ABSOLUTE $|C,1|_k$ SUMMABILITY FACTOR OF IMPROPER INTEGRALS

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ABSTRACT. We introduce $|C,1|_k$ summability for improper integrals and develop a generalized theorem based on absolute Cesáro summability factor of an improper integral under sufficient conditions. We also derive some auxiliary results from the main ones.

1. Introduction

1.1. Summability factor concerning infinite series: Let $\sum_{n=0}^{\infty} a_n$ be an infinite series with sequence of partial sums $s_n = \sum_{n=0}^{n} a_n$ and σ_n be the nth Cesáro means of the series, i.e., $\sigma_n = \frac{1}{n}\sum_{k=0}^{\infty} s_k$. The series $\sum_{n=0}^{\infty} a_n$ is said to be $|C,1|_k$ summable for $k \geqslant 1$ [**3**], if $\sum_{n=1}^{\infty} n^{k-1}|\sigma_n - \sigma_{n-1}|^k < \infty$.

1.2. Summability factor concerning improper integrals: Let f be a real valued continuous function in the interval $[0,\infty)$ and $s(x) = \int_0^x f(t)dt$. Then, the improper integral $\int_0^{\infty} f(t)dt$ is said to be $|C,1|_k$ integrable for $k \geqslant 1$ [**5**], if

$$(1.1) \qquad \int_0^{\infty} x^{k-1}|\sigma'(x)|^k \, dx < \infty.$$

where $\sigma(x)$ is Cesáro mean of $s(x)$ and given by $\sigma(x) = \frac{1}{x}\int_0^x s(t)dt$. The Kronecker identity: $s(x) - \sigma(x) = v(x)$, where $v(x) = \frac{1}{x}\int_0^x tf(t) \, dt$. Condition (1.1) can be written as $\int_0^{\infty} \frac{1}{x}|v(x)|^k \, dx < \infty$.

In 1981, Parashar [**6**] worked on the (N,P_n) & $(K,1,\alpha)$ summable factors and found the minimum set of conditions for an infinite series to be $(K,1,\alpha)$ summable. Borwein and Thorpe [**1**] extended a result concerning the ordinary and absolute summability methods of integral. Patel et al. [**7**] estimated the important results on absolute summability factor for Lacunary Fourier series. Çanak and Totur [**2**] and Totur and Çanak [**8**] worked on the concept of Cesáro summability of integrals and gave very interesting results. In line with the existing studies, result of Mazhar [**4**]

has been extended in the present work with the help of some new generalized conditions and absolute Cesáro $|C, 1|_k$ summable factor for improper integrals.

2. Know results

Özgen [**5**] obtained the following results for improper integrals by considering absolute Cesáro $|C, 1|_k$ summable factors and a positive monotonic nondecreasing function $\gamma(x)$.

THEOREM 2.1. *Let $\gamma(x)$ be a positive monotonic nondecreasing function such that*

$$\lambda(x)\gamma(x) = O(1) \quad as \quad x \to \infty,$$

$$\int_0^x u|\lambda''(u)|\gamma(u)du = O(1),$$

$$\int_0^x \frac{|v(u)|^k}{u} \, du = O(\gamma(x)) \quad as \quad x \to \infty,$$

then the integral $\int_0^\infty f(t)\lambda(t)dt$ is $|C, 1|_k$ summable for $k \geqslant 1$.

3. Main results

The result of Özgen [**5**] has been extended with the help of functions $(\chi(x), \beta(x)$ and $\varepsilon(x))$ and absolute Cesáro $|C, 1|_k$ summability.

THEOREM 3.1. *Let $\chi(x)$ be a positive nondecreasing function and there be two functions $\beta(x)$ and $\varepsilon(x)$ such that*

$$|\varepsilon'(x)| \leqslant \beta(x),$$

$$\beta(x) \to 0 \quad as \quad x \to \infty,$$

$$\int_0^\infty u|\beta'(u)|\chi(u)du < \infty,$$

$$|\varepsilon(x)|\chi(x) = O(1),$$

$$\int_0^x \frac{|v(u)|^k}{u}du = O(\chi(x)),$$

then the improper integral $\int_0^\infty f(t)\varepsilon(t)dt$ is $|C, 1|_k$ integrable for $k \geqslant 1$.

NOTE: Theorem 3.1 can be proved by using the concept that $\int_0^\infty x|\beta'(x)|\chi(x)dx$ is weaker than $\int_0^\infty x|\varepsilon''(x)|\chi(x)dx$ and hence the introduction of the function $\{\beta(x)\}$ is justified.

PROOF. It may be possible to choose the function $\beta(x)$ s.t. $|\varepsilon'(x)| \leqslant \beta(x)$. When $\varepsilon'(x)$ oscillates, $\beta(x)$ may be chosen such that $|\beta(x)| < |\varepsilon''(x)|$. Hence, $\beta'(x) << |\varepsilon''(x)|$, so that $\int_0^\infty x|\beta'(x)|\chi(x)dx < \infty$ is a weaker requirement than $\int_0^\infty x|\varepsilon''(x)|\chi(x)dx < \infty$. $\qquad\square$

4. Proof of the theorem

Let $T(x)$ be the function of nth $(C, 1)$ means of the integral $\int_0^\infty f(t)\varepsilon(t)dt$. The integral is $|C, 1|_k$ integrable, if

$$(4.1) \qquad \int_0^x t^{k-1}|T'(t)|^k dt = O(1) \quad \text{as} \quad x \to \infty,$$

where $T(x)$ is given by

$$T(x) = \frac{1}{x}\int_0^x \int_0^t \varepsilon(u)f(u)du\, dt = \frac{1}{x}\int_0^x \varepsilon(u)f(u)du \int_u^x dt$$

$$= \frac{1}{x}\int_0^x (x-u)\varepsilon(u)f(u)du = \frac{1}{x}\int_0^x \left(1 - \frac{u}{x}\right)\varepsilon(u)f(u)du.$$

On differentiating both sides with respect to x, we get

$$T'(x) = \frac{1}{x^2}\int_0^x u\varepsilon(u)f(u)du = \frac{\varepsilon(x)}{x^2}\int_0^x uf(u)du - \frac{1}{x^2}\int_0^x \varepsilon'(u)\int_0^u f(t)dt\, du$$

$$= \frac{\varepsilon(x)v(x)}{x} - \frac{1}{x^2}\int_0^x u\varepsilon'(u)\left(\frac{1}{u}\int_0^u f(t)dt\right)du$$

$$= \frac{\varepsilon(x)v(x)}{x} - \frac{1}{x^2}\int_0^x u\varepsilon'(u)v(u)du = T_1(x) + T_2(x).$$

Applying Minkowski's inequality,

$$|T_n|^k = |T_1 + T_2|^k < 2^k\left(|T_1|^k + |T_2|^k\right).$$

Applying Hölder's inequality, we have

$$\int_0^x t^{k-1}|T_1(t)|^k dt = \int_0^x t^{k-1}\frac{|v(t)|^k|\varepsilon(t)|^k}{|t|^k}dt$$

$$= \int_0^x \frac{1}{t}|v(t)|^k|\varepsilon(t)|^{k-1}|\varepsilon(t)|dt$$

$$\leqslant \int_0^x \frac{|v(t)|^k}{t}|\varepsilon(t)|dt$$

$$= |\varepsilon(x)|\int_0^x \frac{|v(t)|^k}{t}dt - \int_0^x |\varepsilon'(t)|\int_0^t \frac{|v(u)|^k}{u}du\, dt$$

$$= O(1)|\varepsilon(x)|\chi(x) - \int_0^x \beta(t)\chi(t)dt$$

$$= O(1) - \int_0^\infty |\beta'(x)|dx \int_0^x \chi(u)du$$

$$\leqslant O(1) - \int_0^\infty u\chi(u)|\beta'(u)|du$$

$$= O(1) \quad \text{as} \quad x \to \infty.$$

By virtue of the hypotheses of Theorem 3.1.

$$(4.2) \quad \int_0^x t^{k-1}|T_2(t)|^k dt = \int_0^x t^{k-1}\frac{1}{t^{2k}}\left|\int_0^t u\varepsilon'(u)v(u)du\right|^k dt$$

$$\leqslant \int_0^x \frac{1}{t^2}\left(\int_0^t u^k|\varepsilon'(u)|^k|v(u)|^k du\right)\left(\frac{1}{t}\int_0^t du\right)^{k-1} dt$$

$$= \int_0^x |u\varepsilon'(u)|^{k-1}|u\varepsilon'(u)||v(u)|^k du \int_u^x \frac{dt}{t^2}$$

$$= \int_0^x |u\varepsilon'(u)||v(u)|^k\left(\frac{1}{u}-\frac{1}{x}\right)du$$

$$\leqslant \int_0^x |u\varepsilon'(u)|\frac{|v(u)|^k}{u}du$$

$$= x\varepsilon'(x)|\int_0^x \frac{|v(u)|^k}{u}du - \int_0^x (u|\varepsilon'(u)|)'\int_0^u \frac{|v(t)|^k}{t}dt\,du$$

$$= x|\beta(x)|\chi(x) - \int_0^x |\beta(u)|\chi(u)du - \int_0^x u|\beta'(u)|\chi(u)du$$

$$\leqslant \int_x^\infty u\chi(u)|\beta'(u)|du - \int_0^x |\beta(u)|\chi(u)du - O(1)$$

$$= O(1).$$

On collecting (4.1)–(4.2), we have

$$\int_0^x t^{k-1}|T'(t)|^k dt = O(1) \quad \text{as} \quad x\to\infty.$$

Hence the proof of the theorem is completed.

5. Corollaries

COROLLARY 5.1. *If $\chi(x)$ be a positive monotonic nondecreasing function such that*

$$\varepsilon(x)\chi(x) = O(1) \quad as \quad x\to\infty,$$

$$\int_0^\infty u|\varepsilon''(u)|\chi(u)du = O(1),$$

$$\int_0^x \frac{|v(u)|^k}{u}du = O(\chi(x)) \quad as \quad x\to\infty,$$

then the integral $\int_0^\infty f(t)\varepsilon(t)dt$ is $|C,1|_k$ integrable for $k\geqslant 1$.

COROLLARY 5.2. *Let $\varepsilon(x)$ be a convex function such that $\int \frac{\varepsilon(x)}{x}$ is convergent. If f is bounded $[R,\log n,1]$ with index k, then $\int_0^\infty f(t)\varepsilon(t)dt$ is $|C,1|_k$ summable.*

COROLLARY 5.3. *Let $\varepsilon(x)$ be a convex function such that $\int \frac{\varepsilon(x)}{x}$ is convergent. If f is bounded $[R,\log n,1]$, then $\int_0^\infty f(t)\varepsilon(t)dt$ is $|C,1|$ summable.*

NOTE: The above corollaries can be derived by taking the following assumptions in the main result,

(i) For corollary 5.1, take $|\varepsilon'(x)| = \beta(x)$.
(ii) For corollary 5.2, take $\chi(x) = \log(x)$ and $\varepsilon(x)$ as a convex function.
(iii) For corollary 5.3, take $\chi(x) = \log(x)$, $k = 1$ and $\varepsilon(x)$ as a convex function.

6. Conclusion

An attempt has been made to formulate the problem of generalization of absolute summable factor of improper integrals which make the system stable. The BIBO stability of the impulse response can be achieved by the condition of absolute summable of improper integrals, which is a necessary and sufficient condition, i.e.,

$$\text{BIBO stable} \iff \int_{-\infty}^{\infty} |h(x)| dx < \infty$$

By weakening the conditions and using generalized absolute summable factor, the restrictions of the filter have been reduced and the functions of the filters (like removal of unwanted frequency components, enhancement of the required frequency components, permanently unit power factor, automatic compensation, overcome of unbalancing situation, etc.) have been improved.

Further, this study has a number of direct applications in rectification of signals in digital filters like FIR filter (Finite) and IIR filter (Infinite). Summability techniques are trained to minimize the error. With the use of summability technique, the output of the signals can be made stable, bounded and used to predict the behavior of the input data, the initial situation and the changes in the complete process.

In a nut shell, the generalization of absolute summability methods is a motivation for the researchers interested in theoretical studies of improper integrals.

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