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# THE CONVOLUTION OF FINITE NUMBER OF ANALYTIC FUNCTIONS

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Abstract. We investigate various results associated with the convolution of finite number of analytic functions involving a certain multiplier operator (defined below). Some useful consequences including a result related to the zeta function are also mentioned.

## 1. Introduction and Preliminaries

Let $\mathcal{H}(\mathbb{U})$ denote a class of all analytic functions defined in the open unit disk $\mathbb{U} = \{z \in \mathbb{C} : |z| < 1\}$. For $a \in \mathbb{C}$, $j \in \mathbb{N} = \{1, 2, \dots\}$, let

$$\mathcal{H}[a, j] = \{f \in \mathcal{H}(\mathbb{U}) : f(z) = a + a_j z^j + a_{j+1} z^{j+1} + \cdots\}.$$

We denote a subclass of $\mathcal{H}[0, 1]$ by $\mathcal{A}$ whose members are of the form:

$$f(z) = z + \sum_{k=2}^{\infty} a_k z^k, \quad z \in \mathbb{U}. \tag{1.1}$$

Let $\mathcal{K}$ denote a subclass of $\mathcal{A}$ whose members are convex (univalent) in $\mathbb{U}$ and satisfying

$$\operatorname{Re}\Big(1 + \frac{zf'(z)}{f''(z)}\Big) > 0, \quad z \in \mathbb{U}.$$

For two functions $p, q \in \mathcal{H}(\mathbb{U})$, we say $p$ is *subordinate* to $q$, or $q$ is *superordinate* to $p$ in $\mathbb{U}$ and write $p(z) \prec q(z)$, $z \in \mathbb{U}$, if there exists a Schwarz function $\omega$ analytic in $\mathbb{U}$ with $\omega(0) = 0$ and $|\omega(z)| < 1$, $z \in \mathbb{U}$ such that $p(z) = q(\omega(z))$, $z \in \mathbb{U}$. Furthermore, if the function $q$ is univalent in $\mathbb{U}$, then we have the following equivalence:

$$p(z) \prec q(z) \Leftrightarrow p(0) = q(0) \ \text{ and } \ p(\mathbb{U}) \subset q(\mathbb{U}).$$

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A convolution (or Hadamard product) $*$ of two functions $g_1(z)$ and $g_2(z)$ of the form: $g_1(z) = \sum_{k=0}^{\infty} a_k z^k$ and $g_2(z) = \sum_{k=0}^{\infty} b_k z^k$ is defined by

$$g_1(z) * g_2(z) = (g_1 * g_2)(z) = \sum_{k=0}^{\infty} a_k b_k z^k = (g_2 * g_1)(z).$$

Definition 1.1. Let for $m \in \mathbb{Z} = \{\dots, -2, -1, 0, 1, 2, \dots\}$ and for $\mu > -1$, $\lambda > 0$, a linear operator $\mathcal{J}_{\lambda,\mu}^{m} \colon \mathcal{A} \to \mathcal{A}$ be defined by

$$\begin{cases} \mathcal{J}_{\lambda,\mu}^{m} f(z) = f(z), & m = 0, \\ \mathcal{J}_{\lambda,\mu}^{m} f(z) = \frac{\mu+1}{\lambda} z^{1-\frac{\mu+1}{\lambda}} \int_0^z t^{\frac{\mu+1}{\lambda}-2} \mathcal{J}_{\lambda,\mu}^{m+1} f(t)\, dt, & m = -1, -2, \dots, \\ \mathcal{J}_{\lambda,\mu}^{m} f(z) = \frac{\lambda}{\mu+1} z^{2-\frac{\mu+1}{\lambda}} \frac{d}{dz}\big(z^{\frac{\mu+1}{\lambda}-1} \mathcal{J}_{\lambda,\mu}^{m-1} f(z)\big), & m = 1, 2, \dots. \end{cases}$$

We note that the operator $\mathcal{J}_{\lambda,\mu}^{m}$ is a multiplier operator which was recently introduced and studied in [**9**] (see also [**8**, **13**]).

It is easy to observe that for any $m_1, m_2 \in \mathbb{Z}$ and for $f_1, f_2 \in \mathcal{A}$:

$$\mathcal{J}_{\lambda,\mu}^{m_1} \mathcal{J}_{\lambda,\mu}^{m_2} f(z) = \mathcal{J}_{\lambda,\mu}^{m_1+m_2} f(z), \tag{1.2}$$

$$\mathcal{J}_{\lambda,\mu}^{m_1} f_1(z) * \mathcal{J}_{\lambda,\mu}^{m_2} f_2(z) = \mathcal{J}_{\lambda,\mu}^{m_1+m_2} (f_1 * f_2)(z). \tag{1.3}$$

The series expansion of $\mathcal{J}_{\lambda,\mu}^{m} f(z)$ for $m \in \mathbb{Z}$, $\mu > -1$, $\lambda > 0$ for the function $f$ of the form (1.1) is given by

$$\mathcal{J}_{\lambda,\mu}^{m} f(z) = z + \sum_{k=2}^{\infty} \Big(1 + \frac{\lambda(k-1)}{\mu+1}\Big)^m a_k z^k.$$

The operator $\mathcal{J}_{\lambda,\mu}^{m}$ generalizes some known operators which are exhibited here by the following relationships:

(i) $\mathcal{J}_{\lambda,0}^{m} = \mathcal{D}_{\lambda}^{m}$ ($m \in \mathbb{N}_0$, (Al-Oboudi [**2**])),
(ii) $\mathcal{J}_{1,0}^{m} = \mathcal{D}^{m}$ ($m \in \mathbb{N}_0$, (Sălăgean [**11**])),
(iii) $\mathcal{J}_{\lambda,\mu}^{m} = \mathcal{I}^{m}(\lambda, \mu)$ ($m \in \mathbb{N}_0$, (Cătaş [**3**])),
(iv) $\mathcal{J}_{1,1}^{-\alpha} = \mathcal{P}^{\alpha}$ ($\alpha \in \mathbb{Z}^+$, (Jung et al. [**6**])).

Let $\mathcal{P}[A, B]$ denote a class of functions $p \in \mathcal{H}(\mathbb{U})$ satisfying $p(0) = 1$ and

$$p(z) \prec \frac{1 + Az}{1 + Bz}, -1 \leqslant B \leqslant 1, -1 \leqslant A \leqslant 1, z \in \mathbb{U},$$

which was introduced and studied by Janowski [**5**], and we denote the class $\mathcal{P}[1 - 2\alpha, -1]$ by $\mathcal{P}(\alpha)$ ($\alpha \leqslant 1$).

The object of this paper is to study the convolution of finite number of analytic functions involving the multiplier operator $\mathcal{J}_{\lambda,\mu}^{m}$ ($m \in \mathbb{Z}$). With this aim, some results on subordination are obtained and by applying the $n$-th order Euler-type differential subordination when the functions $f_i$ satisfy the subordination condition $(\mathcal{J}_{\lambda,\mu}^{m} f_i(z))' \prec \frac{1+A_i z}{1+B_i z}$ ($-1 \leqslant B_i < A_i \leqslant 1$; $i = 1, 2, \dots, n$), a sufficiency condition for convexity of the convolution of functions $\mathcal{J}_{\lambda,\mu}^{m} f_i$ ($i = 1, 2, \dots, n$) is derived. Further, on applying the differential subordination, a sharp result on the convolution of functions $(\mathcal{J}_{\lambda,\mu}^{m+1} f_i(z))'$ ($i = 1, 2, \dots, n$) is also obtained.

In order to state and prove our results, we need the following lemmas.

Lemma 1.1. (Hallenbeck and Ruscheweyh [**4**] [**7**, Th. 3.1b, p. 71]) *Let $h$ be convex in $\mathbb{U}$ with $h(0) = a$, $\gamma \neq 0$ and $\operatorname{Re}(\gamma) \geqslant 0$. If $p \in \mathcal{H}[a, n]$ and*

$$p(z) + \frac{1}{\gamma} zp'(z) \prec h(z),$$

*then $p(z) \prec q(z) \prec h(z)$, where*

$$q(z) = \frac{\gamma}{nz^{\gamma/n}} \int_0^z h(t)t^{(\gamma/n)-1}\, dt \quad (z \in \mathbb{U}).$$

*The function $q$ is convex and is the best $(a, n)$-dominant.*

The Gaussian hypergeometric function ${}_2F_1(a, b; c; z)$ is an analytic function in $\mathbb{U}$ and is defined for $a, b, c \in \mathbb{C}$ $(c \neq 0, -1, -2, \dots)$ by

$${}_2F_1(a, b; c; z) = \sum_{n=0}^{\infty} \frac{(a)_n (b)_n}{(c)_n} \frac{z^n}{n!} \quad (z \in \mathbb{U}),$$

where $(a)_n$ is the Pochhammer symbol representing the product

$$(a)_n = a(a + 1) \cdots (a + n - 1),\ n \in \mathbb{N};\ (a)_0 = 1.$$

The following results for the function ${}_2F_1(a, b; c; z)$ are well known.

Lemma 1.2. [**1, 16**] *Let $a, b, c \in \mathbb{C}$ $(c \neq 0, -1, -2, \dots)$, then the Gaussian hypergeometric function ${}_2F_1(a, b; c; z)$ satisfies the following identities:*

(i) ${}_2F_1(a, b; c; z) = \frac{\Gamma(c)}{\Gamma(b)\Gamma(c-b)} \int_0^1 t^{b-1}(1-t)^{c-b-1}(1-tz)^{-a}\, dt$ $(\operatorname{Re}(c) > \operatorname{Re}(b) > 0)$.

(ii) ${}_2F_1(a, b; c; z) = (1 - z)^{-a}\, {}_2F_1(a, c - b; c; \frac{z}{z-1})$.

Lemma 1.3. *[**10**] Let $F, G \in \mathcal{H}(\mathbb{U})$ be any convex univalent functions in $\mathbb{U}$. If $f \prec F$ and $g \prec G$, then $f * g \prec F * G$ in $\mathbb{U}$.*

Lemma 1.4. [**17**] *If $p_i \in \mathcal{P}(\alpha_i)$ $(i = 1, 2;\ \alpha_i \leqslant 1)$, then $p_1 * p_2 \in \mathcal{P}(\alpha_3)$, where $\alpha_3 = 1 - 2(1 - \alpha_1)(1 - \alpha_2)$.*

Lemma 1.5. *[**15**] Let $p \in \mathcal{P}(\alpha)$. Then for given $\alpha$, $0 \leqslant \alpha < 1$,*

$$\operatorname{Re}(p(z)) > 2\alpha - 1 + \frac{2(1 - \alpha)}{1 + |z|} \quad (z \in \mathbb{U}).$$

## 2. Main Results

Making use of Lemma 1.3, we begin by proving the following result which extends Lemma 1.4 for the convolution of a finite number of $n$ functions.

Lemma 2.1. *Let for each $i = 1, 2, \dots, n, -1 \leqslant B_i < A_i \leqslant 1$. If for each $i = 1, 2, \dots, n$; $p_i \in \mathcal{P}[A_i, B_i]$, then for $n \geqslant 2$,*

$$p_1 * p_2 * \cdots * p_n \in \mathcal{P}[A, B], \tag{2.1}$$

*where*

$$B = (-1)^{n+1} \prod_{i=1}^{n} B_i, \quad A - B = \prod_{i=1}^{n} (A_i - B_i). \tag{2.2}$$

*Furthermore,*

$$p_1 * p_2 * \cdots * p_n \in \mathcal{P}(\alpha), \tag{2.3}$$

*where*

$$\alpha = 1 - 2^{n-1} \prod_{i=1}^{n} \Big(1 - \frac{1 - A_i}{1 - B_i}\Big) < 1.$$

Proof. If $p_i \in \mathcal{P}[A_i, B_i]$, then we have

$$p_i(z) \prec \frac{1 + A_i z}{1 + B_i z} \qquad (i = 1, 2, \ldots, n,\ z \in \mathbb{U}). \tag{2.4}$$

Since the functions $\frac{1+A_i z}{1+B_i z}$ $(-1 \leqslant B_i < A_i \leqslant 1)$ for each $i = 1, 2, \ldots, n$ are convex univalent in $\mathbb{U}$, therefore by Lemma 1.3, we get

$$p_1 * p_2 * \cdots * p_n \prec \frac{1 + A_1 z}{1 + B_1 z} * \frac{1 + A_2 z}{1 + B_2 z} * \cdots * \frac{1 + A_n z}{1 + B_n z} = \frac{1 + Az}{1 + Bz},$$

where $A$ and $B$ are given by (2.2). This evidently proves result (2.1). Furthermore, from condition (2.4), we have

$$\operatorname{Re}(p_i(z)) > \frac{1 - A_i}{1 - B_i} =: \alpha_i, \tag{2.5}$$

where $\alpha_i < 1$ for each $i = 1, 2, \ldots, n$ and hence we have

$$p_i(z) \prec \frac{1 + (1 - 2\alpha_i)z}{1 - z} \qquad (i = 1, 2, \ldots, n,\ z \in \mathbb{U}).$$

Thus, on putting $A_i = 1 - 2\alpha_i$ and $B_i = -1$ for each $i = 1, 2, \ldots, n$ in (2.2), we obtain

$$p_1 * p_2 * \cdots * p_n \prec \frac{1}{1 - z}\bigg(1 - \bigg(1 - 2^n \prod_{i=1}^{n}(1 - \alpha_i)\bigg) z\bigg),$$

which upon assigning the values to $\alpha_i$ from (2.5) yields that

$$\operatorname{Re}(p_1 * p_2 * \cdots * p_n) > 1 - 2^{n-1} \prod_{i=1}^{n} \Big(1 - \frac{1 - A_i}{1 - B_i}\Big),$$

where $1 - 2^{n-1} \prod_{i=1}^{n} \big(1 - \frac{1-A_i}{1-B_i}\big) =: \alpha < 1$. This proves the result (2.3). □

We use now the differential subordination in finding a subordination property of the operator $\mathcal{J}^m_{\lambda,\mu}$ and the result is contained in the following:

Theorem 2.1. *If for* $-1 \leqslant D < C \leqslant 1$ *we have* $(\mathcal{J}^{m+1}_{\lambda,\mu} f(z))' \prec \frac{1+Cz}{1+Dz}$, $z \in \mathbb{U}$, *then*

$$(\mathcal{J}^{m}_{\lambda,\mu} f(z))' \prec q(z), \quad z \in \mathbb{U}, \tag{2.6}$$

*where*

$$q(z) = \frac{{}_2F_1(1, 1; \frac{\mu+1}{\lambda} + 1; \frac{Dz}{1+Dz}) + \frac{\mu+1}{\mu+1+\lambda} Cz\, {}_2F_1(1, 1; \frac{\mu+1}{\lambda} + 2; \frac{Dz}{1+Dz})}{1 + Dz}$$

*and* ${}_2F_1$ *is the Gaussian hypergeometric function. The result is best possible.*

Proof. Let $p(z) = (\mathcal{J}^m_{\lambda,\mu} f(z))'$; then $p \in \mathcal{H}[1,1]$ and with the use of the identity

$$\mathcal{J}^{m+1}_{\lambda,\mu} f(z) = \Big(1 - \frac{\lambda}{\mu+1}\Big)\mathcal{J}^m_{\lambda,\mu} f(z) + \frac{\lambda}{\mu+1} z(\mathcal{J}^m_{\lambda,\mu} f(z))',$$

we have

$$(\mathcal{J}^{m+1}_{\lambda,\mu} f(z))' = p(z) + \frac{\lambda}{\mu+1} zp'(z) \prec \frac{1+Cz}{1+Dz}, \tag{2.7}$$

which in view of Lemma 1.1 and a change of variables followed by the use of identities (i) and (ii) of Lemma 1.2 gives

$$\begin{aligned} p(z) &\prec q(z) \\ &= \frac{\mu+1}{\lambda} z^{-\frac{\mu+1}{\lambda}} \int_0^z t^{\frac{\mu+1}{\lambda}-1} \frac{1+Ct}{1+Dt}\, dt, \quad z \in \mathbb{U} \\ &= \frac{\mu+1}{\lambda} \int_0^1 s^{\frac{\mu+1}{\lambda}-1} \frac{1+Csz}{1+Dsz}\, ds \\ &= \frac{\mu+1}{\lambda} \Big[ \int_0^1 \frac{s^{\frac{\mu+1}{\lambda}-1}}{1+Dsz}\, ds + Cz \int_0^1 \frac{s^{\frac{\mu+1}{\lambda}}}{1+Dsz}\, ds \Big] \\ &= \frac{{}_2F_1\big(1,1;\frac{\mu+1}{\lambda}+1;\frac{Dz}{1+Dz}\big) + \frac{\mu+1}{\mu+1+\lambda} Cz\, {}_2F_1\big(1,1;\frac{\mu+1}{\lambda}+2;\frac{Dz}{1+Dz}\big)}{1+Dz}. \end{aligned} \tag{2.8}$$

This establishes the result (2.6) with $q$ as the best subordinate of the subordination (2.7) in $\mathbb{U}$, therefore, the result (2.6) is best possible. This proves Theorem 2.1. $\square$

Applying the cases if $D \neq 0$ and if $D = 0$ to expression (2.8), we get following result (2.9) with the use of (i) and (ii) of Lemma 1.2. Further, if we write expression (2.8) as $q(z) = \int_0^1 \mathcal{G}(s,z)\, d\mu(s)$, where $\mathcal{G}(s,z) = \frac{1+Csz}{1+Dsz}$ $(0 \leqslant s \leqslant 1)$ and $d\mu(s) = \frac{\mu+1}{\lambda} s^{\frac{\mu+1}{\lambda}-1}\, ds$, where $\int_0^1 d\mu(s) = 1$, we obtain for $|z| \leqslant r < 1$:

$$\mathrm{Re}(q(z)) \geqslant \int_0^1 \frac{1-Csr}{1-Dsr}\, d\mu(s) = q(-r),$$

since for $|z| \leqslant r < 1$, we infer that $\mathrm{Re}(\frac{1+Cz}{1+Dz}) \geqslant \frac{1-Cr}{1-Dr}$. By letting now $r \to 1-$, we obtain result (2.10) as follows:

Corollary 2.1. *If for* $-1 \leqslant D < C \leqslant 1$ *we have* $(\mathcal{J}^{m+1}_{\lambda,\mu} f(z))' \prec \frac{1+Cz}{1+Dz}$, $z \in \mathbb{U}$, *then*

$$(\mathcal{J}^m_{\lambda,\mu} f(z))' \prec q(z) = \begin{cases} \frac{1+Cz}{1+Dz} - \frac{\left(\frac{C}{D}-1\right)\left[{}_2F_1\left(1,1;\frac{\mu+1}{\lambda}+1;\frac{Dz}{1+Dz}\right)-1\right]}{1+Dz}, & D \neq 0, \\ 1 + \frac{\mu+1}{\mu+1+\lambda} Cz, & D = 0, \end{cases} \quad z \in \mathbb{U}. \tag{2.9}$$

*Furthermore,*

$$\mathrm{Re}(q(z)) > \rho, \tag{2.10}$$

*where*

$$\rho = \begin{cases} \frac{C}{D} + \frac{\left(1-\frac{C}{D}\right)\, {}_2F_1(1,1;\frac{\mu+1}{\lambda}+1;\frac{D}{D-1})}{1-D}, & D \neq 0, \\ 1 - \frac{\mu+1}{\mu+1+\lambda} C, & D = 0. \end{cases}$$

*The result is best possible.*

Further, on applying a special case when $\lambda = \mu + 1$ in (2.8), we obtain the following result involving the Sălăgean operator $\mathcal{D}^m$ for $m \in \mathbb{N}_0$ with the use of the identity

$$ {}_2F_1\Big(1, 1; 2; \frac{Dz}{1+Dz}\Big) = \frac{(1+Dz)\log(1+Dz)}{Dz} \quad (D \neq 0). $$

Corollary 2.2. *If for $-1 \leqslant D < C \leqslant 1$ we have $(\mathcal{D}^{m+1} f(z))' \prec \frac{1+Cz}{1+Dz}$, $z \in \mathbb{U}$, then*

$$ (\mathcal{D}^m f(z))' \prec \begin{cases} \frac{1+Cz}{1+Dz} - \big(\frac{C}{D} - 1\big)\big[\frac{\log(1+Dz)}{Dz} - \frac{1}{1+Dz}\big], & D \neq 0, \\ 1 + \frac{1}{2}Cz, & D = 0, \end{cases} \quad z \in \mathbb{U}. $$

*The result is best possible.*

Theorem 2.2. *If for $-1 \leqslant D < C \leqslant 1$ we have $\big(\mathcal{J}^{m+1}_{\lambda,\mu} f(z)\big)' \prec \frac{1+Cz}{1+Dz}$, $z \in \mathbb{U}$, then*

$$ \big(\mathcal{J}^m_{\lambda,\mu} f(z)\big)' \in \mathcal{P}(\gamma_1), $$

*where*

$$ \gamma_1 = 1 + \Big(1 - \frac{1-C}{1-D}\Big)\Big[{}_2F_1\Big(1, 1; \frac{\mu+1}{\lambda} + 1; \frac{1}{2}\Big) - 2\Big] < 1. \tag{2.11} $$

Proof. If $\big(\mathcal{J}^{m+1}_{\lambda,\mu} f(z)\big)' \prec \frac{1+Cz}{1+Dz}$, $z \in \mathbb{U}$, then we have

$$ \operatorname{Re}\Big(\Big(\mathcal{J}^{m+1}_{\lambda,\mu} f(z)\Big)'\Big) > \frac{1-C}{1-D} := \gamma \quad (z \in \mathbb{U}), \tag{2.12} $$

where $\gamma < 1$. Noting the fact that $z\big(\mathcal{J}^m_{\lambda,\mu} f(z)\big)' = \mathcal{J}^m_{\lambda,\mu}(zf'(z))$ from Definition 1.1 of the operator $\mathcal{J}^m_{\lambda,\mu}$, we have

$$ \big(\mathcal{J}^m_{\lambda,\mu} f(z)\big)' = \frac{\mu+1}{\lambda} z^{-\frac{\mu+1}{\lambda}} \int_0^z t^{\frac{\mu+1}{\lambda}-1} \big(\mathcal{J}^{m+1}_{\lambda,\mu}\ f(t)\big)'\, dt \quad (z \in \mathbb{U}). \tag{2.13} $$

Hence, on putting $t = uz$, $z \in \mathbb{U}$ and on using Lemma 1.5 with condition (2.12), we obtain that

$$ \begin{aligned} \operatorname{Re}\big((\mathcal{J}^m_{\lambda,\mu} f(z))'\big) &= \frac{\mu+1}{\lambda} \int_0^1 u^{\frac{\mu+1}{\lambda}-1} \operatorname{Re}\big((\mathcal{J}^{m+1}_{\lambda,\mu} f(uz))'\big)\, du, \\ &\geqslant \frac{\mu+1}{\lambda} \int_0^1 u^{\frac{\mu+1}{\lambda}-1} \Big(2\gamma - 1 + \frac{2(1-\gamma)}{1+u|z|}\Big)\, du \\ &> \frac{\mu+1}{\lambda} \int_0^1 u^{\frac{\mu+1}{\lambda}-1} \Big(2\gamma - 1 + \frac{2(1-\gamma)}{1+u}\Big)\, du \\ &= 1 - 2(1-\gamma)\Big[1 - \frac{\mu+1}{\lambda} \int_0^1 \frac{u^{\frac{\mu+1}{\lambda}-1}}{1+u}\, du\Big] \\ &= 1 + (1-\gamma)\Big[{}_2F_1\Big(1, 1; \frac{\mu+1}{\lambda} + 1; \frac{1}{2}\Big) - 2\Big] = \gamma_1, \end{aligned} $$

where $\gamma_1 < 1$. Using now the value of $\gamma$ from (2.12) in the last expression on the right-hand side above, we arrive at the desired result of Theorem 2.2. $\square$

Remark 2.1. In view of Theorem 2.2, we remark that if

$$\big(\mathcal{J}^{m+1}_{\lambda,\mu} f(z)\big)' \prec \frac{1+Cz}{1+Dz} \quad (-1 \leqslant D < C \leqslant 1; z \in \mathbb{U}),$$

then $\frac{1}{z}\mathcal{J}^{-1}_{\lambda,\mu}\big(z(\mathcal{J}^{m+1}_{\lambda,\mu} f(z))'\big) \in \mathcal{P}(\gamma_1)$, where $\gamma_1$ is given by (2.11).

Our next result proves a subordination property of the convolution of functions $\mathcal{J}^{m}_{\lambda,\mu} f_i$ $(i = 1, 2, \dots, n)$.

Theorem 2.3. *Let for each* $i = 1, 2, \dots, n$, $f_i \in \mathcal{A}$. *If*

$$\big(\mathcal{J}^{m}_{\lambda,\mu} f_i(z)\big)' \prec \frac{1+A_i z}{1+B_i z} \quad (-1 \leqslant B_i < A_i \leqslant 1;\ i = 1, 2, \dots, n;\ z \in \mathbb{U}), \tag{2.14}$$

*then*

$$\big(\mathcal{J}^{m}_{\lambda,\mu} f_1 * \mathcal{J}^{m}_{\lambda,\mu} f_2 * \cdots * \mathcal{J}^{m}_{\lambda,\mu} f_n\big)'(z) \prec h(z), \quad z \in \mathbb{U},$$

*where*

$$h(z) = 1 + (A - B)\sum_{k=1}^{\infty} \frac{(-B)^{k-1}}{(k+1)^{n-1}} z^k, \quad z \in \mathbb{U} \tag{2.15}$$

*is convex in* $\mathbb{U}$, *and* $A$, $B$ *are given by* (2.2).

Proof. Observe that if $n = 1$, the result is trivial. Let $n \geqslant 2$ and for $i = 1, 2, \dots, n$, let $p_i(z) = \mathcal{J}^{m}_{\lambda,\mu} f_i(z)/z$, which is analytic in $\mathbb{U}$ with $p_i(0) = 1$. Now using (2.14), we have

$$p_i(z) + z p_i'(z) = \big(\mathcal{J}^{m}_{\lambda,\mu} f_i(z)\big)' \prec \frac{1+A_i z}{1+B_i z} \quad (-1 \leqslant B_i < A_i \leqslant 1;\ i = 1, 2, \dots, n;\ z \in \mathbb{U}).$$

Therefore in view of Lemma 1.1 (for the case if $n = 1$, $\gamma = 1$), we get

$$\frac{\mathcal{J}^{m}_{\lambda,\mu} f_i(z)}{z} \prec q_i(z) = \frac{1}{z}\int_0^z \frac{1+A_i t}{1+B_i t}\, dt, \quad z \in \mathbb{U}, \tag{2.16}$$

where $q_i$ is convex in $\mathbb{U}$ and is the best dominant. Now on applying Lemma 1.3 to subordination (2.16) for $i = 1, 2, \dots, n-1$ and to subordination (2.14) for $i = n$, we get

$$\begin{aligned} &\frac{\mathcal{J}^{m}_{\lambda,\mu} f_1(z)}{z} * \cdots * \frac{\mathcal{J}^{m}_{\lambda,\mu} f_{n-1}(z)}{z} * \big(\mathcal{J}^{m}_{\lambda,\mu} f_n(z)\big)' \\ &\qquad \prec \frac{1}{z}\int_0^z \frac{1+A_1 t}{1+B_1 t}\, dt * \cdots * \frac{1}{z}\int_0^z \frac{1+A_{n-1} t}{1+B_{n-1} t}\, dt * \frac{1+A_n z}{1+B_n z} =: h(z), \end{aligned} \tag{2.17}$$

where $h(z)$ is convex in $\mathbb{U}$ being the convolution of functions which are convex in $\mathbb{U}$ and is given by (2.15). The left-hand side of the above subordination (2.17) is

$$\big(\mathcal{J}^{m}_{\lambda,\mu} f_1(z) * \mathcal{J}^{m}_{\lambda,\mu} f_2(z) * \cdots * \mathcal{J}^{m}_{\lambda,\mu} f_n(z)\big)'.$$

This proves Theorem 2.3. □

We note that the function $h(z)$ given by (2.15) is convex with real coefficients in $\mathbb{U}$ and at $z \in \partial\mathbb{U}$, the series in (2.15) is absolutely convergent if $n > 2$, hence we get the following result directly from Theorem 2.3:

Corollary 2.3. *Let for each* $i = 1, 2, \ldots, n$, $f_i \in \mathcal{A}$ *and*

$$\left(\mathcal{J}^m_{\lambda,\mu} f_i(z)\right)' \prec \frac{1 + A_i z}{1 + B_i z} \quad (-1 \leqslant B_i < A_i \leqslant 1;\ z \in \mathbb{U}).$$

*Then, for* $n > 2$ *(in case* $B_i \neq 0, i = 1, 2, \ldots, n$*):*

$$h(-1) \leqslant \operatorname{Re}\left\{\left(\mathcal{J}^m_{\lambda,\mu} f_1(z) * \mathcal{J}^m_{\lambda,\mu} f_2(z) * \cdots * \mathcal{J}^m_{\lambda,\mu} f_n(z)\right)'\right\} \leqslant h(1),$$

*where* $h(z)$ *is given by* (2.15).

In particular, if $B_i = -1$ for each $i = 1, 2, \ldots, n$, we find then the following result from Theorem 2.3 in terms of the zeta function [**18**, Ex. 5, p. 201]:

Corollary 2.4. *Let for each* $i = 1, 2, \ldots, n$, $f_i \in \mathcal{A}$ *and*

$$(\mathcal{J}^m_{\lambda,\mu} f_i(z))' \prec \frac{1 + A_i z}{1 - z} \quad (-1 < A_i \leqslant 1;\ z \in \mathbb{U}).$$

*Then for* $n > 2$,

$$\begin{aligned}
&\operatorname{Re}\left\{(\mathcal{J}^m_{\lambda,\mu} f_1(z) * \mathcal{J}^m_{\lambda,\mu} f_2(z) * \cdots * \mathcal{J}^m_{\lambda,\mu} f_n(z))'\right\} \\
&\qquad\qquad \geqslant 1 + [(1 - 2^{2-n})\zeta(n-1) - 1] \prod_{i=1}^{n} (A_i + 1), \\
&\operatorname{Re}\left\{(\mathcal{J}^m_{\lambda,\mu} f_1(z) * \mathcal{J}^m_{\lambda,\mu} f_2(z) * \cdots * \mathcal{J}^m_{\lambda,\mu} f_n(z))'\right\} \\
&\qquad\qquad \leqslant 1 + [\zeta(n-1) - 1] \prod_{i=1}^{n} (A_i + 1), \quad z \in \mathbb{U},
\end{aligned}$$

*where* $\zeta$ *is the well known zeta function.*

Remark 2.2. The result obtained on putting $A_i = 1 - 2\alpha_i$ for $\alpha_i < 1$ and $B_i = -1$ for each $i = 1, 2, \ldots, n$ in Theorem 2.3 coincides with the one proved recently in [**12**] which includes the result of Sokół [**14**] (for $n = 3$, and $\zeta(2) = \frac{\pi^2}{6}$).

To prove our next result, we first prove the following lemmas:

Lemma 2.2. *Let* $k$, $n \in \mathbb{N}$ *and* $\lambda_1, \lambda_2, \ldots, \lambda_{n-2}$ *be positive integers depending only upon* $n$ $(n > 2)$ *not on* $k$. *Then*

$$\begin{aligned}
k^n = k(k-1)\cdots(k-n+1) + \lambda_{n-2} k(k-1)\cdots(k-n+2) + \cdots \\
+ \lambda_2 k(k-1)(k-2) + \lambda_1 k(k-1) + k.
\end{aligned} \tag{2.18}$$

Proof. We prove the identity (2.18) for any $k \in \mathbb{N}$ by using induction on $n \in \mathbb{N}$. Since for $n = 2$, we have $k^2 = k(k-1) + k$, then for $n = 3$, we get

$$\begin{aligned}
k^3 = k^2 k &= [k(k-1) + k]k \\
&= \{k(k-1)(k-2) + 2k(k-1)\} + \{k(k-1) + k\} \\
&= k(k-1)(k-2) + 3k(k-1) + k.
\end{aligned} \tag{2.19}$$

This verifies identity (2.18) for $n = 3$ with $\lambda_1 = 3$. Let (2.18) be true for $n = r - 1$ $(> 2)$, that is if for some positive integers $\lambda_1, \lambda_2, \ldots, \lambda_{r-3}$, we have

$$k^{r-1} = k(k-1)\cdots(k-r+2) + \lambda_{r-3}k(k-1)\cdots(k-r+3) + \cdots + \lambda_2 k(k-1)(k-2) + \lambda_1 k(k-1) + k.$$

On multiplying by $k$ (and proceeding similarly as in (2.19)), we get

$$\begin{aligned} k^r &= \{k(k-1)\cdots(k-r+2)(k-r+1) + (r-1)k(k-1)\cdots(k-r+2)\} \\ &\quad + \{\lambda_{r-3}k(k-1)\cdots(k-r+3)(k-r+2) \\ &\qquad + (r-2)\lambda_{r-3}k(k-1)\cdots(k-r+3)\} + \cdots \\ &\quad + \{\lambda_2 k(k-1)(k-2)(k-3) + 3\lambda_2 k(k-1)(k-2)\} \\ &\quad + \{\lambda_1 k(k-1)(k-2) + 2\lambda_1 k(k-1)\} + \{k(k-1) + k\} \\ &= k(k-1)\cdots(k-r+1) + (r-1+\lambda_{r-3})k(k-1)\cdots(k-r+2) + \cdots \\ &\quad + (3\lambda_2 + \lambda_1)k(k-1)(k-2) + (2\lambda_1 + 1)k(k-1) + k. \end{aligned}$$

This verifies identity (2.18) for $n = r$ for some positive integers $(2\lambda_1+1)$, $(3\lambda_2+\lambda_1)$, $\ldots, (r-1+\lambda_{r-3})$ and the desired result is thus proved. □

Remark 2.3. We mention that the identity (2.18) depends only on $n$ $(n > 2)$ not on $k$, that is for the same $n$, it can be written (on replacing $k$ by $k+1$) in the form:

$$\begin{aligned} (k+1)^n &= (k+1)k(k-1)\cdots(k-n+2) \\ &\quad + \lambda_{n-2}(k+1)k(k-1)\cdots(k-n+3) + \cdots \\ &\quad + \lambda_2(k+1)k(k-1) + \lambda_1(k+1)k + (k+1) \\ &= (k+1)[k(k-1)\cdots(k-n+2) \\ &\qquad + \lambda_{n-2}k(k-1)\cdots(k-n+3) + \cdots + \lambda_2 k(k-1) + \lambda_1 k + 1]. \end{aligned}$$

It follows therefore that

$$(k+1)^{n-1} = k(k-1)\cdots(k-n+2) + \lambda_{n-2}k(k-1)\cdots(k-n+3) + \cdots + \lambda_2 k(k-1) + \lambda_1 k + 1. \tag{2.20}$$

Lemma 2.3. *If for each $i = 1, 2, \ldots, n$, $f_i \in \mathcal{A}$ and $\lambda_1, \lambda_2, \ldots, \lambda_{n-2}$, be positive integers depending upon $n$ $(n > 2)$. Then for*

$$\phi_m(z) := \mathcal{J}^m_{\lambda,\mu} f_1(z) * \mathcal{J}^m_{\lambda,\mu} f_2(z) * \cdots * \mathcal{J}^m_{\lambda,\mu} f_n(z), \tag{2.21}$$

$$\phi'_m(z) + \lambda_1 z\phi''_m(z) + \lambda_2 z^2 z\phi'''_m(z) + \cdots + \lambda_{n-2} z^{n-2}\phi^{(n-1)}_m(z) + z^{n-1}\phi^{(n)}_m(z) = \big(\mathcal{J}^m_{\lambda,\mu} f_1(z)\big)' * \big(\mathcal{J}^m_{\lambda,\mu} f_2(z)\big)' * \cdots * \big(\mathcal{J}^m_{\lambda,\mu} f_n(z)\big)'. \tag{2.22}$$

Proof. We prove result (2.22) by taking induction on $n$. If $n = 1$, we directly get $\phi'_m(z) = \big(\mathcal{J}^m_{\lambda,\mu} f_1(z)\big)'$. Next if $n = 2$, then from (2.21), we get

$$z\phi'_m(z) = z\big(\mathcal{J}^m_{\lambda,\mu} f_1(z)\big)' * \mathcal{J}^m_{\lambda,\mu} f_2(z) \text{ and } (z\phi'_m(z))' = \big(\mathcal{J}^m_{\lambda,\mu} f_1(z)\big)' * \big(\mathcal{J}^m_{\lambda,\mu} f_2(z)\big)',$$

and result (2.22) is verified as

$$\phi'_m(z) + z\phi''_m(z) = \big(\mathcal{J}^m_{\lambda,\mu} f_1(z)\big)' * \big(\mathcal{J}^m_{\lambda,\mu} f_2(z)\big)'.$$

If $n = 3$, then from (2.21), we have

$$z(z\phi_m'(z))' = z(\phi_m'(z) + z\phi_m''(z)) = z\big(\mathcal{J}_{\lambda,\mu}^m f_1(z)\big)' * z\big(\mathcal{J}_{\lambda,\mu}^m f_2(z)\big)' * \mathcal{J}_{\lambda,\mu}^m f_3(z),$$

and hence,

$$\begin{aligned}(z(z\phi_m'(z))')' &= (z(\phi_m'(z) + z\phi_m''(z)))' \\ &= \big(\mathcal{J}_{\lambda,\mu}^m f_1(z)\big)' * \big(\mathcal{J}_{\lambda,\mu}^m f_2(z)\big)' * \big(\mathcal{J}_{\lambda,\mu}^m f_3(z)\big)',\end{aligned} \tag{2.23}$$

where the left-hand side of (2.23) is given by

$$\begin{aligned}(z(\phi_m'(z) + z\phi_m''(z)))' &= (z\phi_m'(z))' + (z^2\phi_m''(z))' \\ &= [\phi_m'(z) + z\phi_m''(z)] + [2z\phi_m''(z) + z^2\phi_m'''(z)] \\ &= \phi_m'(z) + 3z\phi_m''(z) + z^2\phi_m'''(z).\end{aligned}$$

This proves result (2.22) with $\lambda_1 = 3$. Let result (2.22) be true for $n = r - 1$ $(> 2)$, that is if we have

$$\begin{aligned}\phi_m'(z) + \lambda_1 z\phi_m''(z) + \lambda_2 z^2\phi_m'''(z) + \cdots + \lambda_{r-3} z^{r-3}\phi_m^{(r-2)}(z) + z^{r-2}\phi_m^{(r-1)}(z) \\ = \big(\mathcal{J}_{\lambda,\mu}^m f_1(z)\big)' * \big(\mathcal{J}_{\lambda,\mu}^m f_2(z)\big)' * \cdots * \big(\mathcal{J}_{\lambda,\mu}^m f_{r-1}(z)\big)',\end{aligned}$$

where $\lambda_1, \lambda_2, \ldots, \lambda_{r-3}$ depending upon $r - 1$ $(> 2)$. Then, if

$$\phi_m(z) := \mathcal{J}_{\lambda,\mu}^m f_1(z) * \mathcal{J}_{\lambda,\mu}^m f_2(z) * \cdots * \mathcal{J}_{\lambda,\mu}^m f_r(z),$$

we get

$$\begin{aligned}(z(\phi_m'(z) + \lambda_1 z\phi_m''(z) + \lambda_2 z^2\phi_m'''(z) + \cdots + \lambda_{r-3} z^{r-3}\phi_m^{(r-2)}(z) + z^{r-2}\phi_m^{(r-1)}(z)))' \\ = \big(\mathcal{J}_{\lambda,\mu}^m f_1(z)\big)' * \big(\mathcal{J}_{\lambda,\mu}^m f_2(z)\big)' * \cdots * \big(\mathcal{J}_{\lambda,\mu}^m f_r(z)\big)'\end{aligned}$$

where the left-hand side is given by

$$\begin{aligned}\big(z\big(\phi_m'(z) + \lambda_1 z\phi_m''(z) + \lambda_2 z^2\phi_m'''(z) + \cdots + \lambda_{r-3} z^{r-3}\phi_m^{(r-2)}(z) + z^{r-2}\phi_m^{(r-1)}(z)\big)\big)' \\ = (z\phi_m'(z))' + (\lambda_1 z^2\phi_m''(z))' + (\lambda_2 z^3\phi_m'''(z))' + \cdots \\ +(\lambda_{r-3} z^{r-2}\phi_m^{(r-2)}(z))' + (z^{r-1}\phi_m^{(r-1)}(z))' \\ = \phi_m'(z) + z\phi_m''(z) + 2\lambda_1 z\phi_m''(z) + \lambda_1 z^2\phi_m'''(z) + 3\lambda_2 z^2\phi_m'''\ (z) + \cdots \\ +(r-2)\lambda_{r-3} z^{r-3}\phi_m^{(r-2)}(z) + \lambda_{r-3} z^{r-2}\ \phi_m^{(r-1)}(z) \\ +(r-1) z^{r-2}\phi_m^{(r-1)}(z) + z^{r-1}\phi_m^{(r)}(z),\end{aligned}$$

which on simplifying proves the result (2.22) for $n = r$ that

$$\begin{aligned}\phi_m'(z) + (1 + 2\lambda_1) z\phi_m''(z) + (\lambda_1 + 3\lambda_2) z^2\phi_m'''(z) + \cdots \\ + (\lambda_{r-3} + r - 1) z^{r-2}\phi_m^{(r-1)}(z) + z^{r-1}\phi_m^{(r)}(z) \\ = \big(\mathcal{J}_{\lambda,\mu}^m f_1(z)\big)' * \big(\mathcal{J}_{\lambda,\mu}^m f_2(z)\big)' * \cdots * \big(\mathcal{J}_{\lambda,\mu}^m f_r(z)\big)'\end{aligned}$$

for some positive integers $(1 + 2\lambda_1), (\lambda_1 + 3\lambda_2), \ldots, (\lambda_{r-3} + r - 1)$. This proves Lemma 2.3. □

THEOREM 2.4. *Let for each $i = 1, 2, \ldots, n$, $f_i \in \mathcal{A}$ such that $\big(\mathcal{J}^m_{\lambda,\mu} f_i(z)\big)' \prec \frac{1+A_i z}{1+B_i z}$ $(-1 \leqslant B_i < A_i \leqslant 1)$ and $\phi_m(z)$ be given by* (2.21). *Then $\phi_m \in \mathcal{K}$ if for $n > 2$,*

$$(A-B)\sum_{k=1}^{\infty}\frac{B^{k-1}}{(k+1)^{n-1}} \leqslant \frac{3}{4}, \tag{2.24}$$

*where $A$ and $B$ are given in terms of $A_i$ and $B_i$ in* (2.2).

PROOF. Let $r(z) = \phi'_m(z)$; then by Lemma 2.3 and (2.1), we get for some positive integers $\lambda_1, \lambda_2, \ldots, \lambda_{n-2}$ depending upon $n$ $(n > 2)$:

$$\begin{aligned} & r(z) + \lambda_1 z r'(z) + \lambda_2 z^2 r''(z) + \cdots + \lambda_{n-2} z^{n-2} r^{(n-2)}(z) + z^{n-1} r^{(n-1)}(z) \\ & \qquad =: \psi(r(z), z r'(z), \ldots, z^{n-1} r^{(n-1)}(z)) \prec \frac{1+Az}{1+Bz}, \quad z \in \mathbb{U}, \end{aligned} \tag{2.25}$$

where $A$ and $B$ are given by (2.2). Again, we denote

$$\psi\big(r(z), z r'(z), \ldots, z^{n-1} r^{(n-1)}(z)\big) =: \psi_m(z),$$

and from Theorem 2.3, we have a possible solution of the above $n$-th order Euler-type differential subordination (2.25) as follows:

$$\phi'_m(z) = r(z) \prec h(z), \quad z \in \mathbb{U}, \tag{2.26}$$

where $h(z)$ is of the form (2.15) and its $r$-th $(r \in \mathbb{N})$ derivative is given by

$$h^{(r)}(z) = (A-B)\sum_{k=1}^{\infty}\frac{k(k-1)\cdots(k-r+1)}{(k+1)^{n-1}}(-B)^{k-1} z^{k-r}.$$

Observe that the positive integers $\lambda_1, \lambda_2, \ldots, \lambda_{n-2}$, appearing in (2.22) are the same as in (2.18), and for these parameters $\lambda_1, \lambda_2, \ldots, \lambda_{n-2}$ and use of the result (2.20) mentioned in Remark 2.3, we get

$$\begin{aligned} & \psi\big(h(z), z h'(z), \ldots, z^{n-1} h^{(n-1)}(z)\big) \\ & \quad = h(z) + \lambda_1 z h'(z) + \lambda_2 z^2 h''(z) + \cdots + \lambda_{n-2} z^{n-2} h^{(n-2)}(z) + z^{n-1} h^{(n-1)}(z) \\ & \quad = 1 + (A-B)\sum_{k=1}^{\infty}(-B)^{k-1} z^k = \frac{1+Az}{1+Bz}, \quad z \in \mathbb{U}. \end{aligned}$$

This verifies the admissibility condition for $h(z)$ in (2.26) to be a solution of the subordination (2.25).

Now the function $\phi_m \in \mathcal{K}$ if

$$1 + \frac{z\phi''_m(z)}{\phi'_m(z)} \prec \frac{1+z}{1-z}, \quad z \in \mathbb{U},$$

or if

$$1 + \frac{z r'(z)}{r(z)} \prec \frac{1+z}{1-z}, \quad z \in \mathbb{U}. \tag{2.27}$$

By [**7**, Theorem 2.6b, p.60], condition (2.27) implies that $r(z) \prec \frac{1}{(1-z)^2}$, $z \in \mathbb{U}$. Thus in view of (2.26), the function $\phi_m \in \mathcal{K}$ if $h(z) \prec \frac{1}{(1-z)^2}$, $z \in \mathbb{U}$, that is if $\inf_{|z|<1} \operatorname{Re}\{h(z)\} = h(-1) \geqslant \frac{1}{4}$, which in view of (2.15) gives

$$h(-1) = 1 - (A - B)\sum_{k=1}^{\infty} \frac{B^{k-1}}{(k+1)^{n-1}} \geqslant \frac{1}{4},$$

which is the given condition (2.24) if $n > 2$. □

Setting $B_i = -1$ for each $i = 1, 2, \ldots, n$ in Theorem 2.4, we get the following result in terms of zeta function [**18**, Ex.5, p.201] $\zeta(n), n > 1$, and $\zeta(2) = \frac{\pi^2}{6}$:

COROLLARY 2.5. *Let for each $i = 1, 2, \ldots, n$, $f_i \in \mathcal{A}$ and $-1 < A_i \leqslant 1$. Also, suppose that*

$$\left(\mathcal{J}^m_{\lambda,\mu} f_i(z)\right)' \prec \frac{1 + A_i z}{1 - z}$$

*for each $i = 1, 2, \ldots, n$, and $\phi_m(z)$ be given by (2.21). Then $\phi_m \in \mathcal{K}$ if for $n > 2$,*

$$\prod_{i=1}^{n} (A_i + 1) \leqslant \frac{3}{4[1 - (1 - 2^{2-n})\zeta(n-1)]}.$$

*In the case, $n = 3$, $\left(\mathcal{J}^m_{\lambda,\mu} f_1 * \mathcal{J}^m_{\lambda,\mu} f_2 * \mathcal{J}^m_{\lambda,\mu} f_3\right)(z) \in \mathcal{K}$ if*

$$(A_1 + 1)(A_2 + 1)(A_3 + 1) \leqslant \frac{9}{12 - \pi^2}.$$

We now prove our next result.

THEOREM 2.5. *Let for each $i = 1, 2, \ldots, n$, $f_i \in \mathcal{A}$ and for $-1 \leqslant D_i < C_i \leqslant 1$,*

$$\left(\mathcal{J}^{m+1}_{\lambda,\mu} f_i(z)\right)' \prec \frac{1 + C_i z}{1 + D_i z}, z \in \mathbb{U}. \tag{2.28}$$

*Let $\psi_m(z) = \left(\mathcal{J}^m_{\lambda,\mu} f_1(z)\right)' * \left(\mathcal{J}^m_{\lambda,\mu} f_2(z)\right)' * \cdots * \left(\mathcal{J}^m_{\lambda,\mu} f_n(z)\right)'$, then*

$$\begin{aligned} &\operatorname{Re}\left(\left(1 - \frac{\lambda}{\mu+1}\right)\psi_m(z) + \frac{\lambda}{\mu+1}(z\psi_m(z))'\right) \\ &\qquad > 1 - 2^{n-1}\prod_{i=1}^{n}\left(1 - \frac{1 - C_i}{1 - D_i}\right)\left[2 - {}_2F_1\left(1, 1; \frac{\mu+1}{\lambda} + 1; \frac{1}{2}\right)\right]^{n-1}. \end{aligned} \tag{2.29}$$

*The result is sharp for certain functions $f_i$ if $D_i = -1$ for each $i = 1, 2, \ldots, n$.*

PROOF. For each $f_i \in \mathcal{A}$ and for $-1 \leqslant D_i < C_i \leqslant 1$, $i = 1, 2, \ldots, n$, the condition (2.28) implies that

$$\operatorname{Re}\left(\left(\mathcal{J}^{m+1}_{\lambda,\mu} f_i(z)\right)'\right) > \frac{1 - C_i}{1 - D_i} =: \beta_i \quad (i = 1, 2, \ldots, n, z \in \mathbb{U}), \tag{2.30}$$

where $\beta_i < 1$ $(i = 1, 2, \ldots, n)$, and hence by Lemma 2.1 it implies that

$$\psi_{m+1} := \left(\mathcal{J}^{m+1}_{\lambda,\mu} f_1(z)\right)' * \left(\mathcal{J}^{m+1}_{\lambda,\mu} f_2(z)\right)' * \cdots * \left(\mathcal{J}^{m+1}_{\lambda,\mu} f_n(z)\right)' \in \mathcal{P}(\delta_0), \tag{2.31}$$

where

$$\delta_0 = 1 - 2^{n-1}\prod_{i=1}^{n}(1-\beta_i) < 1. \tag{2.32}$$

We further denote for $i = 1, 2, \dots,$

$$\delta_i = 1 - (1-\delta_0)\Big[2 - {}_2F_1\Big(1, 1; \frac{\mu+1}{\lambda}+1; \frac{1}{2}\Big)\Big]^i, \tag{2.33}$$

where $\delta_0$ is given by (2.32). Similar to (2.13), we may put

$$\begin{aligned}\psi_m(z) = \frac{\mu+1}{\lambda} z^{-\frac{\mu+1}{\lambda}} \int_0^z t^{\frac{\mu+1}{\lambda}-1}\big(\mathcal{J}^{m+1}_{\lambda,\mu} f_1(t)\big)'\,dt * \dots \\ * \frac{\mu+1}{\lambda} z^{-\frac{\mu+1}{\lambda}} \int_0^z t^{\frac{\mu+1}{\lambda}-1}\big(\mathcal{J}^{m+1}_{\lambda,\mu} f_n(t)\big)'\,dt\end{aligned}$$

and thus, in view of convolution property (1.3), we write

$$\psi_m(z) = \frac{\mathcal{J}^{-n}_{\lambda,\mu}(z(\psi_{m+1}(z)))}{z}. \tag{2.34}$$

Further, we have

$$\Big(1 - \frac{\lambda}{\mu+1}\Big)\psi_m(z) + \frac{\lambda}{\mu+1}(z\psi_m(z))' = \frac{\mathcal{J}_{\lambda,\mu}(z\psi_m(z))}{z} := s(z).$$

Now from (2.34), it follows that

$$s(z) = \frac{\mathcal{J}^{1-n}_{\lambda,\mu}(z(\psi_{m+1}(z)))}{z}.$$

On applying Theorem 2.2 (as in Remark 2.1) that if from (2.31), the function $\psi_{m+1} \in \mathcal{P}(\delta_0)$, then we infer that

$$\frac{\mathcal{J}^{-1}_{\lambda,\mu}(z\psi_{m+1}(z))}{z} \in \mathcal{P}(\delta_1),$$

where $\delta_1 = 1 - (1-\delta_0)\big[2 - {}_2F_1\big(1, 1; \frac{\mu+1}{\lambda}+1; \frac{1}{2}\big)\big] < 1$. By (1.2), we note that

$$\frac{\mathcal{J}^{-1}_{\lambda,\mu}\big(\mathcal{J}^{-1}_{\lambda,\mu}(z\psi_{m+1}(z))\big)}{z} = \frac{\mathcal{J}^{-2}_{\lambda,\mu}(z\psi_{m+1}(z))}{z}.$$

On repeatedly applying the same Theorem 2.3 (as in Remark 2.1) for any $n \geqslant 2$, we find that $s \in \mathcal{P}(\delta_{n-1})$, $\delta_{n-1} < 1$. Thus on putting the value of $\delta_0$ from (2.32) in (2.33) for $i = n-1$ and using (2.30), we obtain the result (2.29). The sharpness of the result follows for the functions $f_i \in \mathcal{A}$ such that for $\psi_{m+1}$ given by (2.31) and for $n \geqslant 2$,

$$\frac{\mathcal{J}^{2-n}_{\lambda,\mu}(z\psi_{m+1}(z))}{z} = \frac{1+(1-2\delta_{n-2})z}{1-z} \quad (\delta_{n-2} < 1; z \in \mathbb{U}).$$

Since for such functions on using Lemma 1.2 and letting $z \to -1$, we get

$$s(z) = \frac{\mu+1}{\lambda}\int_0^1 u^{\frac{\mu+1}{\lambda}-1}\Big[\frac{1+(1-2\delta_{n-2})uz}{1-uz}\Big]\,du$$

$$= 1 + (1 - \delta_{n-2})\Big[2\Big(\frac{\mu+1}{\lambda}\Big)\int_0^1 \frac{u^{\frac{\mu+1}{\lambda}-1}}{1+u}\,du - 2\Big]$$
$$= 1 - (1 - \delta_{n-2})\Big[2 - {}_2F_1\Big(1, 1; \frac{\mu+1}{\lambda}+1; \frac{1}{2}\Big)\Big],$$

which on putting the value of $\delta_{n-2}$ from (2.33) for $i = n-2$ and the value of $\delta_0$ from (2.32), and by (2.30) when $D_i = -1$ for each $i = 1, 2, \dots, n$ gives, as $z \to 1-$,

$$s(z) = 1 - \frac{1}{2}\Big[2 - {}_2F_1\Big(1, 1; \frac{\mu+1}{\lambda}+1; \frac{1}{2}\Big)\Big]^{n-1}\prod_{i=1}^{n}(1 + C_i). \qquad \square$$

By setting $n = 2$ in Theorem 2.5, we obtain a simple case as follows:

Corollary 2.6. *Let for $i = 1, 2$, $f_i \in \mathcal{A}$ and for $-1 \leqslant D_i < C_i \leqslant 1$,*

$$\big(\mathcal{J}^{m+1}_{\lambda,\mu} f_i(z)\big)' \prec \frac{1 + C_i z}{1 + D_i z}, z \in \mathbb{U},$$

*and also, let $F(z) = \mathcal{J}^{m}_{\lambda,\mu}(f_1 * f_2)(z)$. Then*

$$\operatorname{Re}\big(\mathcal{J}^{m+1}_{\lambda,\mu}(zF'(z))\big)' > 1 - \frac{2(C_1 - D_1)(C_2 - D_2)}{(1 - D_1)(1 - D_2)}\Big[2 - {}_2F_1\Big(1, 1; \frac{\mu+1}{\lambda}+1; \frac{1}{2}\Big)\Big].$$

*The result is sharp for functions $f_i$ if $D_i = -1$ for each $i = 1, 2$.*

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