

DEGREES OF THE APPROXIMATION OF INTEGRABLE FUNCTIONS BY SOME SPECIAL MATRIX MEANS OF FOURIER SERIES

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ABSTRACT. We consider the pointwise and normwise approximation of function by some special matrix means of its Fourier series. The results corresponding to the theorem of Łenski and Szal in [5] and the results of Saini and Singh in [6] are shown. Some special cases as corollaries are also formulated.

1. Introduction

Let L^p ($1 \leq p < \infty$) be the class of all 2π -periodic real-valued functions integrable in the Lebesgue sense with p -th power over $Q = [-\pi, \pi]$ with the norm

$$\|f\| = \|f(\cdot)\|_{L^p} = \left(\int_Q |f(t)|^p dt \right)^{1/p} \quad \text{when } 1 \leq p < \infty.$$

Consider the trigonometric Fourier series

$$Sf(x) := \frac{a_0(f)}{2} + \sum_{\nu=1}^{\infty} (a_{\nu}(f) \cos \nu x + b_{\nu}(f) \sin \nu x)$$

with the partial sums $S_k f$. Let $A := (a_{n,k})$ be an infinite lower triangular matrix of real numbers such that $a_{n,k} \geq 0$ when $k = 0, 1, 2, \dots, n$, $a_{n,k} = 0$ when $k > n$, and $\sum_{k=0}^n a_{n,k} = 1$, where $n = 0, 1, 2, \dots$. Further, let for $m = 0, 1, 2, \dots, n$,

$$A_{n,m} = \sum_{k=0}^m a_{n,k} \quad \text{and} \quad \bar{A}_{n,m} = \sum_{k=m}^n a_{n,k}.$$

Let the A -transformation of $(S_k f)$ be given by

$$T_{n,A}f(x) := \sum_{k=0}^n a_{n,k} S_k f(x) \quad (n = 0, 1, 2, \dots).$$

2010 *Mathematics Subject Classification*: 42A24.

Key words and phrases: degree of approximation, Fourier series, matrix means.

Communicated by Gradimir Milovanović.

Following Leindler [4], (see also [7]), a sequence $(a_{n,r})_{r=0}^n$, ($n = 0, 1, 2, \dots$), of nonnegative numbers tending to zero is called the Mean Rest Bounded Variation Sequence, or briefly $a_{n,r} \in \text{MRBVS}$, if it has the property

$$\sum_{r=m+1}^{n-1} |a_{n,r} - a_{n,r+1}| \leq K \frac{1}{m+1} \sum_{m \geq r \geq m/2} a_{n,r},$$

for every positive integer m .

A sequence $(a_{n,r})_{r=0}^n$, ($n = 0, 1, 2, \dots$), of nonnegative numbers will be called the Mean Head Bounded Variation Sequence, or briefly $a_{n,r} \in \text{MHBVS}$, if it has the property

$$\sum_{r=0}^{n-m-1} |a_{n,r} - a_{n,r+1}| \leq K \frac{1}{m+1} \sum_{r=n-m}^n a_{n,r},$$

for all positive integers $m < n$, where the sequence $(a_{n,r})_{r=0}^n$ has only finite nonzero terms and the last nonzero term is $a_{n,n}$.

Let

$$|A|_{n,m} = \begin{cases} A_{n,m}, & \text{when } (a_{n,r})_{r=0}^n \in \text{MRBVS}, \\ \bar{A}_{n,n-m}, & \text{when } (a_{n,r})_{r=0}^n \in \text{MHBVS}. \end{cases}$$

As a measure of approximation of a function f by $T_{n,A}$ mean in the space L^p ($1 \leq p < \infty$) we will use the generalized modulus of continuity of f defined for $\beta \geq 0$ by the formula

$$\omega_\beta f(\delta)_{L^p} := \sup_{0 \leq |t| \leq \delta} \left\{ \left| \sin \frac{t}{2} \right|^{\beta p} \int_0^\pi |\varphi_x(t)|^p dx \right\}^{1/p},$$

where $\varphi_x(t) = f(x+t) + f(x-t) - 2f(x)$. It is clear that for $\beta > \alpha \geq 0$

$$\omega_\beta f(\delta)_{L^p} \leq \omega_\alpha f(\delta)_{L^p},$$

and it is easily seen that $\omega_0 f(\cdot)_{L^p} = \omega f(\cdot)_{L^p}$ is the classical modulus of continuity.

Łenski and Szal [5] have proved the following theorem

THEOREM 1.1. *Let $f \in L^p(\omega)_\beta$, with $\beta < 1 - \frac{1}{p}$, $(a_{n,k})_{k=0}^n \in \text{HBVS}$, (or $(a_{n,k})_{k=0}^n \in \text{RBVS}$) and let ω satisfy*

$$(1.1) \quad \left\{ \int_{\frac{\pi}{n+1}}^\pi \left(\frac{t^{-\gamma} |\varphi_x(t)|}{\omega(t)} \right)^p \sin^{\beta p} \frac{t}{2} dt \right\}^{1/p} = O_x((n+1)^\gamma)$$

with $0 < \gamma < \beta + \frac{1}{p}$,

$$(1.2) \quad \left\{ \int_0^{\frac{\pi}{n+1}} \left(\frac{|\varphi_x(t)|}{\omega(t)} \right)^p \sin^{\beta p} \frac{t}{2} dt \right\}^{1/p} = O_x((n+1)^{-1/p}),$$

$$(1.3) \quad \left\{ \int_0^{\frac{\pi}{n+1}} \left(\frac{\omega(t)}{t \sin^\beta \frac{t}{2}} \right)^q dt \right\}^{1/q} = O\left((n+1)^{\beta + \frac{1}{p}} \omega\left(\frac{\pi}{n+1}\right)\right),$$

where $q = p(p-1)^{-1}$. Then

$$|T_{n,A} f(x) - f(x)| = O_x\left((n+1)^{\beta + \frac{1}{p} + 1} a_n \omega\left(\frac{\pi}{n+1}\right)\right),$$

where

$$a_n = \begin{cases} a_{n,0}, & \text{when } (a_{n,r})_{r=0}^n \in \text{RBVS}, \\ a_{n,n}, & \text{when } (a_{n,r})_{r=0}^n \in \text{HBVS}, \end{cases}$$

for considered x .

Some generalization of this deviation for the sequence $(a_{n,k})_{k=0}^n$ belonging to class HBVS, or RBVS respectively, is also in Krasniqi's paper [2].

THEOREM 1.2. *Let $f \in L^p(\omega)_\beta$ with $\beta < 1 - \frac{1}{p}$, and let ω satisfy (1.1), (1.2) and (1.3), where $q = p/(p-1)$. Then*

$$|T_{n,A}f(x) - f(x)| = O_x \left((n+1)^{\beta + \frac{1}{p} + 1} \sum_{k=0}^n |\Delta a_{n,k}| \omega \left(\frac{\pi}{n+1} \right) \right),$$

for considered x .

Unfortunately, this theorem does not give the best order of approximation. Recently, Saini and Singh [6] have proved the following theorem:

THEOREM 1.3. *Let f be a 2π -periodic function belonging to $\text{Lip}(\omega(t), p)$ -class with $p \geq 1$ and let $A = (a_{n,k})$ be a lower triangular regular matrix with non-negative and non-decreasing (with respect to $0 \leq k \leq n$) entries with $A_{n,0} = 1$. Then the degree of approximation of f by matrix means of its Fourier series is given by*

$$\|T_n(f; \cdot) - f(\cdot)\|_p = O \left(\frac{1}{n+1} \int_{\frac{1}{n+1}}^{\pi} \frac{\omega(t)}{t^{2+1/p}} dt \right),$$

provided $\omega(t)$ is a positive increasing function satisfying the condition

$$\int_0^v \frac{\omega(t)}{t^{1+1/p}} dt = O \left(\frac{\omega(v)}{v^{1/p}} \right), \quad \text{where } 0 < v < \pi.$$

In this paper we will present the estimations of the deviations $T_{n,A}f(\cdot) - f(\cdot)$ under general assumptions and we will show that the obtained degrees of approximations are the best for some subclasses of L^p .

We shall write $J_1 \ll J_2$, if there exists a positive constant C , sometimes depending on some parameters, such that $J_1 \leq C J_2$.

2. Statement of the results

Let us consider a function ω of modulus of continuity type on the interval $[0, 2\pi]$, that is a nondecreasing continuous function having the following properties: $\omega(0) = 0$, $\omega(\delta_1 + \delta_2) \leq \omega(\delta_1) + \omega(\delta_2)$ for any $0 \leq \delta_1 \leq \delta_2 \leq \delta_1 + \delta_2 \leq 2\pi$. It is easy to conclude that the function $\delta^{-1}\omega(\delta)$ is quasi nonincreasing function of δ , that is $\delta_2^{-1}\omega(\delta_2) \leq 2\delta_1^{-1}\omega(\delta_1)$, for $\delta_2 \geq \delta_1 > 0$. Let $L^p(\omega)_\beta = \{f \in L^p : \omega_\beta f(\delta)_{L^p} \leq \omega(\delta)\}$. It is clear that for $\beta > \alpha \geq 0$ we have $L^p(\omega)_\alpha \subset L^p(\omega)_\beta$.

Now we can formulate our main results on the degrees of pointwise approximation of function.

THEOREM 2.1. *Let $f \in L^p$, $(a_{n,k})_{k=0}^n \in \text{MHBVS} \cup \text{MRBVS}$ and $|A|_{n,\tau} = O\left(\frac{\tau}{n+1}\right)$, where $\tau = [\pi/t]$ with $\frac{\pi}{n+1} \leq t \leq \pi$. If ω satisfies the conditions*

$$(2.1) \quad \left\{ \int_0^{\frac{\pi}{n+1}} \left[\frac{|\varphi_x(t)|}{\omega(t)} \sin^\beta \frac{t}{2} \right]^p dt \right\}^{1/p} = O_x((n+1)^{-\frac{1}{p}}),$$

$$(2.2) \quad \left\{ \int_0^{\frac{\pi}{n+1}} \left[\frac{\omega(t)}{t^{\beta+1}} \right]^q dt \right\}^{1/q} = O\left((n+1)^{\beta+\frac{1}{p}} \omega\left(\frac{\pi}{n+1}\right)\right),$$

$$(2.3) \quad \left\{ \int_{\frac{\pi}{n+1}}^\pi \left[\frac{t^{-\gamma} |\varphi_x(t)|}{\omega(t)} \sin^\beta \frac{t}{2} \right]^p dt \right\}^{1/p} = O_x\left((n+1)^{\gamma-\frac{1}{p}}\right)$$

with $0 < \gamma < \beta + \frac{1}{p}$, $\beta > 0$ and $q = p(p-1)^{-1}$, then

$$|T_{n,A}f(x) - f(x)| = O\left((n+1)^\beta \omega\left(\frac{\pi}{n+1}\right)\right)$$

holds for considered x .

THEOREM 2.2. *Let $(a_{n,k})_{k=0}^n \in \text{MHBVS} \cup \text{MRBVS}$ and $|A|_{n,\tau} = O\left(\frac{\tau}{n+1}\right)$, where $\tau = [\pi/t]$ with $\frac{\pi}{n+1} \leq t \leq \pi$, and let $f \in L^p(\omega)_\beta$, ($\beta > 0$), and ω satisfy conditions (2.1), (2.2) and (2.3), with $\frac{1}{p} < \gamma < \beta + \frac{1}{p}$ and $q = p(p-1)^{-1}$. If the function $t^{-\beta}\omega(t)$, ($\beta > 0$), is nondecreasing and concave, then*

$$(n+1)^\beta \omega\left(\frac{\pi}{n+1}\right) \ll \sup_{f \in L^p(\omega)_\beta} |T_{n,A}f(x) - f(x)| \ll (n+1)^\beta \omega\left(\frac{\pi}{n+1}\right).$$

REMARK 2.1. If we consider $\omega(t) = t^\alpha$, with $0 < \beta < \alpha \leq 1 + \beta$, then $t^{-\beta}\omega(t)$ is a nondecreasing and concave function of t .

THEOREM 2.3. *Let $(a_{n,k})_{k=0}^n \in \text{MHBVS} \cup \text{MRBVS}$ with the condition $|A|_{n,\tau} = O\left(\frac{\tau}{n+1}\right)$, where $\tau = [\pi/t]$, ($\frac{\pi}{n+1} \leq t \leq \pi$). If $f \in L^p(\omega)_\beta$, ($\beta > 0$) and ω be such that (2.2) holds, with $\frac{1}{p} < \gamma < \beta + \frac{1}{p}$ and $q = p(p-1)^{-1}$, then*

$$\|T_{n,A}f(\cdot) - f(\cdot)\|_{L^p} = O\left((n+1)^\beta \omega\left(\frac{\pi}{n+1}\right)\right).$$

3. Corollaries

Finally, we give some corollaries as an application of our results.

COROLLARY 3.1. *Under the assumptions of Theorem 2.1 we can obtain a better order of approximation than these in Theorems 1.1 and 1.2.*

COROLLARY 3.2. *From Theorem 2.3, the result of Saini and Singh [6] follows with more general assumptions on the matrix A .*

4. Auxiliary results

We begin this section with some notations following Zygmund [9, Section 5, Chapter II]. It is clear that

$$S_k f(x) = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x+t) D_k(t) dt, \quad T_{n,A}f(x) = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x+t) \sum_{k=0}^n a_{n,k} D_k(t) dt,$$

where

$$D_k(t) = \frac{1}{2} + \sum_{\nu=0}^k \cos \nu t = \frac{\sin \frac{(2k+1)t}{2}}{2 \sin \frac{t}{2}}.$$

Hence

$$T_{n,A}f(x) - f(x) = \frac{1}{\pi} \int_0^\pi \varphi_x(t) \sum_{k=0}^n a_{n,k} D_k(t) dt.$$

Now, we formulate some estimates of the considered Dirichlet kernel.

LEMMA 4.1. (see [9]) *If $0 < |t| \leq \pi/2$, then $|D_k(t)| \leq \frac{\pi}{|t|}$ and for any real t we have $|D_k(t)| \leq k + \frac{1}{2}$.*

LEMMA 4.2. (see [1]) *We have*

$$\left| \sum_{k=0}^n a_{n,k} D_k(t) \right| = \begin{cases} O(t^{-1} \bar{A}_{n,n-2\tau}) & \text{if } (a_{n,k})_{k=0}^n \in \text{MHBVS}, \\ O(t^{-1} A_{n,\tau}) & \text{if } (a_{n,k})_{k=0}^n \in \text{MRBVS}, \end{cases}$$

where $\tau = [\pi/t]$, for $\frac{2\pi}{n} \leq t \leq \pi$, ($n = 2, 3, \dots$).

LEMMA 4.3. *If $t^{-\beta} \omega(t)$ is a concave and nondecreasing function of t , then the function*

$$f_0(x) = \sum_{k=1}^{\infty} \left[k^\beta \omega\left(\frac{1}{k}\right) - (k+1)^\beta \omega\left(\frac{1}{k+1}\right) \right] \cos kx$$

belongs to class $L^p(\omega)_\beta$, with $\beta > 0$.

PROOF. Proof of this lemma goes in the same way as the proof of Lemma 5 from the paper of Totik [8]. Let $\frac{\pi}{m+1} < t \leq \frac{\pi}{m}$. We have

$$\begin{aligned} |\varphi_x^0(t)| &= |f_0(x+t) + f_0(x-t) - 2f_0(x)| \\ &\leq |f_0(x) - f_0(x+t)| + |f_0(x-t) - f_0(x)| = |S_1| + |S_2|. \end{aligned}$$

So we get

$$\begin{aligned} |S_1| &\leq \left| \sum_{k=1}^m \left[k^\beta \omega\left(\frac{1}{k}\right) - (k+1)^\beta \omega\left(\frac{1}{k+1}\right) \right] (\cos kx - \cos k(x+t)) \right| \\ &\quad + \left| \sum_{k=m+1}^{\infty} \left[k^\beta \omega\left(\frac{1}{k}\right) - (k+1)^\beta \omega\left(\frac{1}{k+1}\right) \right] (\cos kx - \cos k(x+t)) \right| \\ &\leq t \left| \sum_{k=1}^m k \left[k^\beta \omega\left(\frac{1}{k}\right) - (k+1)^\beta \omega\left(\frac{1}{k+1}\right) \right] \sin k\alpha \right| + 2(m+1)^\beta \omega\left(\frac{1}{m+1}\right), \end{aligned}$$

by mean theorem, where $x < \alpha < x+t$. Using an Abel transformation for the first term, we obtain

$$\left| \sum_{k=1}^m k^2 \left[k^\beta \omega\left(\frac{1}{k}\right) - (k+1)^\beta \omega\left(\frac{1}{k+1}\right) \right] \frac{\sin k\alpha}{k} \right|$$

$$\begin{aligned}
&\leq \sum_{k=1}^{m-1} \left| k^2 \left[k^\beta \omega\left(\frac{1}{k}\right) - (k+1)^\beta \omega\left(\frac{1}{k+1}\right) \right] \right. \\
&\quad \left. - (k+1)^2 \left[(k+1)^\beta \omega\left(\frac{1}{k+1}\right) - (k+2)^\beta \omega\left(\frac{1}{k+2}\right) \right] \right| \left| \sum_{l=1}^k \frac{\sin l\alpha}{l} \right| \\
&\quad + \left| m^2 \left[m^\beta \omega\left(\frac{1}{m}\right) - (m+1)^\beta \omega\left(\frac{1}{m+1}\right) \right] \right| \left| \sum_{l=1}^m \frac{\sin l\alpha}{l} \right|.
\end{aligned}$$

We can observe that, if $(k^{-\beta}\omega(k))_{k=1}^\infty$ is a concave and nondecreasing sequence, then a sequence $b_k = k^\beta\omega\left(\frac{1}{k}\right) - (k+1)^\beta\omega\left(\frac{1}{k+1}\right)$, ($\beta > 0$), satisfies the conditions

$$(4.1) \quad 0 \leq k^2 b_k \ll k^{1+\beta}\omega\left(\frac{1}{k}\right), \quad \text{and} \quad k^2 b_k - (k+1)^2 b_{k+1} \leq 0.$$

The estimation of the expression $k^2 b_k$ follows immediately from concavity of the sequence $(k^{-\beta}\omega(k))_{k=1}^\infty$. To prove the second one we have to consider the relation (see [8, proof of Lemma 5])

$$\frac{k^2}{k^2 + (k+1)^2} \frac{1}{k} + \frac{(k+1)^2}{k^2 + (k+1)^2} \frac{1}{k+2} < \frac{1}{k+1}.$$

Using properties of the sequence $(k^{-\beta}\omega(k))_{k=1}^\infty$, we have

$$\frac{k^2}{k^2 + (k+1)^2} k^\beta \omega\left(\frac{1}{k}\right) + \frac{(k+1)^2}{k^2 + (k+1)^2} (k+2)^\beta \omega\left(\frac{1}{k+2}\right) \leq (k+1)^\beta \omega\left(\frac{1}{k+1}\right)$$

and hence

$$\begin{aligned}
&k^2 \left[k^\beta \omega\left(\frac{1}{k}\right) - (k+1)^\beta \omega\left(\frac{1}{k+1}\right) \right] \\
&\quad - (k+1)^2 \left[(k+1)^\beta \omega\left(\frac{1}{k+1}\right) - (k+2)^\beta \omega\left(\frac{1}{k+2}\right) \right] \leq 0.
\end{aligned}$$

Thus, back to the proof of Lemma 4.3, using the inequality $\left| \sum_{l=1}^m \frac{\sin l\alpha}{l} \right| \leq 4$, estimations (4.1), we get

$$\begin{aligned}
|S_1| &\leq 4t \sum_{k=1}^{m-1} \left\{ (k+1)^2 \left[(k+1)^\beta \omega\left(\frac{1}{k+1}\right) - (k+2)^\beta \omega\left(\frac{1}{k+2}\right) \right] \right. \\
&\quad \left. - k^2 \left[k^\beta \omega\left(\frac{1}{k}\right) - (k+1)^\beta \omega\left(\frac{1}{k+1}\right) \right] \right\} + 4tm^{1+\beta}\omega\left(\frac{1}{m}\right) \\
&\quad + 2(m+1)^\beta \omega\left(\frac{1}{m+1}\right) \\
&\leq 4tm^2 \left[m^\beta \omega\left(\frac{1}{m}\right) - (m+1)^\beta \omega\left(\frac{1}{m+1}\right) \right] + 4tm^{1+\beta}\omega\left(\frac{1}{m}\right) \\
&\quad + 2(m+1)^\beta \omega\left(\frac{1}{m+1}\right) \\
&\leq 8tm^{1+\beta}\omega\left(\frac{1}{m}\right) + 2(m+1)^\beta \omega\left(\frac{1}{m+1}\right) \ll t^{-\beta}\omega(t).
\end{aligned}$$

Analogously $|S_2| \ll t^{-\beta}\omega(t)$. So we have shown that

$$(4.2) \quad |\varphi_x^0(t)| \ll t^{-\beta}\omega(t).$$

Hence

$$\begin{aligned} \omega_\beta(f_0(\delta)_{L^p}) &= \sup_{0 \leq |t| \leq \delta} \left\{ \left| \sin \frac{t}{2} \right|^{\beta p} \int_0^\pi |\varphi_x^0(t)|^p dx \right\}^{1/p} \\ &\ll \sup_{0 \leq |t| \leq \delta} \left\{ \left| \frac{t}{2} \right|^{\beta p} \int_0^\pi (t^{-\beta}\omega(t))^p dx \right\}^{1/p} \ll \omega(\delta). \end{aligned}$$

Thus we have proved that f_0 belongs to $L^p(\omega)_\beta$. \square

5. Proofs of the results

PROOF OF THEOREM 2.1. Let us start with the obvious relations

$$\begin{aligned} T_{n,A}f(x) - f(x) &= \frac{1}{\pi} \int_0^{\frac{\pi}{n+1}} \varphi_x(t) \sum_{k=0}^n a_{n,k} D_k(t) dt \\ &\quad + \frac{1}{\pi} \int_{\frac{\pi}{n+1}}^\pi \varphi_x(t) \sum_{k=0}^n a_{n,k} D_k(t) dt =: I_1 + I_2 \end{aligned}$$

and $|T_{n,A}f(x) - f(x)| \leq |I_1| + |I_2|$. By the Hölder inequality ($\frac{1}{p} + \frac{1}{q} = 1$), from Lemma 4.1, (2.1) and (2.2) we get

$$\begin{aligned} |I_1| &\ll \int_0^{\frac{\pi}{n+1}} \frac{|\varphi_x(t)|}{t} dt \\ &\leq \left\{ \int_0^{\frac{\pi}{n+1}} \left[\frac{|\varphi_x(t)|}{\omega(t)} \sin^\beta \frac{t}{2} \right]^p dt \right\}^{1/p} \left\{ \int_0^{\frac{\pi}{n+1}} \left[\frac{\omega(t)}{t \sin^\beta \frac{t}{2}} \right]^q dt \right\}^{1/q} \\ &\ll (n+1)^{-\frac{1}{p}} \left\{ \int_0^{\frac{\pi}{n+1}} \left[\frac{\omega(t)}{t^{1+\beta}} \right]^q dt \right\}^{1/q} \ll (n+1)^\beta \omega\left(\frac{\pi}{n+1}\right), \end{aligned}$$

for $\beta > 0$. Using Lemma 4.2, the condition $|A|_{n,\tau} = O\left(\frac{\tau}{n+1}\right)$ and by the Hölder inequality ($\frac{1}{p} + \frac{1}{q} = 1$), for the second term we obtain

$$\begin{aligned} |I_2| &\leq \frac{1}{\pi} \int_{\frac{\pi}{n+1}}^\pi \frac{|\varphi_x(t)|}{t} |A_{n,\tau}| dt \ll (n+1)^{-1} \int_{\frac{\pi}{n+1}}^\pi \frac{|\varphi_x(t)|}{t^2} dt \\ &= (n+1)^{-1} \left\{ \int_{\frac{\pi}{n+1}}^\pi \left[\frac{t^{-\gamma} |\varphi_x(t)|}{\omega(t)} \sin^\beta \frac{t}{2} \right]^p dt \right\}^{1/p} \left\{ \int_{\frac{\pi}{n+1}}^\pi \left[\frac{\omega(t)}{t^{2-\gamma} \sin^\beta \frac{t}{2}} \right]^q dt \right\}^{1/q}. \end{aligned}$$

By quasi monotonicity of the function $t^{-1}\omega(t)$ and (2.3) we have

$$\begin{aligned} |I_2| &\ll (n+1)^{-1} (n+1)^{\gamma - \frac{1}{p}} \left\{ \int_{\frac{\pi}{n+1}}^\pi \left[\frac{\omega(t)}{t^{2+\beta-\gamma}} \right]^q dt \right\}^{1/q} \\ &\ll (n+1)^{\gamma - \frac{1}{p}} \omega\left(\frac{\pi}{n+1}\right) \left\{ \int_{\frac{\pi}{n+1}}^\pi \left[\frac{1}{t^{1+\beta-\gamma}} \right]^q dt \right\}^{1/q} \end{aligned}$$

$$\ll (n+1)^\beta \omega\left(\frac{\pi}{n+1}\right), \quad \text{for } 0 < \gamma < \beta + \frac{1}{p}.$$

Collecting these estimates, we obtain the desired result. \square

PROOF OF THEOREM 2.2. Let us fix a point x and let us consider the class $L^p(\omega)_\beta$, with $\beta > 0$, of all function $f \in L^p$ such that $\omega_\beta f(\delta)_{L^p} \leq \omega(\delta)$, ($0 \leq \delta \leq 2\pi$). In view of Theorem 2.1,

$$\sup_{f \in L^p(\omega)_\beta} |T_{n,A}f(x) - f(x)| \ll (n+1)^\beta \omega\left(\frac{\pi}{n+1}\right).$$

On the other hand, the function

$$f_0(x) = \sum_{k=0}^{\infty} \left[k^\beta \omega\left(\frac{1}{k}\right) - (k+1)^\beta \omega\left(\frac{1}{k+1}\right) \right] \cos kx,$$

by Lemma 4.3, belongs to the class $L^p(\omega)_\beta$, if $t^{-\beta}\omega(t)$ is a concave and nondecreasing function of t . Moreover, f_0 satisfies conditions (2.1) and (2.3) of Theorem 2.1. Indeed, using (4.2) we have

$$\left\{ \int_0^{\frac{\pi}{n+1}} \left[\frac{|\varphi_x^0(t)|}{\omega(t)} \sin^\beta \frac{t}{2} \right]^p dt \right\}^{1/p} \ll \left\{ \int_0^{\frac{\pi}{n+1}} \left[\frac{t^{-\beta}\omega(t)}{\omega(t)} \sin^\beta \frac{t}{2} \right]^p dt \right\}^{1/p} = O_x(n+1)^{-\frac{1}{p}}$$

and

$$\begin{aligned} \left\{ \int_{\frac{\pi}{n+1}}^{\pi} \left[\frac{t^{-\gamma}|\varphi_x^0(t)|}{\omega(t)} \sin^\beta \frac{t}{2} \right]^p dt \right\}^{1/p} &\ll \left\{ \int_{\frac{\pi}{n+1}}^{\pi} \left(\frac{t^{-\gamma}t^{-\beta}\omega(t)}{\omega(t)} \right)^p \sin^{\beta p} \frac{t}{2} dt \right\}^{1/p} \\ &\leq \left\{ \int_{\frac{\pi}{n+1}}^{\pi} t^{-\gamma p} dt \right\}^{1/p} = O_x(n+1)^{\gamma - \frac{1}{p}}, \end{aligned}$$

where γ is such that $\frac{1}{p} < \gamma < \beta + \frac{1}{p}$.

So

$$\begin{aligned} \sup_{f \in L^p(\omega)_\beta} |T_{n,A}f(x) - f(x)| &\geq |T_{n,A}f_0(x) - f_0(x)| = \left| \sum_{k=0}^n a_{n,k} (f_0(x) - S_k f_0(x)) \right| \\ &= \left| \sum_{k=0}^n a_{n,k} \sum_{l=k+1}^{\infty} \left[l^\beta \omega\left(\frac{1}{l}\right) - (l+1)^\beta \omega\left(\frac{1}{l+1}\right) \right] \cos lx \right|. \end{aligned}$$

In a special case, for $x = 0$ we get

$$\sup_{f \in L^p(\omega)_\beta} |T_{n,A}f(0) - f(0)| \geq \sum_{k=0}^n a_{n,k} (k+1)^\beta \omega\left(\frac{1}{k+1}\right) \geq (n+1)^\beta \omega\left(\frac{1}{n+1}\right).$$

When $x = x_0$ we can consider the function $f_{x_0}(\cdot) = f_0(\cdot - x_0)$ instead of $f_0(\cdot)$. Thus our proof is complete. \square

PROOF OF THEOREM 2.3. To prove this theorem we only note that the norms of the expressions on the left hand side of conditions (2.1) and (2.3) have always the

same orders as their right-hand side. Really, for the function $f \in L^p(\omega)_\beta$, ($\beta > 0$), we have

$$\begin{aligned} \left\| \left\{ \int_0^{\frac{\pi}{n+1}} \left(\frac{|\psi(t)|}{\omega(t)} \right)^p \sin^{\beta p} \frac{t}{2} dt \right\}^{1/p} \right\|_{L^p} &\leq \left\{ \int_0^{\frac{\pi}{n+1}} \frac{\|\psi(t) \sin^\beta \frac{t}{2}\|_{L^p}^p}{\omega^p(t)} dt \right\}^{1/p} \\ &\leq \left\{ \int_0^{\frac{\pi}{n+1}} \frac{\omega_\beta f(t)_{L^p}^p}{\omega^p(t)} dt \right\}^{1/p} = O((n+1)^{-\frac{1}{p}}), \end{aligned}$$

and for the second one

$$\begin{aligned} \left\| \left\{ \int_{\frac{\pi}{n+1}}^\pi \left(\frac{t^{-\gamma} |\psi(t)|}{\omega(t)} \right)^p \sin^{\beta p} \frac{t}{2} dt \right\}^{1/p} \right\|_{L^p} &\leq \left\{ \int_{\frac{\pi}{n+1}}^\pi \frac{t^{-\gamma p} \|\psi(t) \sin^\beta \frac{t}{2}\|_{L^p}^p}{\omega^p(t)} dt \right\}^{1/p} \\ &\leq \left\{ \int_{\frac{\pi}{n+1}}^\pi \frac{t^{-\gamma p} \omega_\beta f(t)_{L^p}^p}{\omega^p(t)} dt \right\}^{1/p} = O((n+1)^{\gamma - \frac{1}{p}}), \quad \text{where } \gamma > \frac{1}{p}. \end{aligned}$$

Hence our proof is complete. \square

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(Received 16 01 2018)