's inequality (1.1). The main purpose of this paper is to establish some reverse Hilbert's type inequalities by using the technique of real analysis, which provides new estimates on inequalities of these type.

2. Some lemmas

Lemma 2.1. [**6**, p.39] *If* x *and* y *are positive real numbers and* $0 < d < 1$, *then* $d \cdot x^{d-1}(x-y) \leqslant x^d - y^d \leqslant d \cdot y^{d-1}(x-y)$, *with equality if and only if* $x = y$.

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Lemma 2.2. [**1**] *If* $a_1 \geqslant a_2 \geqslant \ldots \geqslant a_n > 0$ *and* $b_1 \geqslant b_2 \geqslant \ldots \geqslant b_n > 0$, *then*

$$\left(\sum_{k=1}^{n} a_k^2\right)\left(\sum_{k=1}^{n} b_k^2\right) \leqslant c \sum_{k=1}^{n} a_k b_k, \tag{2.1}$$

with equality if and only if $a_1 = a_2 = \cdots = a_n$ *and* $b_1 = b_2 = \cdots = b_n$, *and where*

$$c = \max\left\{b_1 \sum_{k=1}^{n} a_k, a_1 \sum_{k=1}^{n} b_k\right\}.$$

Lemma 2.3 (Jensen's inequality [**12**]). *If* $f(x)$ *is continuous and convex function and* p_i $(i = 1, 2, \ldots, n)$ *are nonnegative real numbers (not all* 0*), then*

$$f\left(\frac{\sum_{i=1}^{n} p_i x_i}{\sum_{i=1}^{n} p_i}\right) \leqslant \frac{\sum_{i=1}^{n} p_i f(x_i)}{\sum_{i=1}^{n} p_i}, \tag{2.2}$$

with equality if and only if $x_1 = \cdots = x_n$.

This inequality is reversed if $f(x)$ *is a concave function.*

3. Main results

The following inequality involving series of nonnegative terms was established in [**14**] without equal sign condition.

Theorem 3.1. [**14**] *Let* $p \geqslant 1, q \geqslant 1$ *and* $\{a_m\}$ *and* $\{b_n\}$ *be two non-negative sequences of real numbers defined for* $m = 1, \ldots, k$ *and* $n = 1, \ldots, r$, *where* k, r *are natural numbers. Let* $A_m = \sum_{s=1}^{m} a_s$ *and* $B_n = \sum_{t=1}^{n} b_t$. *Then*

$$\sum_{m=1}^{k}\sum_{n=1}^{r} \frac{A_m^p B_n^q}{m+n} \leqslant C(p,q,k,r)\left(\sum_{m=1}^{k} (k-m+1)\big(a_m A_m^{p-1}\big)^2\right)^{1/2} \times\left(\sum_{n=1}^{r} (r-n+1)\big(b_n B_n^{q-1}\big)^2\right)^{1/2}, \tag{3.1}$$

where $C(p,q,k,r) = \frac{1}{2}pq(kr)^{1/2}$.

In the paper, we first establish a reverse Hilbert type inequality of (3.1) with the equality condition.

Theorem 3.2. *Let* $0 < p < 1$, $0 < q < 1$ *and* $\{a_m\}$ *and* $\{b_n\}$ *be two non-negative and decreasing sequences of real numbers defined for* $m = 1, 2, \ldots, k$ *and* $n = 1, 2, \ldots, r$ *where* k, r *are natural numbers and define* $A_m = \sum_{s=1}^{m} a_s$, $B_n = \sum_{t=1}^{n} b_t$ *and* $A_0 = B_0 = 0$. *Then*

$$\sum_{m=1}^{k}\sum_{n=1}^{r} A_m^p B_n^q \geqslant C'(p,q,a_1,b_1)\left(\sum_{m=1}^{k} (k-m+1)(a_m A_m^{p-1})^2\right) \times\left(\sum_{n=1}^{r}(r-n+1)(b_n B_n^{q-1})^2\right), \tag{3.2}$$

with equality if and only if $a_1 = \cdots = a_m = 0$ *and* $b_1 = \cdots = b_n = 0$, *and where* $C'(p,q,a_1,b_1) = pqa_1^{-p}b_1^{-q}$.

Proof. From the inequality in Lemma 2.1, we obtain

$$\sum_{m=0}^{k-1} pA_{m+1}^{p-1}(A_{m+1}-A_m) \leqslant \sum_{m=0}^{k-1}(A_{m+1}^p - A_m^p).$$

From the equality condition of Lemma 2.1, the above equality holds if and only if $A_1 = \cdots = A_k = 0$, it follows the equality holds if and only if $a_1 = \cdots = a_k = 0$. Hence

$$A_k^p \geqslant p \sum_{m=0}^{k-1} a_{m+1} A_{m+1}^{p-1}, \tag{3.3}$$

with equality if and only if $a_1 = \cdots = a_k = 0$. By replacing m with s first, and then replacing k with m in (3.3), we have

$$A_m^p \geqslant p \sum_{s=1}^{m} a_s A_s^{p-1} \ m = 1, 2, \ldots, k, \tag{3.4}$$

with equality if and only if $a_1 = \cdots = a_m = 0$.

Similarly

$$B_n^q \geqslant q \sum_{t=1}^{n} b_t B_t^{q-1} \ n = 1, 2, \ldots, r, \tag{3.5}$$

with equality if and only if $b_1 = \cdots = b_n = 0$. From (3.4) and (3.5), and in view of Lemma 2.2, we have

(3.6)

$$\begin{aligned} A_m^p B_n^q &\geqslant pq \sum_{s=1}^{m} a_s A_s^{p-1} \sum_{t=1}^{n} b_t B_t^{q-1} \\ &= pq \Bigg(\sum_{s=1}^{m} \big(a_s A_s^{p-1}\big) \times 1 \Bigg) \Bigg(\sum_{t=1}^{n} \big(b_t B_t^{q-1} \times 1\big) \Bigg) \\ &\geqslant pq \frac{1}{\max\left\{ \sum_{s=1}^{m} a_s A_s^{p-1}, m a_1 A_1^{p-1} \right\}} \frac{1}{\max\left\{ \sum_{t=1}^{n} b_t B_t^{q-1}, n b_1 B_1^{q-1} \right\}} \\ &\qquad \times \sum_{s=1}^{m} \big(a_s A_s^{p-1}\big)^2 \cdot \Bigg(\sum_{s=1}^{m} 1^2 \Bigg) \cdot \sum_{t=1}^{n} \big(b_t B_t^{q-1}\big)^2 \cdot \Bigg(\sum_{t=1}^{n} 1^2 \Bigg). \end{aligned}$$

From equality conditions of (3.4), (3.5) and Lemma 2.2, we find that the equality in (3.6) holds if and only if $a_1 = \cdots = a_m = 0$, $b_1 = \cdots = b_n = 0$, and $a_1 A_1^{p-1} = \cdots = a_m A_m^{p-1}$, and $b_1 B_1^{q-1} = \cdots = b_n A_n^{q-1}$. Combining these equality conditions, we get that equality in (3.6) holds if and only if $a_1 = \cdots = a_m = 0$ and $b_1 = \cdots = b_n = 0$. On the other hand, noting that

$$\max\left\{ \sum_{s=1}^{m} a_s A_s^{p-1}, m a_1 A_1^{p-1} \right\} = m a_1^p, \quad \max\left\{ \sum_{t=1}^{n} b_t B_t^{q-1}, n b_1 B_1^{q-1} \right\} = n b_1^q,$$

we obtain

$$A_m^p B_n^q \geqslant pq a_1^{-p} b_1^{-q} \sum_{s=1}^{m} \left(a_s A_s^{p-1}\right)^2 \sum_{t=1}^{n} \left(b_t B_t^{q-1}\right)^2, \tag{3.7}$$

with equality if and only if $a_1 = \cdots = a_m = 0$ and $b_1 = \cdots = b_n = 0$. Taking the sum on both sides of (3.7) over n from 1 to r first and then the sum over m from 1 to k, we obtain

$$\begin{aligned}
\sum_{m=1}^{k}\sum_{n=1}^{r} A_m^p B_n^q &\geqslant pq a_1^{-p} b_1^{-q} \sum_{m=1}^{k} \left(\sum_{s=1}^{m} \left(a_s A_s^{p-1}\right)^2\right) \sum_{n=1}^{r} \left(\sum_{t=1}^{n} \left(b_t B_t^{q-1}\right)^2\right) \\
&= C'(p.q, a_1, b_1) \bigg(\sum_{s=1}^{k} (a_s A_s^{p-1})^2 \sum_{m=s}^{k} 1\bigg)\bigg(\sum_{t=1}^{r} (b_t B_t^{q-1})^2 \sum_{n=t}^{r} 1\bigg) \\
&= C'(p.q, a_1, b_1) \bigg(\sum_{s=1}^{k} (a_s A_s^{p-1})^2 (k-s+1)\bigg)\bigg(\sum_{t=1}^{r} (b_t B_t^{q-1})^2 (r-t+1)\bigg) \\
&= C'(p.q, a_1, b_1) \bigg(\sum_{m=1}^{k} (a_m A_m^{p-1})^2 (k-m+1)\bigg)\bigg(\sum_{n=1}^{r} (b_n B_n^{q-1})^2 (r-n+1)\bigg),
\end{aligned}$$

with equality if and only if $a_1 = \cdots = a_m = 0$ and $b_1 = \cdots = b_n = 0$. □

Remark 3.1. From (3.1) in Theorem 3.1, we may estimate the product

$$\left(\sum_{m=1}^{k} (k-m+1)\left(a_m A_m^{p-1}\right)^2\right)^{1/2} \left(\sum_{n=1}^{r} (r-n+1)\left(b_n B_n^{q-1}\right)^2\right)^{1/2} \tag{3.8}$$

and can get a lower bound. Namely

$$\begin{aligned}
&\left(\sum_{m=1}^{k} (k-m+1)\left(a_m A_m^{p-1}\right)^2\right)^{1/2} \left(\sum_{n=1}^{r} (r-n+1)\left(b_n B_n^{q-1}\right)^2\right)^{1/2} \\
&\qquad\qquad \geqslant 2(pq)^{-1}(kr)^{-1/2} \sum_{m=1}^{k}\sum_{n=1}^{r} \frac{A_m^p B_n^q}{m+n}.
\end{aligned}$$

On the other hand, dividing both sides of (3.2) by $pq a_1^{-p} b_1^{-q}$ and then rooting both sides of (3.2), we have

$$\begin{aligned}
&\left(\sum_{m=1}^{k} (k-m+1)(a_m A_m^{p-1})^2\right)^{1/2} \left(\sum_{n=1}^{r} (r-n+1)(b_n B_n^{q-1})^2\right)^{1/2} \\
&\qquad\qquad \leqslant (pq)^{-1/2} (a_1^p b_1^q)^{1/2} \left(\sum_{m=1}^{k}\sum_{n=1}^{r} A_m^p B_n^q\right)^{1/2},
\end{aligned}$$

with equality if and only if $a_1 = \cdots = a_m = 0$ and $b_1 = \cdots = b_n = 0$. This is just an upper bound of product (3.8).

The following inequality, involving series of nonnegative terms, was also established in [**14**].

THEOREM 3.3. [**14**] *Let* $\{a_m\}$, $\{b_n\}, A_m, B_n$ *be defined as in Theorem* 3.1. *Let* $\{p_m\}$ *and* $\{q_n\}$ *be positive sequences for* $m = 1, \dots, k$ *and* $n = 1, \dots, r$, *where* k, r *are natural numbers. Define* $P_m = \sum_{s=1}^m p_s$ *and* $Q_n = \sum_{t=1}^n q_t$. *Let* ϕ *and* ψ *be nonnegative, convex, sub-multiplicative functions defined on* $\mathbb{R}_+ = [0, +\infty)$. *Then*

$$\sum_{m=1}^k \sum_{n=1}^r \frac{\phi(A_m)\psi(B_n)}{m+n} \leqslant M(k,r)\Big(\sum_{m=1}^k (k-m+1)\Big(p_m\phi\Big(\frac{a_m}{p_m}\Big)\Big)^2\Big)^{1/2} \times\Big(\sum_{n=1}^r (r-n+1)\Big(q_n\psi\Big(\frac{b_n}{q_n}\Big)\Big)^2\Big)^{1/2} \tag{3.9}$$

where

$$M(k,r) = \frac{1}{2}\Big(\sum_{m=1}^k \Big(\frac{\phi(P_m)}{P_m}\Big)^2\Big)^{1/2}\Big(\sum_{n=1}^r \Big(\frac{\psi(Q_n)}{Q_n}\Big)^2\Big)^{1/2}.$$

Next, we establish a reverse Hilbert type inequality of (3.9).

THEOREM 3.4. *Let* $\{a_m\}, \{b_n\}, A_m, B_n$ *be defined as in Theorem* 3.2. *Let* $\{p_m\}$ *and* $\{q_n\}$ *be two positive and decreasing sequences for* $m = 1, 2, \dots, k$ *and* $n = 1, 2, \dots, r$ *where* k, r *are natural numbers and define* $P_m = \sum_{s=1}^m p_s$, $Q_n = \sum_{t=1}^n q_t$. *Let* ϕ *and* ψ *be two nonnegative, increasing, concave and super-multiplicative functions defined on* $\mathbb{R}_+$. *Then*

$$\sum_{m=1}^k \sum_{n=1}^r (\phi(A_m)\psi(B_n))^{1/2} \geqslant \frac{M(p_1, q_1, a_1, b_1, k, r)}{\alpha\beta} \times\Bigg(\sum_{m=1}^k (k-m+1)\Big(p_m\phi\Big(\frac{a_m}{p_m}\Big)\Big)^2\Bigg)\Bigg(\sum_{n=1}^r (r-n+1)\Big(q_n\psi\Big(\frac{b_n}{q_n}\Big)\Big)^2\Bigg),$$

where

$$M(p_1, q_1, a_1, b_1, k, r) = \Bigg(p_1q_1\phi\Big(\frac{a_1}{p_1}\Big)\psi\Big(\frac{b_1}{q_1}\Big)\Bigg)^{1/2}\sum_{m=1}^k \frac{\phi(P_m)}{P_m}\sum_{n=1}^r \frac{\psi(Q_n)}{Q_n},$$

$$\alpha = \max\Bigg\{\Big(\frac{\phi(P_1)}{P_1}\Big)^{1/2}\Bigg(\sum_{s=1}^m \Bigg(p_s\phi\Big(\frac{a_s}{p_s}\Big)\Bigg)^2\Bigg)^{1/2},\ p_1\phi\Big(\frac{a_1}{p_1}\Big)\sum_{m=1}^k \Bigg(\frac{\phi(P_m)}{P_m}\Bigg)^{1/2}\Bigg\},$$

$$\beta = \max\Bigg\{\Big(\frac{\psi(Q_1)}{Q_1}\Big)^{1/2}\Bigg(\sum_{t=1}^n \Bigg(q_t\psi\Big(\frac{b_t}{q_t}\Big)\Bigg)^2\Bigg)^{1/2},\ q_1\psi\Big(\frac{b_1}{q_1}\Big)\sum_{n=1}^r \Bigg(\frac{\psi(Q_n)}{Q_n}\Bigg)^{1/2}\Bigg\}.$$

PROOF. From Lemmas 2.1 and 2.2, and since ϕ is a super-multiplicative function, we obtain

$$\phi(A_m) = \phi\left(\frac{P_m\sum_{s=1}^m p_s a_s/p_s}{\sum_{s=1}^m p_s}\right) \geqslant \phi(P_m)\phi\left(\frac{\sum_{s=1}^m p_s a_s/p_s}{\sum_{s=1}^m p_s}\right) \geqslant \frac{\phi(P_m)}{P_m}\sum_{s=1}^m p_s\phi\Big(\frac{a_s}{p_s}\Big) \geqslant \frac{m\phi(P_m)}{\alpha' P_m}\sum_{s=1}^m \Bigg(p_s\phi\Big(\frac{a_s}{p_s}\Big)\Bigg)^2, \tag{3.10}$$

where $\alpha' = \max\big\{\sum_{s=1}^{m} p_s\phi\big(\frac{a_s}{p_s}\big), mp_1\phi\big(\frac{a_1}{p_1}\big)\big\}$. Since $\{a_m\}$ and $\{q_m\}$ are two non-negative and decreasing sequences, then $\{\frac{a_m}{q_m}\}$ is also non-negative and decreasing sequence, and noting that ϕ is a nonnegative and increasing function, hence

$$\alpha' = mp_1\phi\Big(\frac{a_1}{p_1}\Big). \tag{3.11}$$

Similarly

$$\psi(B_n) \geqslant \frac{n\phi(Q_n)}{\beta' Q_n}\sum_{t=1}^{n}\Big(q_t\psi\Big(\frac{b_t}{q_t}\Big)\Big)^2, \quad \text{where} \quad \beta' = nq_1\psi\Big(\frac{b_1}{q_1}\Big). \tag{3.12}$$

From (3.10), (3.11), and (3.12), we get

$$\begin{aligned}(\phi(A_m)\psi(B_n))^{1/2} &\geqslant \Big(p_1q_1\phi\Big(\frac{a_1}{p_1}\Big)\psi\Big(\frac{b_1}{q_1}\Big)\Big)^{1/2}\\ &\times\Big(\frac{\phi(P_m)}{P_m}\Big)^{1/2}\Big(\sum_{s=1}^{m}\Big(p_s\phi\Big(\frac{a_s}{p_s}\Big)\Big)^2\Big)^{1/2}\Big(\frac{\phi(Q_n)}{Q_n}\Big)^{1/2}\Big(\sum_{t=1}^{n}\Big(q_t\psi\Big(\frac{b_t}{q_t}\Big)\Big)^2\Big)^{1/2}.\end{aligned}$$

Taking the sum over n from 1 to r first and then the sum over m from 1 to k, and by using the inequality in Lemma 2.2, we obtain

$$\begin{aligned}\sum_{m=1}^{k}\sum_{n=1}^{r}(\phi(A_m)\psi(B_n))^{1/2} &\geqslant \Big(p_1q_1\phi\Big(\frac{a_1}{p_1}\Big)\psi\Big(\frac{b_1}{q_1}\Big)\Big)^{1/2}\\ &\quad\times\sum_{m=1}^{k}\Big(\Big(\frac{\phi(P_m)}{P_m}\Big)^{1/2}\Big(\sum_{s=1}^{m}\Big(p_s\phi\Big(\frac{a_s}{p_s}\Big)\Big)^2\Big)^{1/2}\Big)\\ &\quad\times\sum_{n=1}^{r}\Big(\Big(\frac{\psi(Q_n)}{Q_n}\Big)^{1/2}\Big(\sum_{t=1}^{n}\Big(q_t\psi\Big(\frac{b_t}{q_t}\Big)\Big)^2\Big)^{1/2}\Big)\\ &\geqslant \alpha^{-1}\beta^{-1}\Big(p_1q_1\phi\Big(\frac{a_1}{p_1}\Big)\psi\Big(\frac{b_1}{q_1}\Big)\Big)^{1/2}\sum_{m=1}^{k}\frac{\phi(P_m)}{P_m}\times\sum_{m=1}^{k}\Big(\sum_{s=1}^{m}\Big(p_s\phi\Big(\frac{a_s}{p_s}\Big)\Big)^2\Big)\\ &\quad\times\sum_{n=1}^{r}\frac{\psi(Q_n)}{Q_n}\times\sum_{n=1}^{r}\Big(\sum_{t=1}^{n}\Big(q_t\psi\Big(\frac{b_t}{q_t}\Big)\Big)^2\Big)\\ &= \alpha^{-1}\beta^{-1}M(p_1,q_1,a_1,b_1,k,r)\Big(\sum_{s=1}^{k}\Big(p_s\phi\Big(\frac{a_s}{p_s}\Big)\Big)^2(k-s+1)\Big)\\ &\quad\times\Big(\sum_{t=1}^{r}\Big(q_t\psi\Big(\frac{b_t}{q_t}\Big)\Big)^2(r-t+1)\Big)\\ &= \alpha^{-1}\beta^{-1}M(p_1,q_1,a_1,b_1,k,r)\Big(\sum_{m=1}^{k}\Big(p_m\phi\Big(\frac{a_m}{p_m}\Big)\Big)^2(k-m+1)\Big)\\ &\quad\times\Big(\sum_{n=1}^{r}\Big(q_n\psi\Big(\frac{b_n}{q_n}\Big)\Big)^2(r-n+1)\Big),\end{aligned}$$

where

$$M(p_1,q_1,a_1,b_1,k,r)=\Big(p_1q_1\phi\Big(\frac{a_1}{p_1}\Big)\psi\Big(\frac{b_1}{q_1}\Big)\Big)^{1/2}\sum_{m=1}^{k}\frac{\phi(P_m)}{P_m}\sum_{n=1}^{r}\frac{\psi(Q_n)}{Q_n},$$

$$\alpha=\max\bigg\{\Big(\frac{\phi(P_1)}{P_1}\Big)^{1/2}\Big(\sum_{s=1}^{m}\Big(p_s\phi\Big(\frac{a_s}{p_s}\Big)\Big)^2\Big)^{1/2},\ p_1\phi\Big(\frac{a_1}{p_1}\Big)\sum_{m=1}^{k}\Big(\frac{\phi(P_m)}{P_m}\Big)^{1/2}\bigg\},$$

$$\beta=\max\bigg\{\Big(\frac{\psi(Q_1)}{Q_1}\Big)^{1/2}\Big(\sum_{t=1}^{n}\Big(q_t\psi\Big(\frac{b_t}{q_t}\Big)\Big)^2\Big)^{1/2},\ q_1\psi\Big(\frac{b_1}{q_1}\Big)\sum_{n=1}^{r}\Big(\frac{\psi(Q_n)}{Q_n}\Big)^{1/2}\bigg\}.\quad\square$$

REMARK 3.2. From (3.9), we may estimate the product

$$\Big(\sum_{m=1}^{k}(k-m+1)\Big(p_m\phi\Big(\frac{a_m}{p_m}\Big)\Big)^2\Big)^{1/2}\Big(\sum_{n=1}^{r}(r-n+1)\Big(q_n\psi\Big(\frac{b_n}{q_n}\Big)\Big)^2\Big)^{1/2}. \tag{3.13}$$

and can get a lower bound. Namely

$$\Big(\sum_{m=1}^{k}(k-m+1)\Big(p_m\phi\Big(\frac{a_m}{p_m}\Big)\Big)^2\Big)^{1/2}\Big(\sum_{n=1}^{r}(r-n+1)\Big(q_n\psi\Big(\frac{b_n}{q_n}\Big)\Big)^2\Big)^{1/2} \geqslant \frac{1}{M(k,r)}\sum_{m=1}^{k}\sum_{n=1}^{r}\frac{\phi(A_m)\psi(B_n)}{m+n}$$

On the other hand, dividing both sides of (3.2) by $\frac{M(p_1,q_1,a_1,b_1,k,r)}{\alpha\beta}$ and then rooting both sides of (2.1), we have

$$\Big(\sum_{m=1}^{k}(k-m+1)\Big(p_m\phi\Big(\frac{a_m}{p_m}\Big)\Big)^2\Big)^{1/2}\Big(\sum_{n=1}^{r}(r-n+1)\Big(q_n\psi\Big(\frac{b_n}{q_n}\Big)\Big)^2\Big)^{1/2} \leqslant \frac{(\alpha\beta)^{1/2}}{M^{1/2}(p_1,q_1,a_1,b_1,k,r)}\Big(\sum_{m=1}^{k}\sum_{n=1}^{r}(\phi(A_m)\psi(B_n))^{1/2}\Big)^{1/2}.$$

This is just an upper bound of product (3.13).

The following inequality involving series of nonnegative terms was established in [**14**] without equal sign condition.

THEOREM 3.5. [**14**] *Let $\{a_m\}$ and $\{b_n\}$ be as defined in Theorem 3.2. Define $A_m=\frac{1}{m}\sum_{s=1}^{m}p_sa_s$ and $B_n=\frac{1}{n}\sum_{t=1}^{n}q_tb_t$, for $m=1,\dots,k$ and $n=1,\dots,r$, where k,r are natural numbers. Let ϕ and ψ be nonnegative and convex functions defined on $\mathbb{R}_+=[0,+\infty)$. Then*

$$\sum_{m=1}^{k}\sum_{n=1}^{r}\frac{mn}{m+n}\phi(A_m)\psi(B_n)\leqslant\frac{1}{2}(kr)^{1/2}\Big(\sum_{m=1}^{k}(k-m+1)(\phi(a_m))^2\Big)^{1/2} \times\Big(\sum_{n=1}^{r}(r-n+1)(\psi(b_n))^2\Big)^{1/2}. \tag{3.14}$$

In the following, we establish a reverse Hilbert type inequality of (3.14) with equality condition.

THEOREM 3.6. *Let $\{a_m\}$ and $\{b_n\}$ be is in Theorem 3.2 and define $A_m = \frac{1}{m}\sum_{s=1}^{m} a_s$, and $B_n = \frac{1}{n}\sum_{t=1}^{n} b_t$, for $m = 1, 2, \ldots, k$ and $n = 1, 2, \ldots, r$ where k, r are natural numbers. Let ϕ and ψ be two nonnegative, increasing and concave functions defined on R_+. Then*

$$\sum_{m=1}^{k}\sum_{n=1}^{r}\frac{\phi(A_m)\psi(B_n)}{(m^2+n^2)^{-1}} \geqslant 2(\phi(a_1)\psi(b_1))^{-1}\left(\sum_{m=1}^{k}(k-m+1)(\phi(a_m))^2\right) \times\left(\sum_{n=1}^{r}(n-t+1)(\psi(b_n))^2\right) \tag{3.15}$$

with equality if and only if $a_1 = \cdots = a_m = 0$, $b_1 = \cdots = b_n = 0$ and $m = n$.

PROOF. By the hypotheses and Jensen's inequality, we have

$$\phi(A_m) = \phi\left(\frac{1}{m}\sum_{s=1}^{m} a_s\right) \geqslant \frac{1}{m}\sum_{s=1}^{m}\phi(a_s), \tag{3.16}$$

with equality if and only if $a_1 = \cdots = a_m$. By using Lemma 2.2, we have

$$\phi(A_m) \geqslant \frac{1}{\max\{\sum_{s=1}^{m}\phi(a_s), m\phi(a_1)\}}\sum_{s=1}^{m}(\phi(a_s))^2 = \frac{1}{m\phi(a_1)}\sum_{s=1}^{m}(\phi(a_s))^2. \tag{3.17}$$

From the equality conditions of (3.16) and the inequality Lemma 2.2, we find that the equality in (3.17) holds if and only if $a_1 = \cdots = a_m$ and $\phi(a_1) = \cdots = \phi(a_m)$. Combining these equality conditions, it follows the equality in (3.17) holds if and only if $a_1 = \cdots = a_m = 0$.

Similarly

$$\psi(B_n) \geqslant \frac{1}{n\psi(b_1)}\sum_{t=1}^{n}(\psi(b_t))^2, \tag{3.18}$$

with equality if and only if $b_1 = \cdots = b_n = 0$. From (3.17), (3.18) and by using the elementary inequality $2mn \leqslant m^2 + n^2$, we obtain

$$\begin{aligned}\phi(A_m)\psi(B_n) &\geqslant \frac{1}{mn\phi(a_1)\psi(b_1)}\sum_{s=1}^{m}(\phi(a_s))^2\sum_{t=1}^{n}(\psi(b_t))^2 \\ &\geqslant \frac{2}{(m^2+n^2)\phi(a_1)\psi(b_1)}\sum_{s=1}^{m}(\phi(a_s))^2\sum_{t=1}^{n}(\psi(b_t))^2\end{aligned} \tag{3.19}$$

with equality if and only if $a_1 = \cdots = a_m = 0$, $b_1 = \cdots = b_n = 0$ and $m = n$. Dividing both sides of (3.19) by $(m^2 + n^2)^{-1}$ first and then taking the sum over n from 1 to r and the sum over m from 1 to k, we obtain

$$\sum_{m=1}^{k}\sum_{n=1}^{r}\frac{\phi(A_m)\psi(B_n)}{(m^2+n^2)^{-1}} \geqslant \frac{2}{\phi(a_1)\psi(b_1)}\sum_{m=1}^{k}\left(\sum_{s=1}^{m}(\phi(a_s))^2\right)\cdot\sum_{n=1}^{r}\left(\sum_{t=1}^{n}(\psi(b_t))^2\right)$$

$$\geqslant \frac{2}{\phi(a_1)\psi(b_1)}\bigg(\sum_{s=1}^{k}(\phi(a_s))^2(k-s+1)\bigg)\bigg(\sum_{t=1}^{r}(\psi(b_t))^2(r-t+1)\bigg)$$
$$= \frac{2}{\phi(a_1)\psi(b_1)}\bigg(\sum_{m=1}^{k}(\phi(a_m))^2(k-m+1)\bigg)\bigg(\sum_{n=1}^{r}(\psi(b_n))^2(n-t+1)\bigg),$$

with equality if and only if $a_1 = \cdots = a_m = 0$, $b_1 = \cdots = b_n = 0$ and $m = n$. □

REMARK 3.3. From (3.14), we can estimate the product

$$\bigg(\sum_{m=1}^{k}(k-m+1)\big(\phi(a_m)\big)^2\bigg)^{1/2}\bigg(\sum_{n=1}^{r}(r-n+1)\big(\psi(b_n)\big)^2\bigg)^{1/2}. \tag{3.20}$$

and can get the lower bound. Namely

$$\bigg(\sum_{m=1}^{k}(k-m+1)\big(\phi(a_m)\big)^2\bigg)^{1/2}\bigg(\sum_{n=1}^{r}(r-n+1)\big(\psi(b_n)\big)^2\bigg)^{1/2} \geqslant 2(kr)^{-1/2}\sum_{m=1}^{k}\sum_{n=1}^{r}\frac{mn\phi(A_m)\psi(B_n)}{m+n}.$$

On the other hand, dividing both sides of (3.15) by $\frac{2}{\phi(a_1)\psi(b_1)}$ and then rooting both sides of (3.15), we have

$$\bigg(\sum_{m=1}^{k}(k-m+1)\big(\phi(a_m)\big)^2\bigg)^{1/2}\bigg(\sum_{n=1}^{r}(r-n+1)\big(\psi(b_n)\big)^2\bigg)^{1/2} \leqslant \frac{\sqrt{2}}{2}(\phi(a_1)\psi(b_1))^{1/2}\bigg(\sum_{m=1}^{k}\sum_{n=1}^{r}\frac{\phi(A_m)\psi(B_n)}{(m^2+n^2)^{-1}}\bigg)^{1/2}$$

with equality if and only if $a_1 = \cdots = a_m = 0$, $b_1 = \cdots = b_n = 0$ and $m = n$. This is just an upper bound of product (3.20).

The following inequality involving series of nonnegative terms was also established in [**14**] without equal sign condition.

THEOREM 3.7. [**14**] *Let* $\{a_m\}, \{b_n\}, \{p_m\}, \{q_n\}, P_m, Q_n$ *be is in Theorem* 3.3 *and define* $A_m = (1/P_m)\sum_{s=1}^{m} p_s a_s$, $B_n = (1/Q_n)\sum_{t=1}^{n} q_t b_t$, *for* $m = 1, 2, \ldots, k$ *and* $n = 1, 2, \ldots, r$ *where* k, r *are the natural numbers. Let* ϕ *and* ψ *be as in Theorem* 3.5. *Then*

$$\sum_{m=1}^{k}\sum_{n=1}^{r}\frac{P_mQ_n\phi(A_m)\psi(B_n)}{m+n} \leqslant \frac{1}{2}(kr)^{1/2} \times\bigg(\sum_{m=1}^{k}(k-m+1)\Big(p_m\phi(a_m)\Big)^2\bigg)^{1/2}\bigg(\sum_{n=1}^{r}(r-n+1)\Big(q_n\psi(b_n)\Big)^2\bigg)^{1/2}. \tag{3.21}$$

Finally, we establish a reverse Hilbert type inequality of (3.21) with equality condition.

Theorem 3.8. *Let* $\{a_m\}, \{b_n\}, \{p_m\}, \{q_n\}, P_m, Q_n$ *be is in Theorem* 3.4 *and define* $A_m = (1/P_m)\sum_{s=1}^{m} p_s a_s$, *and* $B_n = (1/Q_n)\sum_{t=1}^{n} q_t b_t$, *for* $m = 1, 2, \dots, k$ *and* $n = 1, 2, \dots, r$ *where* k, r *are the natural numbers. Let* ϕ *and* ψ *be as in Theorem* 3.6. *Then*

$$\sum_{m=1}^{k}\sum_{n=1}^{r} P_m Q_n \phi(A_m)\psi(B_n) \geqslant \frac{1}{p_1 q_1 \phi(a_1)\psi(b_1)} \times \left(\sum_{m=1}^{k}(k-m+1)(p_m\phi(a_m))^2\right)\left(\sum_{n=1}^{r}(r-n+1)(\psi(q_n b_n))^2\right), \tag{3.22}$$

with equality if and only if $a_1 = \cdots = a_m = 0$, $p_1 = \cdots = p_m = 0$, $b_1 = \cdots = b_n = 0$ *and* $q_1 = \cdots = q_n = 0$.

Proof. By the hypotheses and Jensen's inequality (2.2), we have

$$\phi(A_m) = \phi\Big(\frac{1}{P_m}\sum_{s=1}^{m} p_s a_s\Big) \geqslant \frac{1}{P_m}\sum_{s=1}^{m} p_s \phi(a_s), \tag{3.23}$$

with equality if and only if $a_1 = \cdots = a_m$. By using Lemma 2.2, we have

$$\begin{aligned}\phi(A_m) &\geqslant \frac{m}{P_m \max\{\sum_{s=1}^{m} p_s\phi(a_s), m p_1 \phi(a_1)\}}\sum_{s=1}^{m}(p_s\phi(a_s))^2 \\ &= \frac{1}{P_m p_1 \phi(a_1)}\sum_{s=1}^{m}(p_s\phi(a_s))^2.\end{aligned} \tag{3.24}$$

From the equality conditions of (3.23) and inequality Lemma 2.2, we find that the equality in (3.24) holds if and only if $a_1 = \cdots = a_m$ and $p_1\phi(a_1) = \cdots = p_m\phi(a_m)$. Combining these equality conditions, it follows that the equality in (3.24) holds if and only if $a_1 = \cdots = a_m = 0$ and $p_1 = \cdots = p_m = 0$.

Similarly

$$\psi(B_n) \geqslant \frac{1}{Q_n q_1 \psi(b_1)}\sum_{t=1}^{n}(q_t\psi(b_t))^2, \tag{3.25}$$

with equality if and only if $b_1 = \cdots = b_n = 0$ and $q_1 = \cdots = q_n = 0$.

From (3.24) and (3.25), we have

$$P_m Q_n \phi(A_m)\psi(B_n) \geqslant \frac{1}{p_1 q_1 \phi(a_1)\psi(b_1)}\sum_{s=1}^{m}(p_s\phi(a_s))^2\sum_{s=1}^{m}(q_t\psi(b_t))^2,$$

with equality if and only if $a_1 = \cdots = a_m = 0$, $p_1 = \cdots = p_m = 0$, $b_1 = \cdots = b_n = 0$ and $q_1 = \cdots = q_n = 0$.

Taking the sum over n from 1 to r first and then the sum over m from 1 to k, we obtain

$$\sum_{m=1}^{k}\sum_{n=1}^{r} P_m Q_n \phi(A_m)\psi(B_n) \geqslant \frac{1}{p_1 q_1 \phi(a_1)\psi(b_1)} \times \sum_{m=1}^{k}\bigg(\sum_{s=1}^{m}(p_s\phi(a_s))^2\bigg)\sum_{n=1}^{r}\bigg(\sum_{t=1}^{n}(q_t\psi(b_t))^2\bigg)$$
$$= \frac{1}{p_1 q_1 \phi(a_1)\psi(b_1)}\bigg(\sum_{s=1}^{k}(k-s+1)(p_s\phi(a_s))^2\bigg)\bigg(\sum_{t=1}^{r}(r-t+1)(\psi(q_t b_t))^2\bigg)$$
$$= \frac{1}{p_1 q_1 \phi(a_1)\psi(b_1)}\bigg(\sum_{m=1}^{k}(k-m+1)(p_m\phi(a_m))^2\bigg)\bigg(\sum_{n=1}^{r}(r-n+1)(\psi(q_n b_n))^2\bigg)$$

with equality if and only if $a_1 = \cdots = a_m = 0$, $p_1 = \cdots = p_m = 0$, $b_1 = \cdots = b_n = 0$ and $q_1 = \cdots = q_n = 0$. □

REMARK 3.4. From (3.21) in Theorem 3.7, we can estimate the product

$$\bigg(\sum_{m=1}^{k}(k-m+1)\big(p_m\phi(a_m)\big)^2\bigg)^{1/2}\bigg(\sum_{n=1}^{r}(r-n+1)\big(q_n\psi(b_n)\big)^2\bigg)^{1/2}. \tag{3.26}$$

and can get the lower bound. Namely

$$\bigg(\sum_{m=1}^{k}(k-m+1)\big(p_m\phi(a_m)\big)^2\bigg)^{1/2}\bigg(\sum_{n=1}^{r}(r-n+1)\big(q_n\psi(b_n)\big)^2\bigg)^{1/2} \geqslant 2(kr)^{-1/2}\sum_{m=1}^{k}\sum_{n=1}^{r}\frac{P_m Q_n \phi(A_m)\psi(B_n)}{m+n}.$$

On the other hand, dividing both sides of (3.22) by $\frac{1}{p_1 q_1\phi(a_1)\psi(b_1)}$ and then rooting both sides of (3.22), we have

$$\bigg(\sum_{m=1}^{k}(k-m+1)\big(p_m\phi(a_m)\big)^2\bigg)^{1/2}\bigg(\sum_{n=1}^{r}(r-n+1)\big(q_n\psi(b_n)\big)^2\bigg)^{1/2} \geqslant (p_1 q_1\phi(a_1)\psi(b_1))^{1/2}\bigg(\sum_{m=1}^{k}\sum_{n=1}^{r} P_m Q_n\phi(A_m)\psi(B_n)\bigg)^{1/2}$$

with equality if and only if $a_1 = \cdots = a_m = 0$, $p_1 = \cdots = p_m = 0$, $b_1 = \cdots = b_n = 0$ and $q_1 = \cdots = q_n = 0$.

This is just an upper bound of product (3.26).

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